University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 8

Due 11:59pm Thursday, November 21, 2024

1. **Brain-warmers.**

   (a) **BCH practice.** Suppose that $[A, B] = \mathrm{ad}_A(B) = \alpha B$. Find an expression for $\log\left(e^{-A}e^{A+B}\right)$.

   (b) Show that the *adjoint* representation matrices

   $$\left(T^A\right)_{BC} \equiv -\mathbf{i}f_{ABC}$$

   furnish a $\dim \mathsf{G}$-dimensional representation of the Lie algebra

   $$[T^A, T^B] = \mathbf{i}f_{ABC}T^C \quad .$$

   Hint: commutators satisfy the Jacobi identity

   $$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

   (c) Show that if $(T_A)_{ij}$ are generators of a Lie algebra in some unitary representation R, then so are $-(T_A)^\star_{ij}$. Convince yourselves that these are the generators of the complex conjugate representation $\bar{R}$.

2. $\mathsf{so}(4)$**.**

   Show that $\mathsf{so}(4) = \mathsf{so}(3) \oplus \mathsf{so}(3)$.

3. **The rest of the Lie algebra in Cartan-Weyl form.**

   (a) Use the Jacobi identity to show that $|[E_\alpha, E_\beta]\rangle$ has weight $\alpha + \beta$, and hence $[E_\alpha, E_\beta] = NE_{\alpha+\beta}$ for some constant N.

   (b) Can you conclude from this that if α is a root, 2α is not a root?