University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 8 – Solutions

Due 11:59pm Thursday, November 21, 2024

1. **Brain-warmers.**

   (a) **BCH practice.** Suppose that $[A, B] = \text{ad}_A(B) = \alpha B$. Find an expression for $\log\left(e^{-A}e^{A+B}\right)$.

   $\text{ad}_A(B) = \alpha B$ says that B is an eigenvector of the adjoint action of A, so $\text{ad}^n_A(B) = \alpha^n B$, and $e^{-t\text{ad}_A}B = e^{-\alpha t}B$. So using the BCH formula

   $$e^{-A}e^{A+B} = \mathcal{T}e^{\int_0^1 dt e^{-t\text{ad}_A}B} \tag{1}$$

   we get

   $$\log\left(e^{-A}e^{A+B}\right) = \int_0^1 dt e^{-\alpha t}B = \frac{1-e^{-\alpha}}{\alpha}B. \tag{2}$$

   A useful check on the algebra is that if $\alpha \to 0$, this becomes B.

   (b) Show that the *adjoint* representation matrices

   $$\left(T^A\right)_{BC} \equiv -\mathbf{i}f_{ABC}$$

   furnish a dim G-dimensional representation of the Lie algebra

   $$[T^A, T^B] = \mathbf{i}f_{ABC}T^C \quad .$$

   Hint: commutators satisfy the Jacobi identity

   $$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

   The structure constants f_{ABC} are part of the definition of the Lie algebra – in *any* representation, the generators satisfy $[T^A, T^B] = \mathbf{i}f^{AB}_C T^C$. This is a property of the algebra, not of any particular representation. The Jacobi identity follows from this fact, by taking the commutator of the BHS with T^D. Reshuffling this identity gives the desired equation (up to a sign which may be flipped by redefining $T \to -T$).

   (c) Show that if $(T_A)_{ij}$ are generators of a Lie algebra in some unitary representation R, then so are $-(T_A)^\star_{ij}$. Convince yourselves that these are the generators of the complex conjugate representation $\bar{R}$.

   We have $[T_A, T_B] = \mathbf{i}f_{ABC}T_C$, so $([T_A, T_B])^\star = -\mathbf{i}f_{ABC}T^\star_C$ (the structure constants are real for a unitary rep) so $[T^\star_A, T^\star_B] = -\mathbf{i}f_{ABC}T^\star_C$, so $[-T^\star_A, -T^\star_B] = \mathbf{i}f_{ABC}(-T^\star_C)$.

The representation operators in the rep R are $e^{\mathbf{i}\alpha^A T^A}$, with α^A real and T^A hermitian. In the rep $\bar{R}$, they are $e^{-\mathbf{i}\alpha^A (T^A)^\star}$, so the generators in this rep are indeed $-(T^A)^\star_{ij}$.

2. $\mathsf{so}(4)$.

   Show that $\mathsf{so}(4) = \mathsf{so}(3) \oplus \mathsf{so}(3)$.

   $J^i_\pm \equiv \frac{1}{2}\left(J^i \pm K^i\right)$ satisfy

   $$4[J^i_\pm, J^j_\mp] = [J^i, J^j] \pm [J^i, K^j] \mp [K^i, J^j] - [K^i, K^j] = \mathbf{i}\epsilon_{ijk}J^k \pm \mathbf{i}\epsilon_{ijk}K^k \pm \mathbf{i}\epsilon_{jik}K^k - \mathbf{i}\epsilon_{ijk}J^k = 0.$$

   and

   $$4[J^i_\pm, J^j_\pm] = [J^i, J^j] \pm [J^i, K^j] \pm [K^i, J^j] + [K^i, K^j] = 4\mathbf{i}\epsilon_{ijk}J^k.$$

3. **The rest of the Lie algebra in Cartan-Weyl form.**

   (a) Use the Jacobi identity to show that $|[E_\alpha, E_\beta]\rangle$ has weight $\alpha + \beta$, and hence $[E_\alpha, E_\beta] = N E_{\alpha+\beta}$ for some constant N.

   $$[H_i, [E_\alpha, E_\beta]] = -[E_\alpha, [E_\beta, H_i]] - [E_\beta, [H_i, E_\alpha]] = [E_\alpha, \beta_i E_\beta] - [E_\beta, \alpha_i E_\alpha] = (\alpha_i + \beta_i)[E_\alpha, E_\beta].$$

   (b) Can you conclude from this that if α is a root, 2α is not a root?

   $0 = [E_\alpha, E_\alpha] \propto E_{2\alpha}$. But it could just be that the coefficient of proportionality is zero.