University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 6

Due 11:59pm Thursday, November 7, 2024

1. **Fun with the Frobenius-Schur indicator.**

   (a) Compute the Frobenius-Schur indicator for the **2**-dimensional irrep of S_3 using the matrices you constructed on the homework 3 problem 6. If you get zero, can you find the basis where it is real?

   (b) Consider the 2-dimensional representation of Q_8. Using the character table, decide if it is complex, real or pseudoreal.

   Write down its representation matrices in terms of Pauli matrices and check your answer.

   (c) Let $\sigma_h \equiv$ the number of square roots g of the element $h = g^2$ in the group G. Then by writing the Frobenius-Schur indicator for an irrep as

$$\eta_a \equiv \frac{1}{|G|} \sum_{g \in G} \chi_a(g^2) = \frac{1}{|G|} \sum_{h=g^2 \in G} \sigma_h \chi_a(h) \tag{1}$$

   and using character orthogonality, show that

$$\sigma_h = \sum_{\text{irreps},a} \eta_a \chi_a(h). \tag{2}$$

2. **Fun with Frobenius reciprocity.**

   (a) Take each irrep of S_3 and decompose it into irreps of $\mathbb{Z}_3 \subset S_3$ (for example using the characters).

   (b) For each of the irreps of $\mathbb{Z}_3$, construct the induced representation of S_3 (you did this already for the trivial representation) and decompose it into irreps of S_3 (for example using the characters).

   (c) Check Frobenius reciprocity.

   (d) [Bonus problem] Verify Frobenius reciprocity for $A_4 \subset S_4$.

3. **A_5 is simple.**

   (a) Show that an invariant subgroup $H \subset G$ is a union of conjugacy classes of G.

(b) [Bonus problem] Find the number of elements of the conjugacy classes n_C of A_5. The answer is $n_C = 1, 15, 20, 12, 12$.

(c) Show that A_5 is simple by checking for possible sums of n_Cs (necessarily including 1) that divide $5!/2$.

4. **Induced representations.**

(a) Check that the definition of induced representation in the notes actually produces a representation of G in the sense that $D(g_1 g_2) |n, x\rangle = D(g_1)D(g_2) |n, x\rangle$ for all $g_{1,2} \in G$ and $|n, x\rangle \in W \times V_{G/H}$.

(b) Show that the character of the induced representation $\mathrm{Ind}_H^G(W)$ is

$$\chi_{\mathrm{Ind}_H^G(W)}(g) = \mathrm{Ind}_H^G[\chi_W](g)$$

where the operation on the RHS is defined in the lecture notes.

(c) Prove Frobenius reciprocity:

$$\left\langle \psi, \mathrm{Res}_H^G[\phi] \right\rangle_H = \left\langle \mathrm{Ind}_H^G[\psi], \phi \right\rangle_G .$$