Physics 220: Symmetry in Physics
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0.1 Introductory remarks

Compromise. Historically this class is quite bimodal: when it is taught by a condensed matter physicist it is mostly about finite groups and very detailed applications to crystal structure; when it is taught by a high energy physicist it is mostly about lie groups and lie algebras.

I work on both condensed matter physics and high energy physics (or maybe neither, depending on whom you ask), and so I am going to try to find some compromise which will keep everyone happy. The subject is important enough in all fields that there is something in it for everyone.

Your input about choice of topics and emphasis is encouraged and appreciated. I am developing this course as we go, and I am happy to try to take requests.

Some motivating problems. We begin with a little motivation. Consider the following problem. Assign to each vertex of an octahedron an initial temperature. The edges of the octahedron are conductors of heat, so as time passes, the thermal energy diffuses. (More explicitly, the temperatures satisfy

$$a(t+\Delta t)_i = a(t)_i + \lambda \sum_{\langle ij \rangle} (a(t)_j - a(t)_i) \equiv a(t)_i + \lambda \left(\sum_j H_{ij} \left(a(t)_j - a(t)_i \right) \right) ,$$

where $\langle ij \rangle$ indicates the four pairs of neighbors of a site, and H_{ij} is the *adjacency matrix*

$$H_{ij} = \begin{cases} 1, & \text{if } \langle ij \rangle \text{ is a link} \\ 0, & \text{else} \end{cases} . \tag{0.1}$$

of the graph with vertices at the corners of the octahedron.) What is the final temperature and how long does it take to reach equilibrium? (If you are not impressed by the octahedron, consider instead the buckyball.) Notice that this is a question in classical physics.

Actually, to remove the mysteries of heat conduction, consider instead the following version [Kirillov, p. 269]: In the lobby of a building is a model of a cube, each face labelled with a number 1 to 6. As a practical joke, one of the workers in the building replaces each number by the average of the numbers of the neighboring faces. Suppose she does this each day for 30 days. Approximately what are the numbers on the faces at the end? And more interestingly, what is the error in your estimate?

These first two problems are the same since faces of the cube are in 1-1 correspondence with vertices of the octahedron.

Really the same problem arises in the following other classical situation: consider a collection of point masses arranged in an octahedron, and connected by springs. Find the normal modes.

And the problem is most direct in the following quantum situation: consider a particle hopping on a collection of orbitals placed at the vertices of a octahedron:

$$H = \sum_{\langle ij \rangle} |i\rangle\langle j|$$

where $|i\rangle$ is the state of a particle sitting at vertex i and $\langle ij \rangle$ indicates vertices sharing a link. Find the eigenstates of H. (Notice that the matrix elements of H in the position basis are $\langle i| H |j\rangle = H_{ij}$, the adjacency matrix defined above in (0.1).

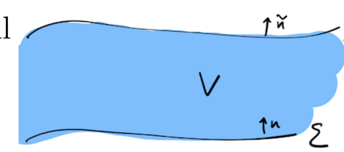
If for no other reason than making simple the answers to questions like this, it is worth learning some group theory, and more specifically the theory of group representations.

However, in addition to explaining how to use group representation theory to solve such problems, in this class I would like to try to take a broad perspective on symmetry in physics.

Let me mention some other places in physics where the ideas in this course are important, some of which we'll discuss explicitly (I'm not sure yet which). The motivating example I gave above involved six sites; when there are instead a macroscopic number of sites, it becomes even more important to try to decompose the Hamiltonian into blocks using symmetries. Group theory is extremely useful in thinking about the physics of crystalline solids – they are complicated, but by definition have a lot of symmetry, which we must use to our advantage to understand them. It is often useful for solving statistical mechanics problems, which can sometimes reduce to combinatorics. Group theory (as opposed to symmetry) is a crucial ingredient in the construction of the Standard Model of particle physics and its possible extensions to higher energies. It is extremely useful for taking advantage of simplifications of physics happening in symmetric spacetimes, such as euclidean space (a good approximation for physics happening slowly enough on short-enough length scales), or Minkowski spacetime (a good approximation for physics happening on short-enough length scales), or de Sitter spacetime (a good approximation for physics happening on cosmological length scales). It is the basis for a classification of phases of matter and transitions between them called the Landau Paradigm. Symmetry can provide rare instances of exact statements about strongly-coupled systems of many degrees of freedom ("strongly-coupled" means that there is no small parameter in which to perturbate) in the form of *anomalies* (or sometimes 't Hooft anomalies); here is a simply-stated example called the Lieb-Schulz-Mattis theorem: in a chain of spin one-half degrees of freedom with spin rotation symmetry and translation symmetry, there cannot be a unique gapped groundstate – the system must either be gapless or break a symmetry.

Recently we have learned that sometimes the study of symmetry in physics even goes beyond group theory (I have in mind *e.g.* this and this). Let me say a few high-level words about this before we start from scratch.

I would like to try to stress the close connection between symmetry and topology. The basic idea is the following. Noether's theorem relates continuous symmetries to conserved quantities. Think about the notion of a conserved quantity in a spacetime picture. Usually[1] the thing that's conserved can be written as an integral over a spatial slice of a local density, $Q = \int_\Sigma n_\mu j^\mu$, where Σ is some codimension-one (that means I have to specify one coordinate, locally, to determine a point on Σ) surface, and n^μ is its normal vector. (The simplest case is when the normal is just ∂_t and $Q = \int_{\text{space}} j^0$.) That Q is conserved is the statement that it does not depend on the choice of Σ – I can smoothly deform the surface to a later time (or even only locally to a later time), and I still get the same answer for the integral



$$\int_{\tilde{\Sigma}} \tilde{n}_\mu j^\mu - \int_\Sigma n_\mu j^\mu = \int_{\tilde{\Sigma}-\Sigma} n_\mu j^\mu = \int_{\partial V} n \cdot j = \int_V \partial_\mu j^\mu = 0.$$

(In the second step we assumed there is no current flowing out at infinity.) At the last step we used current conservation $\partial_\mu j^\mu = 0$. So we can deform Σ as much as we want, as long as we never cross a source or sink of charge, where $\partial_\mu j^\mu \neq 0$. Such a source would act a like a hole in spacetime around which we cannot deform our surface. The conclusion is that Q is a *topological invariant*.

In the case of a quantum field theory, Q is an operator. It is an operator associated with a codimension-one surface Σ which only depends on the topological class of Σ in the spacetime minus the set of points where charge can appear or disappear. It is tempting then to streamline Noether's theorem to say that symmetries are associated with "topological surface operators". (Recently people have found that sometimes such

[1]This word is always dangerous in physics and I apologize for using it in my haste to quickly get to the point in this discussion. There are two dangers here. (1) The first is that here I am talking about systems with degrees of freedom spread over space, like in a material, or like the electromagnetic field in vacuum, or more generally a *field theory*. Quantum field theory (QFT) is not a prerequisite for this class, and I will not assume sophisticated knowledge of QFT. You all know about electricity and magnetism, though, and I expect that you will have seen some other context where degrees of freedom are spread over space, such as inside a magnet. (2) More generally saying that something is 'usually' true is dangerous because it implies some kind of measure on the space of physics situations and because it means that it's not always true. Here the extra assumption is that the theory is local, for example that it is defined by an action which is a single integral over space and time. More on this in a bit.

topological surface operators aren't associated with groups! In particular, when we try to multiply them, rather than $ab = c$ we get something like $ab = c + d$!)

Noether's theorem also has a converse: given a topological surface operator Q (assume it's hermitian or replace it with $Q + Q^\dagger$), it generates a symmetry: $U = e^{\mathbf{i}\alpha Q}$ is a unitary operator which commutes with the time evolution operator.

This idea has some natural generalizations. For example, what about topological operators associated with codimension-p surfaces, X_{D-p}? Such a thing would come from a conserved current with $p+1$ indices, antisymmetrized – a $p+1$-form:

$$\int_{X_{D-p}} j_{\nu_1 \cdots n_p} n_1^{\mu_1} \cdots n_p^{\mu_p} = \int_{X_{D-p}} \star j \tag{0.2}$$

($n_1 \cdots n_p$ are the normals to X_{D-p}) satisfying $\partial^\mu j_{\mu\nu_1\cdots n_p} = 0$. A field theory with such an object is said to have a p-form symmetry. (In the second step in (0.2) I've used the Hodge star $(\star j)_{\mu_1 \cdots \mu_{D-p}} = \epsilon_{\mu_1 \cdots \mu_D} j^{\mu_{D-p+1} \cdots \mu_D}$. and differential form notation. Don't worry if this is unfamiliar right now.) What we've thought of as symmetry all along is the special case called 0-form symmetry.

This appearance of differential forms reminds me to mention the following. There are many other connections between groups and topology which are relevant to physics. An important one is the use of *groups* themselves as topological invariants. In the quest for topological invariants of various geometric data, such as the shape of space, or a space of maps from one space into another, many of them turn out to be themselves groups. Here I have in mind homology groups, cohomology groups, homotopy groups, cobordism groups. These topological invariants are useful in (parts of) physics all the time (some examples which spring to mind are: when classifying phases of matter, when identifying topological defects in a given phase of matter, when compactifying string theory), so I feel pretty justified talking about some of them in a class about group theory in physics! We'll see how far we get.

Group theory is useful in field theory in many ways. A common situation in field theory is: we have a list of degrees of freedom, and a symmetry group, and we would like to know what interactions are allowed.

Backing up a little, what is a field theory? It's a physical system where the degrees of freedom are fields $\phi(x, t)$ – functions of space and time, which may have some extra decorations like vector indices. The interactions are governed by an action of the form

$$S[\phi] = \int d^d x dt \mathcal{L}(\phi(x), \partial\phi(x)).$$

This is not the most general functional of ϕ; rather, I have assumed the property of *locality*. A good example to keep in mind (where we know the answer) is Maxwell's theory, where

$$S[A_\mu] = \int d^3x dt \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right), \quad \vec{B} \equiv \vec{\nabla} \times \vec{A}, \vec{E} \equiv -\partial_t \vec{A} + \vec{\nabla} A_0.$$

(If we regard the variables as quantum mechanical, then it is a quantum field theory.)

A common situation where field theory arises is in studying a chunk of matter, like a magnet. A magnet is a material whose hamiltonian preserves some subgroup G of the spin rotation symmetry. The magnetization $\vec{\phi}(x,t)$ is a local property of the material which says which way (if any) the spins are pointing near the spacetime point (x,t), so it is a field. It transforms under the spin rotations (and other symmetries, such as spatial symmetries, like symmetries of the lattice making up the material) in a definite way. This constrains the form of the action for $\vec{\phi}$.

A similar situation obtains in particle physics. There we discover in various ways particles that are the quanta of fields (as the photon is the quantum of the EM field A_μ), and carry various symmetry properties – which we infer from which reactions we see and which ones don't happen. A priori we don't know how they interact with each other, and we would like to write an action for these fields; constraints from symmetries are indispensable.

0.2 Conventions

For some of us, eyesight is a valuable commodity. In order not to waste it, I will often denote the Pauli spin operators by

$$\mathbf{X} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{Y} \equiv \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} \quad \mathbf{Z} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(rather than $\sigma^{x,y,z}$).

$\equiv$ means 'equals by definition'. $A \stackrel{!}{=} B$ means we are demanding that $A = B$. $A \stackrel{?}{=} B$ means A probably doesn't equal B.

The convention that repeated indices are summed is always in effect unless otherwise indicated.

A useful generalization of the shorthand $\hbar \equiv \frac{h}{2\pi}$ is

$$đk \equiv \frac{\mathrm{d}k}{2\pi}.$$

I will also write $\not{\delta}(q) \equiv (2\pi)^d \delta^d(q)$.

I try to be consistent about writing Fourier transforms as

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{ikx} \tilde{f}(k) \equiv \int đ^d k \; e^{ikx} \tilde{f}(k) \equiv f(x).$$

WLOG $\equiv$ without loss of generality.
IFF $\equiv$ if and only if.
RHS $\equiv$ right-hand side. LHS $\equiv$ left-hand side. BHS $\equiv$ both-hand side.
IBP $\equiv$ integration by parts.
$+\mathcal{O}(x^n) \equiv$ plus terms which go like x^n (and higher powers) when x is small.
iid $\equiv$ independent and identically distributed.

We work in units where $\hbar$ and k_B are equal to one unless otherwise noted.

Please tell me if you find typos or errors or violations of the rules above.

0.3 Sources

Zee, *Group Theory in a Nutshell*.

Georgi, *Lie Algebras in Particle Physics*, editions 1 & 2. Particle-physics centric, but very clear and worthwhile. The first edition has no finite groups.

Stone and Goldbart, *Mathematics for Physics, a guided tour for graduate students*. The discussion of group theory is in two very concise chapters – 74 pages long, total. For our purposes, it takes a too-geometric approach in the bit about Lie groups. It has a good list of examples, balanced in application.

Fulton and Harris, *Representation Theory, A First Course*. This is a real math book, but it is quite accessible and clear and full of examples. (It is a Springer Yellow Book, so you can get it electronically through the UCSD library at the link above.)

Ramadevi and Dubey, *Group Theory for Physicists*. Most applications are to chemistry, but it is clear and concise.

W. K. Tung, *Group Theory in Physics*. Lots of good stuff about spatial symmetries, and about the relations between special functions and noncompact Lie groups.

Brian Hall, *Lie Groups, Lie Algebras, and Representations*. A math book which is usefully focused on matrix lie groups.

Kirillov, *Elements of the Theory of Representations*. A bit heavier mathematically.

Cvitanovic, *Group Theory*. Birdtracks!

1 Groups

1.1 Basic notions

Definition of a group: a group G is a set of elements with a multiplication law,
$\cdot : \begin{array}{rcl} G \times G & \to & G \\ (g_1, g_2) & \mapsto & g_2 \cdot g_1 \end{array}$ and the following properties

1. The product is **associative**: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.
2. There is an **identity** element e: $e \cdot g = g = g \cdot e$ for all $g \in G$.
3. Every element g has an **inverse**: $g^{-1} \cdot g = e$, $g \cdot g^{-1} = e$.

Associativity is inevitable if we regard the elements of the group as transformations acting on some set of objects: at any intermediate step of the composition of operations, there is a well-defined object.

The order of operations may matter: $g_1 \cdot g_2 \stackrel{?}{=} g_2 \cdot g_1$. If all the products commute, the group is called *abelian*, else *non-abelian*. Notice in thinking about the group as a set of operations and the product as composition, we have to choose a convention for the order of composition. If we act first with g_1 and then with g_2, we will write $g_2 \cdot g_1$. As a perhaps-counterintuitive consequence of the fact that we write from left to right, time goes to the left.

The number of elements, $|G|$, is called the *order* of the group. It may be finite or infinite. The elements may or may not be countable, and the labels on the group elements may be discrete or continuously variable. Furthermore, an infinite group may be *compact* or non-compact. The useful definition of compact for our purposes is: a group G is compact if we can sum (integrate) over the elements in a G-invariant way and get a finite answer. For discrete groups this is $\sum_{g \in G} 1$; for infinite groups we require a G-invariant integration measure. You'll see why we care about summing the elements so much soon.

I will now stop writing the dot in the multiplication law: $g \cdot h \equiv gh$.

On the homework you'll show that we gain nothing by relaxing the demand that the left inverse is the same as the right inverse, or the demand that the 'left identity' $ge = e$ is the same as the 'right identity' $eg = e$.

A finite group can be specified by its multiplication table. For example, here is the group Q_8 of quaternions, a group with $|Q_8| = 8$ elements, defined by the rules

$\mathbf{i}^2 = \mathbf{k}^2 = \mathbf{j}^2 = -1, \mathbf{ij} = \mathbf{k}$ and its images under $\mathbf{i} \to \mathbf{j} \to \mathbf{k}$:

Q_8	1	-1	$\mathbf{i}$	$-\mathbf{i}$	$\mathbf{j}$	$-\mathbf{j}$	$\mathbf{k}$	$-\mathbf{k}$
1	1	-1	$\mathbf{i}$	$-\mathbf{i}$	$\mathbf{j}$	$-\mathbf{j}$	$\mathbf{k}$	$-\mathbf{k}$
-1	-1	1	$-\mathbf{i}$	$\mathbf{i}$	$-\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{i}$	-1	1	$\mathbf{k}$	$-\mathbf{k}$	$-\mathbf{j}$	$\mathbf{j}$
$-\mathbf{i}$	$-\mathbf{i}$	$\mathbf{i}$	1	-1	$-\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{j}$	$-\mathbf{k}$	$\mathbf{k}$	-1	1	$\mathbf{i}$	$-\mathbf{i}$
$-\mathbf{j}$	$-\mathbf{j}$	$\mathbf{j}$	$\mathbf{k}$	$-\mathbf{k}$	1	-1	$-\mathbf{i}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$-\mathbf{k}$	$\mathbf{j}$	$-\mathbf{j}$	$-\mathbf{i}$	$\mathbf{i}$	-1	1
$-\mathbf{k}$	$-\mathbf{k}$	$\mathbf{k}$	$-\mathbf{j}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{i}$	1	-1

To find ab, look at the row with a and the column with b. Notice that $ab \neq ba$ in general, so this group is non-abelian.

Sudoku rule. $ax = bx$ implies $a = b$. This is just because x is always invertible. This means that each element of the group must appear exactly once in each row and each column of the multiplication table.

1.2 Examples of groups and where they come from

Equivalence of groups. Two groups are equivalent (isomorphic) if there is a map between their group elements that preserves the multiplication rule (this is called a group homomorphism) which is bijective ($\equiv$ one-to-one and onto).

In this subsection I am going to give a bunch of examples of groups, and examples of ways that groups are defined. Sometimes (very) different definitions lead to isomorphic abstract groups.

Integers $\mathbb{Z}$ under addition. $(k, l) \mapsto k + l$. 0 is the identity. This group is discrete and infinite. (Integers under multiplication is not a group because 0 has no multiplicative inverse.)

Integers modulo n under addition. $(k, l) \mapsto (k+l)_n$. (By $(a)_n$ I mean a modulo n.) This group, called $\mathbb{Z}_n$, is discrete and finite. We can make the product look more like multiplication by letting $\omega = e^{2\pi \mathbf{i}/n}$ be an nth root of unity and identifying k mod n with ω^k: $\omega^k \omega^l = \omega^{k+l}$. Multiplying a complex number by ω^k is a rotation in the complex plane by an angle $2\pi k/n$.

Infinite not-discrete groups. By "not-discrete" here I mean that the elements can be labelled by a continuously variable parameter – a real number.[2]

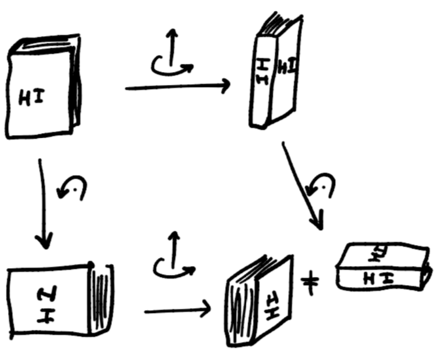
[2] In a previous version of these notes I used the term "continuous group" here. This was a mistake,

We can make an infinite, not-discrete group by taking $n \to \infty$: $\{e^{\mathbf{i}\phi}, \phi \in [0, 2\pi)\} = \mathsf{U}(1)$. Multiplication by such a phase makes continuous rotations in the complex plane; decomposing into real and imaginary parts, $e^{\mathbf{i}\phi} = \cos\phi + \mathbf{i}\sin\phi$, this is the group of rotations in 2d, $\mathsf{SO}(2)$. $\mathsf{U}(1)$ is infinite and not-discrete and compact, since $\int_0^{2\pi} d\phi = 2\pi$.

Another infinite not-discrete group is $\mathbb{R}$ under addition. This is infinite and not-discrete but unlike $\mathsf{U}(1)$, it is non-compact.

An infinite not-discrete group which is also a smooth manifold, and whose product is also smooth enough, is called a *Lie group*. So $\mathsf{U}(1) = \mathsf{SO}(2)$ and $\mathbb{R}$ are our first examples of Lie groups. We will not think about not-discrete groups which are not Lie groups[3].

So far all the examples (besides Q_8) are abelian. A non-abelian example is the group of rotations in $\mathbb{R}^3$, $\mathsf{SO}(3)$ (infinite, not discrete, compact). A simpler non-abelian example (finite, discrete) is the subgroup of rotations with angle $\pi/2$. This is the group of rotational symmetries of the cube, and has 12 elements; it is sometimes called O (for octahedral) but we will learn to call it A_4.



To see that O is not abelian, consider its action on a book at right. Compare the result of successive $\pi/2$ rotations about $\hat{z}$ and $\hat{x}$ versus the opposite order of operations. Notice that a book is a good choice to illustrate this because no elements of O map the book to itself.

[End of Lecture 1]

New groups from old. Given two groups, we can make another, the tensor product: $G_1 \times G_2$ is

$$\{(g_1, g_2), g_1 \in G_1, g_2 \in G_2\}, \text{ with product } (g_1, g_2) \cdot (g_1', g_2') = (g_1 g_1', g_2 g_2'). \tag{1.1}$$

since this name is sometimes used (as a synonym for "topological group") to mean a topological space with a group action which is continuous (with respect to the given topology). Annoyingly, if we choose the discrete topology (declare that every set is open), then even finite groups are "continuous groups" by this definition. The distinction I am trying to make is between discrete groups and groups with uncountably many elements. Of the latter (in general, scary) class we will only talk about Lie groups.

[3]I haven't actually defined what I mean by a smooth manifold and I don't want to. It means that locally it looks like $\mathbb{R}^n$ for some n (the dimension) and that we can do calculus all we want.

The question of to what extent we can relax the smoothness assumption (and still end up with the same set of examples) is the subject of Hilbert's 5th problem, and the answer is 'yes, to a large extent'; see here.

A subset H of a group G which is also a group (under the product in G) is, naturally, a *subgroup*. We write this $H \subset G$, overloading the subset notation by context. For example, for each element $g \in G$ of a finite group, $\{e, g, g^2 \cdots g^k\}$ is a subgroup isomorphic to $\mathbb{Z}_k$ for some k.

Matrix Lie groups. Consider the set of $n \times n$ matrices M with $\det M = 1$, under matrix multiplication. This is a group because $\det M_1 M_2 = \det M_1 \det M_2$. This is $\mathsf{SL}(n, F)$, where $F = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ is the ring where the matrix entries live. If we don't demand that $\det M = 1$, but just that M is invertible, we get the group $\mathsf{GL}(n, F)$. These are non-compact, since the set of solutions of these conditions runs off to infinity.

$\mathsf{O}(n)$ is defined as the group of $n \times n$ matrices O with real entries satisfying $O^T O = 1\!\!1$, aka

$$O^i{}_k \delta_{ij} O^j{}_l = \delta_{kl}. \tag{1.2}$$

This condition means that under $v^i \mapsto O^i{}_j v^j$, O preserves the length of vectors:

$$\| v \|^2 = v^T v \mapsto v^T O^T O v = v^T v = \| v \|^2, \quad \text{or} \quad \| v \|^2 = v^i \delta_{ij} v^j \mapsto v^k O^i{}_k \delta_{ij} O^j{}_l v^l \overset{(1.2)}{=} \| v \|^2 .$$

If we further demand that $\det(O) = 1$, we get the group $\mathsf{SO}(n)$ of rotations in $\mathbb{R}^n$. $\mathsf{O}(n)$ and $\mathsf{SO}(n)$ and infinite, continuous, compact. The difference is that $\mathsf{O}(n)$ contains reflections, like $(x, y, z) \to (-x, y, z)$ which can have determinant -1. The elements with determinant -1 are not continuously connected to those with determinant $+1$, but are all connected to each other by rotations (note that $(x, y, z) \to (-x, -y, z)$ is a rotation), so $\mathsf{O}(n)$ has two connected components (each of which has the structure of $\mathsf{SO}(n)$).[4]

[4] You may wonder whether $\mathsf{O}(n)$ was then the same as $\mathsf{SO}(n) \times \mathbb{Z}_2$, where $\times$ is tensor product, defined in (1.1). They are certainly the same as sets, since we've just seen that $\mathsf{O}(n) = \mathsf{SO}(n) \dot{\cup} P \mathsf{SO}(n)$ where $\dot{\cup}$ is disjoint union, and P is an element with $\det P = -1$. (If $G = \{g_1, g_2 \cdots\}$, then by hG I mean the set of elements $\{hg_1, hg_2 \cdots\}$ – a coset.) However, in order for them to be isomorphic as groups, we require that the element P commute with everybody else (that $P \in Z(\mathsf{O}(n))$, the *center* of $\mathsf{O}(n)$). For odd n, an element with determinant -1 is just an overall reflection (like $(x, y, z) \to -(x, y, z)$) which does commute with everyone. For even $n > 2$, an element with determinant -1 is something like $(x, y, z, w) \to (-x, y, z, w)$ which does not commute with rotations.

For even n, $\mathsf{O}(n)$ is isomorphic to a *semi-direct product* of $\mathsf{SO}(n)$ and $\mathbb{Z}_2$ – in symbols, $\mathsf{O}(n) \simeq \mathbb{Z}_2 \ltimes \mathsf{SO}(n)$. $G_1 \ltimes G_2$ is not a well-defined operation like the tensor product (1.1); it merely connotes that the *set* of elements is the same as $G_1 \times G_2$, but the action of the second factor does not commute with the action of the first. In this case, the product that reproduces the $\mathsf{O}(n)$ group law is

$$((-1)^{p_1}, M_1) \cdot ((-1)^{p_2}, M_2) = ((-1)^{p_1 + p_2}, P^{-p_2} M_1 P^{p_2} M_2). \tag{1.3}$$

(To see this, write any $g \in \mathsf{O}(n)$ in a canonical form $g = P^p M$, with $p = 0, 1$ and $\det M = 1$. This corresponds to the element $((-1)^p, M)$ in $\mathbb{Z}_2 \ltimes \mathsf{SO}(n)$. Then the product of two elements in $\mathsf{O}(n)$ is

$$g_1 g_2 = P^{p_1} M_1 P^{p_2} M_2 = P^{p_1} P^{p_2} P^{-p_2} M_1 P^{p_2} M_2$$

Consider instead $(n+m)\times(n+m)$ matrices $L^\mu{}_\nu$ with real entries satisfying instead of (1.2) the relation $L^T\eta L = \eta$, aka $L^\mu{}_\rho \eta_{\mu\nu} L^\nu{}_\sigma = \eta_{\rho\sigma}$, where $\eta_{\mu\nu} = \begin{pmatrix} \mathbb{1}_{n\times n} & 0 \\ 0 & -\mathbb{1}_{m\times m}\end{pmatrix}$. This is $\mathsf{O}(n,m)$; for $n=1, m=d$, this is the Lorentz group in d space dimensions. For $n>0, m>0$, $\mathsf{O}(n,m)$ is infinite, continuous, but non-compact, since the condition is like the equation for a hyperbola.

Here's another important family of matrix groups, which plays a starring role in quantum mechanics. Consider the group of transformations $|\psi\rangle \to U|\psi\rangle$ on an n-dimensional Hilbert space that preserve the inner product:

$$\langle\phi|\psi\rangle \stackrel{!}{=} \langle\phi| U^\dagger U |\psi\rangle \quad \forall \phi, \psi.$$

This requires $U^\dagger U = \mathbb{1}$, which condition defines the unitary group $\mathsf{U}(n)$ (which is infinite, continuous, compact). (If we pick a basis of our Hilbert space, $\mathcal{H} = \text{span}\{|i\rangle, i = 1..N\}$, then the matrix elements of the unitary operator U, $U_{ij} \equiv \langle i| U |j\rangle$ comprise a unitary matrix.)

The final class of matrix groups may be familiar from classical mechanics: The symplectic groups are $\mathsf{Sp}(n) = \{M | M^T \epsilon M = \epsilon\}$, where ϵ is an antisymmetric matrix, like the Poisson bracket which pairs canonical coordinates (q,p) on phase space.

You might think this list could go on forever. Remarkably, it is possible to classify compact Lie groups into just these infinite classes plus a set of 5 sporadic ('exceptional') groups $E_{6,7,8}, F_4, G_2$ (and tensor products (see (1.1)) of these). For a little while we will focus on finite groups.

The symmetric group, S_n. The set of permutations of n objects forms a group called S_n. One way to denote the permutation that takes the list $\{1,2,\cdots,n\}$ to the list $\{\pi_1,\pi_2,\cdots,\pi_n\}$ is $\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n\end{pmatrix}$. Each column, read from top to bottom, indicates where each element ends up. In this notation, the identity is $e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n\end{pmatrix}$, and the inverse permutation of π above is $\pi^{-1} = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \\ 1 & 2 & \cdots & n\end{pmatrix}$. The order of S_n is $|S_n| = n!$.

Imagine following a particular entry in the list through repeated actions of (the same) π. After at most n operations, the entry comes back to its original position. The set of entries it is mapped into during this journey, its orbit under π^k, is called a

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where in the last step we put it back in the canonical form.) (1.3) is only the same as (1.1) if $P^{-p_2} M_1 P^{p_2} = M_1, \forall M_1 \in \mathsf{SO}(n)$. But as we've seen, there is no such P which commutes with all the elements of $\mathsf{SO}(n)$ for even n.

cycle. Each entry appears in exactly one cycle. An alternative notation for permutation instead indicates the cycles, grouped by parentheses:

$$e = (1)(2)\cdots(n), \quad \begin{pmatrix} 1\ 2\ 3\cdots \\ 2\ 1\ 3\cdots \end{pmatrix} = (12), \quad \begin{pmatrix} 1\ 2\ 3\ 4\ 5\cdots \\ 3\ 4\ 5\ 2\ 1\cdots \end{pmatrix} = (135)(24).$$

Notice that we can omit cycles of length one. When I say 'the cycle decomposition' of a permutation, I will mean a decomposition into non-overlapping ('disjoint') cycles which share no elements.

Every permutation can be written as a product of (possibly-overlapping) two-cycles. This is just because we can achieve any rearrangement by a series of interchanges of pairs. For example (just to clarify the notation: on the left and far right I am using cycle notation, the second expression is the notation we introduced first which displays the input and output of the permutation on the top and bottom row, and in the third expression I extend that notation in a natural way to include an intermediate step along the way from top to bottom),

$$(132) = \begin{pmatrix} 1\ 2\ 3 \\ 3\ 2\ 1 \end{pmatrix} \begin{pmatrix} 1\ 2\ 3 \\ 1\ 3\ 2 \end{pmatrix} = \begin{pmatrix} 1\ 2\ 3 \\ 1\ 3\ 2 \\ 3\ 1\ 2 \end{pmatrix} = (13)(32).$$

Notice that this expression is *not* the cycle decomposition of the group element, since 3 appears in two different cycles. Here are some simple rules for composition of permutations:

- Cyclic permutations within a cycle don't change the cycle: $(12) = (21)$ and $(123) = (231) = (312)$ (but not (132)).
- Exchanges square to the identity $(12)(12) = e$.
- $(12)(23) = (123)$ – a product of overlapping two-cycles gets joined at a shared element.
- $(12)(234) = (1234)$ – the same is true for longer cycles.
- $(12)(34) = (34)(12)$ – non-overlapping cycles commute.

Embedding in S_n. Here's a reason that S_n is an important family of groups. What I've called the Sudoku rule above (sometimes grandiosely called the 'rearrangement lemma') implies that any group G of order n is equivalent to a subgroup of S_n. Here's why: put the elements of G in some arbitrary order: $(g_1, g_2, \cdots g_n)$. Multiplication by any element h acts on this list by $(hg_1, hg_2, \cdots hg_n) = (g_{\pi_1}, g_{\pi_2} \cdots g_{\pi_n})$. But by the Sudoku rule this is the same set in a different order – a permutation $\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$. This fact (due to Cayley) misled mathematicians into overemphasizing the importance of S_n and held back the study of representation theory by decades. But it does have some simple useful consequences.

Perhaps-surprising claim: The π we get from the construction in the previous paragraph is not an arbitrary permutation. The cycle lengths in the permutation π must all be the same. Suppose the contrary, *e.g.* $\pi = (123)(45)$. Then $\pi^2 = (132)(4)(5)$. So π^2 would have cycles of length 1. But this means that some element of G is mapped to itself by multiplication by h^2 (not the identity) which is forbidden by the Sudoku rule. The same would happen if π had cycles of length l_1, l_2; if $l_1 \neq l_2$ then π^{l_2} violates the Sudoku rule.

A consequence of this is that if the order of the group is a prime number p, it is isomorphic to the subgroup of S_n generated by $(1 \cdots p)$, the cyclic permutation. This shows that it is equivalent to $\mathbb{Z}_p$, the cyclic group.

Point groups. A big source of groups and even moreso of nomenclature is the discussion of symmetries of lattices and 3d objects like platonic solids. They have intimidating names like C_{3v} and $\mathsf{T}, \mathsf{O}, \mathsf{I}$ which I can never remember and don't approve of. These names refer to their definitions as symmetries of particular objects, but they are in fact isomorphic to ordinary things like $\mathbb{Z}_n$ and S_n and their products.

Fundamental group of a topological space. Consider a topological space, X, that is, a set of points with some notion of which points are next to each other. We can associate a group ($\pi_1(X)$, its fundamental group, or first homotopy group) to it as follows: Consider continuous maps f from the circle S^1 into X that start and end at some 'base point' f_0. The image is a loop in X. The product of f and g is defined by appending f to g: parametrize the circle as $\theta \in [0, 2\pi)$,

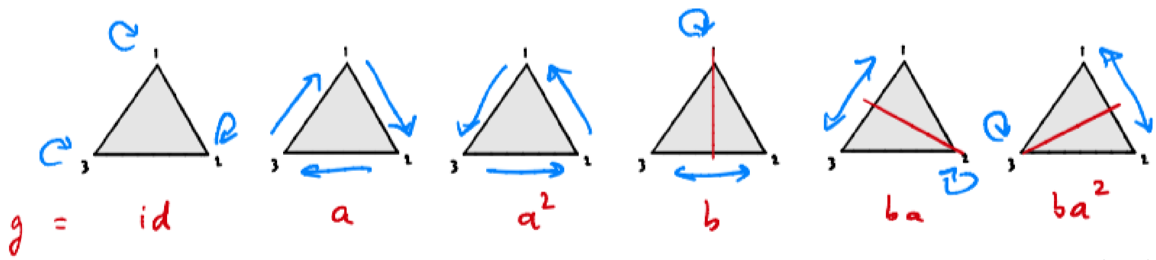
$$f \cdot g(\theta) \equiv \begin{cases} f(2\theta), & \theta < \pi \\ g(2(\theta - \pi)), & \theta > \pi \end{cases}.$$

One more step: we regard two loops as equivalent if they can be continuously deformed into each other through other loops (in which case they are said to be *homotopic*). $\pi_1(X)$ is the group of equivalence classes. The identity of the group is the constant map $f(\theta) = f_0$. The inverse of f is just f run backwards, $f^{-1}(\theta) = f(2\pi - \theta)$, which unwinds the loop. For example, $\pi_1(S^1) = \mathbb{Z}$, where the integer is the winding number. A less trivial example is $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$. $\mathbb{RP}^2$ can be defined as the disk with antipodal points identified[5]. A path that goes from one side to the other cannot be deformed

[5] A more physical definition is: the configuration space of a two-headed arrow in three dimensions

to nothing, but a path which does this twice can be deformed away. (Why doesn't $\pi_1(X)$ depend on the choice of base point? If we can find a path γ in X from x_0 to x_1, then any loop α with base point x_0 ($[\alpha] \in \pi_1(X, x_0)$) can be mapped to the the loop $\gamma\alpha\gamma^{-1}$, which gives a representative in $\pi_1(X, x_1)$, so $\pi_1(X)$ is independent of the choice of base point if X is *path-connected*.) I've tried to be very brief here; for a lot more on this huge subject see section 8 of these great notes by Justin Roberts or chapter 1 of Hatcher.

Generators and relations. Writing out the group multiplication table quickly gets tiresome as $|G|$ grows. A more compact way to present a group $G = \langle g_1, g_2 ... | r_1(g), r_2(g)...\rangle$ is through its generators g_i and relations r_a . For example, $\mathbb{Z}_n = \langle a|a^n = e\rangle$, and $S_3 = \langle r, s|r^3 = e, s^2 = e, rs = sr^2\rangle$, where $r = (123)$ and $s = (12)$. This expression means the group made from all powers and products of the elements before the | (and their inverses and the identity), subject to the relations after the |. A new example is the dihedral group, $D_n = \langle a, b|a^n = e, b^2 = e, (ab)^2 = e\rangle$. This is the symmetry group of the regular n-gon. The first two relations are there because a is a $2\pi/n$ rotation and b is a reflection. The third relation can be rewritten as $bab^{-1} = a^{-1}$, which expresses the fact that a rotation viewed in the mirror goes in the opposite direction. Here it[6] is for $n = 3$:



(1.4)

in which case it is the same as S_3.

This way of describing a group is called a *presentation* of the group, and is sometimes much more efficient. For example, if there are k generators but no relations at all, the group is called the *free group* on k generators.

Consider this example:

$$\Gamma_{\text{English}} \equiv \langle a, b, c, \cdots, z|A = B \text{ if the words } A \text{ and } B \text{ are homophones}\rangle.$$

Here the multiplication law is just concatenation of the 26 generators. Any non-native English speaker will not be surprised to hear that the authors of the linked paper

(of fixed position). A two-headed arrow is the order parameter (called a director field) for a nematic phase (*e.g.* of a liquid crystal, like a liquid of little rods which line up with each other).

[6]Actually what I am drawing here is the *group action* of D_3 on the vertices of the equilateral triangle. More on this notion and its utility in §1.4.

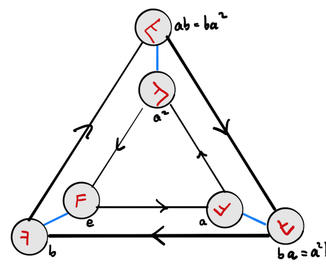
prove that this is actually the group with one element (and the same for French, but not Japanese or Greek). The result for English (French) is proved in French (English).

[End of Lecture 2]

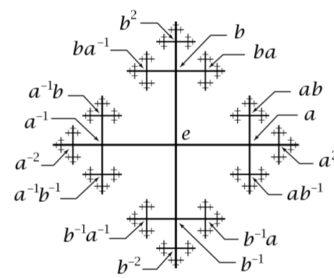
Presentations are not unique. Another presentation of D_4 is $\langle b, c | b^2 = c^2 = e, bcbc = cbcb \rangle$. Here c is a reflection through a diagonal of the square.

Cayley diagram of a group presentation. A useful device, at least for visualizing discrete groups, is the Cayley diagram or Cayley graph of a group presentation. It is a graph whose vertices correspond to elements of the group, and whose edges correspond to generators of the presentation (sometimes people color the edges associated to different generators different colors). Each relation implies a loop. So for example, $\mathbb{Z}_n = \langle x | x^n = 1 \rangle$ is a cyclic graph, a circular chain of n nodes.

The Cayley diagram for the first presentation of $D_n = \langle a, b | a^n = e, b^2 = e, (ab)^2 = e \rangle$ is two n-gons (associated with a^k and ba^k, $k = 0..n-1$, respectively) connected at their corresponding edges, as at right for $D_3 = S_3$. The Cayley diagram for the other presentation is uglier.

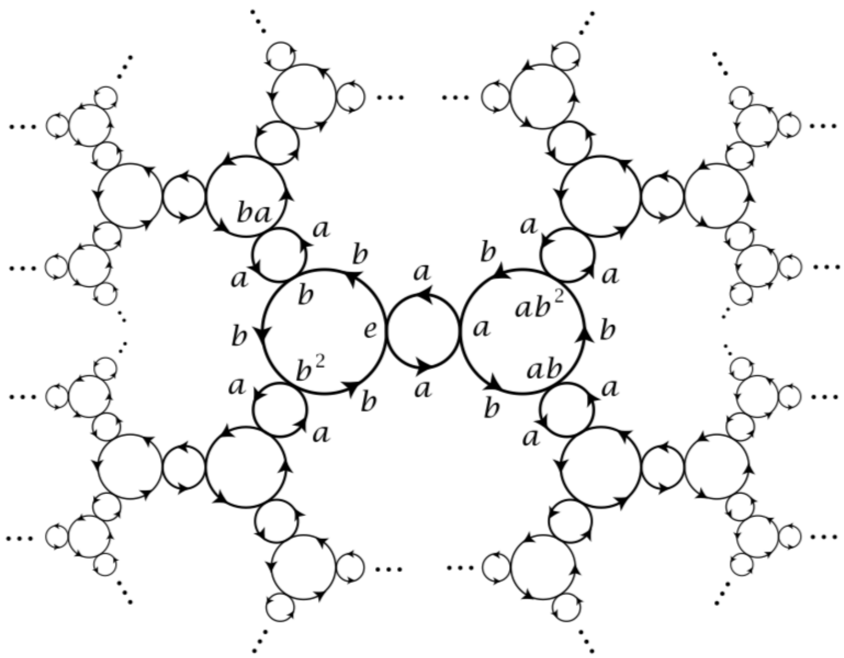


The Cayley diagram for the free group on q elements is the q-Bethe lattice, shown at right for $q = 2$ (from Hatcher). It has no relations and hence no loops.



Here is the Cayley diagram for the infinite group $\langle a, b | a^2 = e, b^3 = e \rangle$ (just like

$S_3 = D_3$ but with no third relation), also from Hatcher:



For any discrete group G, a topological space X_G whose fundamental group is $\pi_1(X_G) = G$ can be constructed from the Cayley graph. The idea is to start from the Cayley graph of G and attach a 2-cell (a disk) into the loop associated with each relation, and then take a quotient of the resulting space by the right action of G. For more on what I mean by the 'action of G,' wait for §1.4. See Hatcher page 77.

1.3 Subgroups and conjugacy classes

There are two useful ways to chop up a group, into cosets and conjugacy classes.

Cosets. One is to look at its subgroups. Any group G has subgroups $\{e\}$ and G. These are trivial, any other subgroup is called a *proper subgroup*. Given a subgroup $H = \{h_1, \cdots, h_n\} \subset G$ (of order $|H| = n$), and $g_1 \in G$, the (left) *coset* is

$$g_1 H \equiv \{g_1 h_1, \cdots, g_1 h_n\}.$$

For any other element $g_2 \neq g_1$, either $g_1 H \cap g_2 H = \emptyset$ or $g_1 H = g_2 H$ (if they intersect at all, $g_1 h_1 = g_2 h_2$ for some h_1, h_2, then $g_1 = g_2 h_2 h_1^{-1} \in g_2 H$, so $g_1 H = g_2 H$). Each $g_i H$ has $n = |H|$ distinct elements, and they are all distinct (if $gh_1 = gh_2$ then $h_1 = h_2$). Therefore

$$G = g_1 H \dot{\cup} g_2 H \dot{\cup} \cdots g_k H \quad \text{for some } k \in \mathbb{Z}$$

(where I use $\dot{\cup}$ to denote disjoint union). This means $|H|k = |G|$, the order of any subgroup divides the order of the group (this result is due to Lagrange). $k = |G|/|H|$ is called the *index* of the subgroup, and the set of cosets is called G/H.

An *invariant* or *normal* subgroup H satisfies $g^{-1}Hg = H, \forall g \in G$. This notation means $g^{-1}hg \in H, \forall h \in H$. If H is normal, then the set of cosets G/H is a group, the *quotient group* with product defined as follows: choose a representative $g_i h_i$ of each coset; then $(g_i H)(g_j H) \equiv$ the coset $g_k H$ containing $g_i h_i g_j h_j$. For a normal subgroup, $g_i h_i g_j h_j = g_i (g_j g_j^{-1}) h_i g_j h_j = g_i g_j \underbrace{g_j^{-1} h_i g_j}_{=h_k} h_j = g_i g_j h_k h_j \in g_i g_j H$, so this is independent of our choice of representative, and in fact $g_i H g_j = g_i g_j H$.

G is *simple* if it has no invariant subgroups. There exists a classification of finite simple groups. There are 18 infinite families (cyclic $\mathbb{Z}_n$, alternating A_n, and 16 others) plus 26 sporadic groups, including the monster group, which has $\sim 10^{54}$ elements (and deep connections to 2d conformal field theory, called *moonshine*). It only has 194 conjugacy classes, though. What's a conjugacy class?

Conjugacy classes. The second way to chop up a group is into conjugacy classes. Two elements $g_1, g_2 \in G$ are *conjugate* if $g_1 = g^{-1} g_2 g$ for some $g \in G$. Conjugacy defines an equivalence relation (it is reflexive $g \sim g$, symmetric $g_1 \sim g_2 \implies g_2 \sim g_1$, and transitive $g_1 \sim g_2, g_2 \sim g_3 \implies g_1 \sim g_3$), so we may divide the group into equivalence (or conjugacy) classes, $C_g = \{kgk^{-1}, k \in G\}$: $G = C_e \dot{\cup} C_{g_1} \dot{\cup} \cdots \dot{\cup} C_{g_l}$, where $\dot{\cup}$ denotes disjoint union. The identity is always its own conjugacy class. In an abelian group, each class has one element. More generally, the number of elements of C_g divides $|G|$, since $Z_g \equiv \{h \in G | h^{-1} g h = g\} \subset G$ is a subgroup (called the *centralizer* of g, the set of elements of G that commute with g), and the number of elements of C_g is $|G|/|Z_g|$. Why: $C_g = \{kgk^{-1}, k \in G\}$. But if and only if $k \in Z_g$, it doesn't generate a new element of the conjugacy class. (This is a special case of the "orbit-stabilizer theorem" which we'll use again and prove a little more carefully in §1.4.)

In words and pictures: two elements $g_{1,2}$ are conjugate $g_1 = k^{-1} g_2 k$ if g_2 looks like g_1 to an observer who has been transformed by k.

Examples: two rotations in $\mathsf{SO}(3)$ are conjugate if they are rotations by the same angle $[R(\hat{n}, \theta)] = [R(\hat{n}', \theta)]$, possibly about different axes. (The element by which we conjugate rotates $\hat{n}$ to $\hat{n}'$. More generally, two elements of a matrix group are conjugate if they have the same eigenvalues (the eigenvalues of $R(\hat{n}, \theta)$ are determined by θ).

Conjugacy classes of S_n are specified by the cycle structure. All elements of with k_j j-cycles are conjugate to each other in S_n. For example, conjugating (23) by $(12) = (12)^{-1}$ gives (using the rules above for multiplying cycles)

$$(12)(23)(12) = (123)(12) = (312)(12) = (31)(12)(12) = (31).$$

$$(12)(234)(12) = (1234)(12) = (3412)(12) = (341)(12)(12) = (341).$$

The effect is merely to interchange the locations of elements 1 and 2 in the cycle structure, $(1)(23) \overset{(12)}{\mapsto} (2)(13)$, and $(1)(234) \overset{(12)}{\mapsto} (2)(134)$. The same is true if 1 and 2 are in the same cycle

$$(12)(123)(12) = (12)(12)(23)(12) = (32)(21) = (321) = (213).$$

Since any permutation is a product of cycles of length two, any conjugation is simply a rearrangement of the elements within a fixed cycle structure. (This fact will be easier to understand by thinking about representations.)

Given that cycles label conjugacy classes of S_n, let's think about how to label cycle decompositions. A cycle decomposition of an element of S_n specifies a *partition* of $n = \sum_j jk_j$, a set of numbers that add up to n. A useful notation for this is the *Young diagram* (or Ferrer diagram in the older literature). If there is a j-cycle, draw a column of boxes of height j; left-justify the columns, and order them by decreasing j. So for example, the conjugacy class containing $(12)(34)(5)$ in S_5 can be labelled $\ydiagram{3,2}$, and the class with 5 1-cycles (the identity) is $\ydiagram{5}$. The total number of boxes is n. The number of elements of the conjugacy class $\{k_j\}$ is $\frac{n!}{\prod_j j^{k_j} k_j!}$. The factors in the denominator come from the fact that the cyclic order within a cycle doesn't matter (j^{k_j}), and that the order of cycles of the same length doesn't matter $(k_j!)$.

1.4 Group actions (and statistical mechanics)

Suppose we have an action of a finite group G on a finite set X. A bit formally, this means a map $\begin{array}{rl} \mathsf{G} \times X & \to X \\ (g, x) & \mapsto gx \end{array}$ that respects the group properties (*e.g.*, $(hg)x = h(gx)$)[7]. Sometimes it is useful to write $g(x)$ for the action of g on x. This formal definition matches the intuitive idea of, say, rotations of the vertices of a cube, or permutations of a set of elements. We've already seen an example in the illustration (1.4).

Orbits. Each element $x \in X$ is part of an *orbit* of G, $Gx \equiv \{gx, g \in \mathsf{G}\}$. The number of elements in the orbit of x need not be G because some elements of G may map x to itself. The set of such $g \in \mathsf{G}$ which fix x is a subgroup, G_x, called the *stabilizer* subgroup of x. (Check it.) Being a subgroup, its order divides $|\mathsf{G}|$, and the size of the orbit of x is therefore

$$|\mathsf{G}x| = |\mathsf{G}|/|\mathsf{G}_x| \ , \tag{1.5}$$

[7] Alternatively, a group action of G on a set of order $|X| = n$ is a group homomorphism from $\mathsf{G} \to S_n$. In particular, every gx must be undo-able (so gx can't be gy for $x \neq y \in X$), and the identity must map each point in x to itself.

(an integer!). (This number is sometimes called the *index* of the subgroup.) The fancy way to say this fact is that elements of the orbit of x, Gx, are in 1-1 correspondence with cosets of G by elements of the stabilizer subgroup G_x. This fact is sometimes called the 'orbit-stabilizer theorem'[8]

Notice that $Gx = G(gx)$, $\forall g \in \mathsf{G}$. By the Sudoku property, two orbits are either identical or non-overlapping. Denote the set of distinct orbits X/G. So we can decompose X into a disjoint union of these distinct orbits:

$$X = \cup_{A \in X/\mathsf{G}} \{x \in A\}.$$

Example. Any group has an action on itself in various ways, such as $h \to gh$ (left action by g) or $h \to ghg^{-1}$ (conjugation by g). Consider the case where $X = \mathsf{G}$ and its action is by conjugation $x \to gxg^{-1}$. In this context, we've seen all of the above before. Claim: the orbits are conjugacy classes. The stabilizer subgroup G_x is the centralizer group $Z(x)$. Elements of a conjugacy class C_x are in 1-1 correspondence with cosets of G by $Z(x)$. The fixed point set of an element g is $\{x \in \mathsf{G} | gxg^{-1} = x\} = \{x \in \mathsf{G} | gx = xg\} = Z(g)$ is *also* the centralizer of g.

The Lemma that is not Burnside's.[9] How many orbits are there?

$$\text{Claim:} \quad \#\text{ of orbits} \equiv |X/\mathsf{G}| = \frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} |X^g| \tag{1.6}$$

where $X^g \equiv \{x \in X | gx = x\}$ is the set of fixed points of g.

[End of Lecture 3]

Proof: First, the fact that X is a disjoint union of its orbits under G means that

$$|X/\mathsf{G}| = \sum_{A \in X/\mathsf{G}} 1 = \sum_{A \in X/\mathsf{G}} \sum_{x \in A} \frac{1}{|A|} = \sum_{x \in X} \frac{1}{|\mathsf{G}x|}.$$

Now we use (1.5) to write

$$|X/\mathsf{G}| = \sum_{x \in X} \frac{|\mathsf{G}_x|}{|\mathsf{G}|}.$$

[8]Here is a proof that $|\mathsf{G}| = k|\mathsf{G}_x| = |\mathsf{G}x||\mathsf{G}_x|, \forall x \in X$, which also shows the bijection I referred to above between elements of the conjugacy class and cosets by the centralizer. Let $\mathsf{G}x = \{x_i \cdots x_k\} \ni x_1 = x$. For all g, $gx = x_i$ for some $i = 1..k = |\mathsf{G}x|$. So let $\mathsf{G}_i = \{g \in \mathsf{G} | g(x) = x_i\}$. Then we can decompose $\mathsf{G} = \mathsf{G}_1 \dot\cup \mathsf{G}_2 \dot\cup \cdots \dot\cup \mathsf{G}_k$ (with $\mathsf{G}_1 = \mathsf{G}_x$, the stabilizer group of x). I claim that all of these objects have the same size, $|\mathsf{G}_i| = |\mathsf{G}_x|$. Pick $h \in \mathsf{G}_i$, which means $hx = x_i$. Here's a bijection: $\phi : \begin{matrix} \mathsf{G}_x & \to & \mathsf{G}_i \\ g & \mapsto & hg \equiv \phi(g) \end{matrix}$. $\phi(g) = hg \in \mathsf{G}_i$ since $hgx = hx = x_i$. The inverse of ϕ is just $\phi^{-1} : \begin{matrix} \mathsf{G}_i & \to & \mathsf{G}_x \\ g & \mapsto & h^{-1}g \end{matrix}$. Therefore $|\mathsf{G}| = k|\mathsf{G}_x| = |\mathsf{G}x||\mathsf{G}_x|$.

[9]I'd like to thank Jin-Long Huang for introducing me to this fact. He used it to excellent effect for a paper we wrote involving the structure of entanglement in 3d states with topological order.

Finally,

$$\sum_{x\in X}|\mathsf{G}_x| = |\{(g,x)\in \mathsf{G}\times X | gx = x\}| = \sum_{g\in\mathsf{G}}|X^g|$$

and we arrive at (1.6). ■

In the case where $X = \mathsf{G}$ and the action is by conjugation, not-Burnside's Lemma gives a (not-too surprising) formula for the number of conjugacy classes

$$\#\text{ of conjugacy classes} = \frac{1}{|\mathsf{G}|}\sum_{g\in\mathsf{G}}|X^g| = \frac{1}{|\mathsf{G}|}\sum_{g\in\mathsf{G}}|Z(g)|.$$

Overkill. Consider n objects of which we want to select k without worrying about the order. Let's figure out how many ways there are to do this using the not-Burnside lemma. Let $X = S_n$; by declaring that we take the first k objects of $(\pi_1, \pi_2, \cdots \pi_n)$, this is the set of ways to pick out k objects while keeping track of the order of all n objects. There is an action of $\mathsf{G} = S_k \times S_{n-k}$ on X, where the element of S_k (S_{n-k}) permutes the first k (last $n-k$) items. This is the 'gauge group' – two choices are equivalent if they are related by the action of G. To apply not-Burnside's lemma to this choice of G and X we need to count fixed points. But the only element with fixed points is (e, e), which fixes all $n!$ elements. So the number of orbits (the number of ways to choose k elements of n irrespective of order) is

$$\frac{1}{|S_k \times S_{n-k}|}\sum_{(\sigma,\pi)\in S_k\times S_{n-k}}|X^{(\sigma,\pi)}| = \frac{1}{k!(n-k)!}(n! + 0 + \cdots + 0) = \binom{n}{k}.$$

You are not surprised by this conclusion. This same technology can be used to count much less trivial things, such as: how many non-isomorphic graphs are there with k vertices?

Weighted not-Burnside lemma. A generalization of the not-Burnside lemma allows us to keep track of more information about the things we're counting than just the number of orbits. [I recommend these two lectures on this subject, which this discussion follows.] It's usually described in terms of coloring a set, but it can just as well be described in terms of statistical mechanics models: Suppose we have a system of spins (up or down, + or −) living on the vertices of a square, but – here is a wrinkle – we regard configurations related by a rotation of the square as identical. (Don't worry about exactly why; one reason would be if the spins were identical bosons pinned to the corners of the square.) So we could apply the ordinary not-Burnside lemma with $\mathsf{G} = \{\text{rotations of } \square\}$, $X = \{\text{spin configurations on } \square\}$ to find

$$\#\text{of orbits} = \frac{1}{4}\left(|X^0| + |X^{90}| + |X^{180}| + |X^{270}|\right) = \frac{1}{4}\left(2^4 + 2 + 4 + 2\right) = 6.$$

This agrees with an explicit enumeration:

$$\begin{array}{cccccc} ++ & ++ & ++ & +- & +- & -- \\ ++ & +- & -- & -+ & -- & -- \end{array} \tag{1.7}$$

But you can see that there is more structure here: we have

- 1 orbit with 4 + and 0 −,
- 1 orbit with 3 + and 1 −,
- 2 orbits with 2 + and 2 −,
- 1 orbit with 1 + and 3 −,
- 1 orbit with 0 + and 4 −.

Suppose we were doing equilibrium statistical mechanics at fixed temperature. Then we could weigh these configurations by a Boltzmann weight that depends on the magnetization:

$$Z = \sum_{\text{configs}} e^{-\beta H(\text{config})}.$$

The tricky point here is that according to our definition, a configuration (a physical state of the system) is an *orbit* of the rotation group. Notice that the number of + and − are constant on each orbit. This is a crucial property: the Hamiltonian on spin configurations is *gauge invariant* – it is invariant under the group operation by which we want to identify configurations.

So let's introduce fugacities p and m for + and − respectively. Then an orbit with n_+ +s and n_- −s contributes $p^{n_+}m^{n_-}$ to the partition sum. First let's do it by hand. Looking at the list (1.7) above, the partition sum is

$$Z = 1p^4m^0 + 1p^3m^1 + 2p^2m^2 + 1p^1m^3 + 1p^0m^4. \tag{1.8}$$

(In combinatorics, this partition function is called a weight sum, or generating function.)

Here's the theorem: given an action of G on X, and a G-invariant weight function $W(x)$ (that is, W depends only on the orbit of x by G),

$$Z \equiv \sum_{\text{orbits}, O \in X/\mathsf{G}} W(O) = \frac{1}{|\mathsf{G}|} \sum_{g\in\mathsf{G}} \sum_{x \in X^g} W(x).$$

Notice that when $W(x) = 1$ this reduces to the ordinary non-Burnside lemma. Here's the proof: Since the weight is G-invariant, each subset of orbits with a given weight W

itself carries an action of G. Use the ordinary not-Burnside lemma in each such sector with weight W, and then add the results, weighted by W.

Let's check that it reproduces the result (1.8).

$$Z = \frac{1}{4}\left(\sum_{x\in X^0} W(x) + \sum_{x\in X^{90}} W(x) + \sum_{x\in X^{180}} W(x) + \sum_{x\in X^{270}} W(x) \right).$$

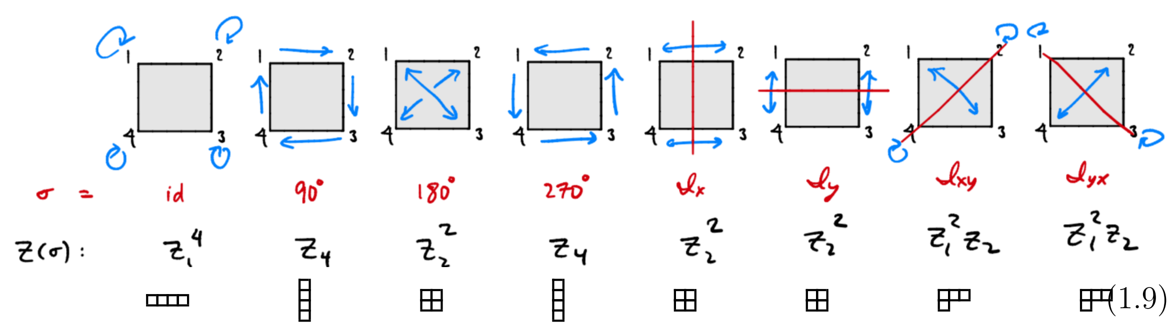
Every config is fixed by the identity, so the four spins are independently summed: $\sum_{x\in X^0} W(x) = (p+m)^4$. A 90° or 270° rotation require all the spins to be the same, so $\sum_{x\in X^{90}} W(x) = p^4 + m^4 = \sum_{x\in X^{270}} W(x)$. A 180° rotation requires pairs of antipodal spins to be the same, so $\sum_{x\in X^{180}} W(x) = (p^2+m^2)^2$. Adding these together, you can check that this reproduces Z.

Polya's enumeration theorem. A useful sharpening of the weighted not-Burnside lemma, which streamlines its application, is the Polya enumeration theorem. It requires introducing a bit of technology first. When a group acts on a set of $n = |X|$ elements, as we've been discussing, each of its elements is mapped to a permutation $\sigma \in S_n$, since clearly it must permute the elements of X. This permutation therefore has a cycle decomposition, with c_i cycles of length i. Let $z(\sigma) \equiv z_1^{c_1} z_2^{c_2} \cdots z_n^{c_n}$ for some formal variables z_i. (These c_i are the same as what I called k_j earlier.) The *cycle index* or *cycle indicator* of the action of G on X is

$$Z(\mathsf{G}, X) \equiv \frac{1}{|\mathsf{G}|} \sum_{\sigma\in\mathsf{G}} z(\sigma),$$

a bit like a partition function.

An example is extremely helpful. Take $\mathsf{G} = D_4$ and $X = \{\text{vertices of } \square\}$. There are 8 elements:



$$(1.9)$$

so

$$Z(D_4, \square) = \frac{1}{8}\left(z_1^4 + 2z_1^2 z_2 + 3z_2^2 + 2z_4\right).$$

It is a fun exercise to compute the cycle index for *e.g.* the action of a group on itself by left multiplication.

So far, this has been a definition. Now, if G acts on a set X, then G also acts on a spin system living on X. That is, we could put a little variable that takes k different values at each element of X. Call the set of configurations of these variables Y (in the combinatorics literature, they are called colorings of X). So an element of Y is $y = \{(x, s(x))|x \in X\}$, where $s(x) \in \{1 \cdots k\}$ is a spin configuration, or an assignment of a color to each element of X. The canonical action of $\sigma \in \mathsf{G}$ on Y is:

$$\bar{\sigma}\left(y\right) \equiv \bar{\sigma}\left(\{(x, s(x))|x \in X\}\right) \equiv \{(\sigma(x), s(x))|x \in X\} \in Y. \tag{1.10}$$

Note that on the RHS here we do not write $s(\sigma(x))$, which would be just a relabelling of the spins.

Now suppose we want to enumerate orbits of G on Y. For example, suppose we want to compute the kind of partition function we evaluated above, where we introduce a fugacity for each color, or suppose we want to enumerate how many different necklaces we can make with some number of different colors of beads... (many more examples on the homework).

Polya enumeration theorem:

$$\sum_{\text{orbits}, O \in Y/\mathsf{G}} W(O) = Z(\mathsf{G}, X)|_{z_i = b_1^i + \cdots b_k^i},$$

where b_i is the fugacity for the ith color (that is, $W(y) = b_1^{\#\text{ of } x \text{ s.t. } s(x)=1} b_2^{\#\text{ of } x \text{ s.t. } s(x)=2} \cdots$).

Proof: By the weighted not-Burnside lemma, the LHS is

$$\sum_{\text{orbits } O \in Y/\mathsf{G}} W(O) = \frac{1}{|\mathsf{G}|} \sum_{\sigma \in \mathsf{G}} \sum_{y \in Y^{\bar{\sigma}}} W(y).$$

If $y \in Y^{\bar{\sigma}}$ is a fixed point of $\bar{\sigma}$ it means the following, by (1.10):

$$y = \{(x, s(x))|x \in X\} = \{(\sigma(x), s(x))|x \in X\}. \tag{1.11}$$

Since σ is a permutation, we can relabel the elements of y as

$$y = \{(x, s(x))|x \in X\} = \{(\sigma(x), s(\sigma(x))|x \in X)\}.$$

Therefore (1.11) says $s(x) = s(\sigma(x))$ for all $x \in X$. This is the very sensible statement that *all the elements of a given cycle of σ have the same color*. And elements participating in different cycles are colored independently. Therefore

$$\sum_{y \in Y^{\bar{\sigma}}} W(y) = (b_1 + \cdots + b_k)^{c_1} (b_1^2 + \cdots + b_k^2)^{c_2} \cdots (b_1^n + \cdots + b_k^n)^{c_n} = z(\sigma)|_{z_i \to b_1^i + \cdots + b_k^i}.$$

■

Here's an example. Let's enumerate configurations of the 3-state Potts model (this just means $k = 3$) on the square, modulo all symmetries of the square, D_4. A physical realization is: we have some spin-1 (three states) identical bosons which for some reason formed a square-shaped molecule, and we want to know its partition function in a Zeeman field (and spin-spin interactions are negligible for some reason). Using the Polya enumeration theorem and (1.9),

$$Z = Z(D_4, \square) = \frac{1}{8}\left(z_1^4 + 2z_1^2 z_2 + 3z_2^2 + 2z_4\right)|_{z_i \to R^i + G^i + B^i}$$

where I called R, G, B the fugacities of the three spin states (for spin-one particles in a Zeeman field, $R = e^{-\beta h}, G = 1, B = e^{+\beta h}$). This is an homage to the most canonical application of this theorem, which is counting necklaces. The number of configurations (necklaces with four beads) with 2 red, 1 green and 1 blue is the coefficient of $R^2 GB$ in this polynomial, which (I recommend Mathematica's Coefficient command) is 2.

For (perhaps too many) more applications of this line of thought, see hw 2.

[End of Lecture 4]

2 Representations

A *representation* R of a group G associates to each $g \in \mathsf{G}$ a linear operator $D_R(g)$ on some vector space V (sometimes called the *carrier space*), such that

- $D_R(g_1)D_R(g_2) = D_R(g_1 g_2)$
- and in particular $D_R(e) = \mathbb{1}$, the identity operator on V

i.e., more succinctly, the map $R : g \mapsto D_R(g)$ is a group homomorphism, from G to $\mathsf{GL}(n, \mathbb{C})$), where $\mathbb{C}$ is the complex numbers. Here $n = \dim V$, the dimension of V as a vector space over $\mathbb{C}$, is the *dimension* of the representation. We'll always be interested in vector spaces over $\mathbb{C}$ (though sometimes it will be interesting to ask when we can make the matrices real). As we'll see, a given group can have many representations, of various dimensions.

There are several strong motivations for thinking about representations. Most importantly, representation theory is arguably the main point of contact between group theory and quantum physics, where the vector space in question is generally the Hilbert space of a physical system, $V = \mathcal{H}$. And the linear operators in question realize transformations of the system (like translations or rotations or anything else you can think of) which must take $\mathcal{H} \to \mathcal{H}$. A special role is played by *unitary representations*, where all of the $D(g)$ are unitary $D(g)D(g)^\dagger = \mathbb{1}$, since these preserve the norms of states[10].

More generally, though, by studying its representations, we can study groups using *linear* algebra. As usual in QM, a linear operator D in a particular basis $\mathcal{H} = \text{span}_i\{|i\rangle\}$ determines a matrix, with elements

$$(D)_{ij} = \langle i| D |j\rangle .$$

Therefore, in a basis, the group operation becomes matrix multiplication:

$$(D(g_1g_2))_{ij} = (D(g_1)D(g_2))_{ij} = \langle i| D(g_1) \underbrace{\mathbb{1}}_{=\sum_{k=1}^n |k\rangle\langle k|} D(g_2) |j\rangle = \sum_k (D(g_1))_{ik} (D(g_2))_{kj} .$$

[10]You might object that in QM we often don't care about the overall phase of the wavefunction, and hence we shouldn't be too worried if the group law is only satisfied up to a phase:

$$D_R(g_1)D_R(g_2) = e^{\mathbf{i}\phi(g_1,g_2)} D_R(g_1g_2)$$

instead of the rule given above. Indeed. Such a thing is called a *projective representation* and there is a lot to say about them. Be patient and wait for §2.7.

Some examples. Every group has the *trivial representation*, which I'll denote by $\mathbf{1}$: $D(g) = 1, \forall g \in \mathsf{G}$. It is well-named.

Here is a nontrivial 1d representation of $\mathbb{Z}_n = \langle g | g^n = 1 \rangle$: $D_1(e) = 1, D_1(g) = \omega, D_1(g^2) = \omega^2 \cdots$, where $\omega \equiv e^{2\pi \mathbf{i}/n}$. For $n > 2$, another one has $D_2(g) = \omega^2$.

Any group has its *regular representation*, defined as follows. Associate an orthonormal basis with the elements of the group, so the carrier space is $\mathcal{H}_\mathsf{G} \equiv \text{span}\{|g\rangle\,,\ g \in \mathsf{G}\}$. Then we can define, quite naturally,

$$D_{\text{reg}}(g_1)\,|g_2\rangle \equiv |g_1 g_2\rangle$$

where the argument of the ket on the RHS is the group product. (Notice that we could have instead defined it by right multiplication.) Check that this is a representation. Its dimension is $|\mathsf{G}|$. For example, for $\mathbb{Z}_3$, the matrix elements are

$$D_{\text{reg}}(e) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad D_{\text{reg}}(g) = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \quad D_{\text{reg}}(g^2) = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}. \tag{2.1}$$

Any action of G on a finite set X, $|X| = n$ gives a representation of G of dimension n. We could describe this by introducing a Hilbert space $\text{span}\{|x\rangle\,, x \in X\}$ spanned by orthonormal states labelled by elements of X. The representation operators are then $D(g)\,|x\rangle = |g(x)\rangle$.

Both of the previous examples are a special kind of representation called a permutation representation, where the matrices D_{ij} have a single nonzero entry in each row and column.

New reps from old. Given two representations $R_{1,2}$ of G, there are two ways to make a new representation. The *direct sum* $R_1 \oplus R_2$ acts on the direct sum of the carrier spaces, and has matrices

$$D_{R_1 \oplus R_2}(g) = \begin{pmatrix} D_{R_1}(g) & 0 \\ 0 & D_{R_2}(g) \end{pmatrix}.$$

The '0's in this matrix are $\dim R_1 \times \dim R_2$ and $\dim R_2 \times \dim R_1$ matrices. The dimensions add: $\dim(R_1 \oplus R_2) = \dim R_1 + \dim R_2$

The *direct product* $R_1 \otimes R_2$ lives on the direct product of the carrier spaces, and the matrices are

$$\langle i\alpha|\, D_{R_1 \otimes R_2}\, |j\beta\rangle = (D_{R_1 \otimes R_2})_{i\alpha, j\beta} \equiv (D_{R_1} \otimes D_{R_2})_{i\alpha, j\beta} \equiv (D_{R_1})_{ij}\, (D_{R_2})_{\alpha\beta}$$

where I use different alphabets to index the basis states of the two spaces to emphasize that they can be completely unrelated. Notice that it is useful to think of $i\alpha$ or $j\beta$ here as a single object, a multi-index. $\dim(R_1 \otimes R_2) = \dim R_1 \cdot \dim R_2$.

Representations of S_n. We'll have a lot to say about representations of S_n. To get started, let

$$\mathrm{sign}(\pi) \equiv (-1)^\pi \equiv (-1)^{k_2+k_4+\cdots} = (-1)^{\#\text{ of } even\text{-length cycles of } \pi}$$

be the sign (or signature) of the permutation π. Notice that $\mathrm{sign}(\pi_1)\mathrm{sign}(\pi_2) = \mathrm{sign}(\pi_1\pi_2)$ (just check that the rules for combining cycles in §1.2 preserve the number of even-length cycles mod two, since they can only annihilate in pairs), so sign is a 1d representation of S_n.

By the way, the kernel of the map $\begin{matrix} S_n \to & \mathbb{Z}_2 \\ \pi \mapsto & \mathrm{sign}(\pi) \end{matrix}$ specifies a subgroup of S_n [11]. That is, the subset of even permutations $\{\pi \in S_n | \mathrm{sign}(\pi) = +1\} \equiv A_n$, the *alternating group*[12]. It is an invariant subgroup, and the quotient group S_n/A_n is (not coincidentally) just $\mathbb{Z}_2$.

Another important representation of S_n is called the *defining* (or *fundamental*) representation, which is n-dimensional. The carrier space is $\mathcal{H} \equiv \mathrm{span}\{|j\rangle, j = 1\cdots n\}$, where the $|j\rangle$ are orthonormal. The definition of $D(\pi)$ is deceptively simple:

$$D(\pi)|j\rangle = |\pi_j\rangle .$$

For example, for $n = 3$, the action of the element (123) is $D(123)|1\rangle = |2\rangle, D(123)|2\rangle = |3\rangle, D(123)|3\rangle = |1\rangle$. Similarly $D(12)|1\rangle = |2\rangle, D(12)|2\rangle = |1\rangle, D(12)|3\rangle = |3\rangle$. The matrix elements are

$$D(e)_{ij} = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}_{ij} \quad D(123)_{ij} = \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}_{ij} \quad D(321)_{ij} = \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}_{ij} \tag{2.2}$$
$$D(12)_{ij} = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}_{ij} \quad D(23)_{ij} = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}_{ij} \quad D(31)_{ij} = \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}_{ij}$$

Now here's a simple proof that the cycle structure of a permutation is a property of its conjugacy class: in the defining representation, conjugation is a similarity transformation – it is a just a change of basis. For example, conjugating by $D(12)$ implements the basis change that switches basis vectors 1 and 2, as you can see from

[11] By 'kernel' of a group homomorphism, I mean the set of elements that map to the identity; this is always a subgroup. In fact, it is always a normal subgroup. Exercise.

[12] A *simple* group is a group with no proper invariant subgroups. The fact that A_n is simple for $n \geq 5$ implies that there is no solution of the general quintic equation by radicals. This result, which played an important role in the development of understanding of group theory on this planet, is due to Ruffini, Abel and Galois.

(2.2). Moreover, since we can make any permutation by a sequence of interchanges (order-two elements), and by conjugation by interchanges we can make every possible rearrangement of the elements within a cycle, each conjugacy class consists of all possible permutations with the same cycle structure.

Equivalence. When are two reps different? Notice that we didn't specify a basis of V. So changing basis of V shouldn't change representation, though it will certainly change the representation matrices. So, since, for any invertible matrix S, the replacement $D(g) \mapsto D'(g) = S^{-1}D(g)S, \quad \det S \neq 0$ (a similarity transformation, aka change of basis) gives a representation with the same multiplication rules, we declare D' and D equivalent. (It's crucial that we use the same S for every g!)

2.1 Irreps

Reducibility. A representation is *reducible* if it has an *invariant subspace*. An invariant subspace is $W \subset V$ such that $D(g)\,|w\rangle \in W, \forall\,|w\rangle \in W, g \in \mathsf{G}$. The existence of such a subspace means we can make a smaller representation out of just the action on W, as follows. Let P_W be the projector onto $W \subset V$. This means $P_W^2 = P_W$. The statement that W is an invariant subspace can be written

$$P_W D(g) P_W = D(g) P_W, \quad \forall g \in \mathsf{G}. \tag{2.3}$$

But this says that the projected operators $D_W(g) \equiv P_W D(g) P_W$ form a representation.

For example, in the regular representation, the vector $|u\rangle \equiv \sum_{g\in\mathsf{G}} |g\rangle$ is mapped to itself by all of the $D(g)$s. For the $\mathbb{Z}_3$ example in (2.1), in the given basis, this is the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and the (rank one) projector $P_W = |u\rangle\langle u|$ onto this subspace has the matrix representation $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. This shows that for any group,

$$R_{\text{regular}} = \underline{1} \oplus R_{|\mathsf{G}|-1} \ ,$$

where $\underline{1}$ denotes the trivial 1d representation and the other summand is some representation of dimension $|\mathsf{G}| - 1$. So the regular rep is reducible.

If a representation is not reducible, it is called an *irreducible representation*, or *irrep*. As Stone and Goldbart say, the irreps of a group are the atoms – the elementary particles – of representation theory, and the irreps of commonly-occurring groups are going to become our good friends.

A representation is *completely reducible* (I believe it is sometimes called *decomposable* if it can be decomposed (by a similarity transformation S) into a direct sum of irreps:

$$S^{-1}D(g)S = \begin{pmatrix} D_1(g) & 0 & 0 \\ 0 & D_2(g) & 0 \\ 0 & 0 & \ddots \end{pmatrix}$$

(the same S for every g!), *i.e.* $D = D_1 \oplus D_2 \oplus \cdots$. For example, there is a basis transformation that takes the $\mathbb{Z}_3$ regular representation to

$$D'(e) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad D'(g) = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}, \quad D'(g^2) = \begin{pmatrix} 1 & & \\ & \omega^2 & \\ & & \omega \end{pmatrix}, \quad \omega = e^{2\pi \mathbf{i}/3}, \tag{2.4}$$

which shows that for $\mathbb{Z}_3$, $R_{\text{regular}} = \underline{1} \oplus \underline{1}_1 \oplus \underline{1}_2$, a sum of three 1d irreps. You can find S just by diagonalizing the original $D(g)$s. Since they commute ($\mathbb{Z}_3$ is abelian), they can be simultaneously diagonalized by one S. (Note that this argument shows that all irreps of any abelian group must be one-dimensional.)

Now why do I make a big deal about completely reducible? Some matrices can't be diagonalized – the alternative is that the matrices have a block structure in their Jordan normal form, like

$$\begin{pmatrix} D_1(g) & B \\ 0 & D_2(g) \end{pmatrix}.$$

An example where this happens is the following 2d rep of the group of integers under addition: $D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. It is reducible because $D(x)P = P$ with $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (which in turn implies $PD(x)P = P = D(x)P$, the condition for the projector onto an invariant subspace (2.3)), but it is not completely reducible (*i.e.* indecomposable) because $D(x)(\mathbb{1} - P) \neq \mathbb{1} - P$ – the transformations preserve the subspace but not its complement.

This pathology is possible because $\mathbb{Z}$ is non-compact. This annoyance is ruled out for unitary representations, since unitary matrices are normal (commute with their dagger) and hence can be diagonalized.

Theorem: any unitary rep is completely reducible.

Proof: If the rep is reducible at all, there's a projector P such that $PD(g)P = D(g)P, \forall g \in \mathsf{G}$. But the adjoint of this equation is $PD(g)^\dagger P = PD(g)^\dagger, \forall g \in \mathsf{G}$. For a unitary rep, $D(g)^\dagger = D(g)^{-1} = D(g^{-1})$, so this says $PD(h)P = PD(h), \forall h = g^{-1} \in \mathsf{G}$. This is true for each g iff $(\mathbb{1} - P)D(g)(\mathbb{1} - P) = D(g)(\mathbb{1} - P)$, which says that $\mathbb{1} - P$ always also projects onto an invariant subspace. ■

And

Theorem: for a *compact group*, every representation is equivalent to a unitary representation. Recall that compact means that we can do G-invariant sums and get finite answers; in particular this includes any finite group.

Proof: The proof is satisfyingly constructive: Let $S \equiv \sum_{g\in\mathsf{G}} D(g)^\dagger D(g)$. This operator is hermitian ($S^\dagger = S$) and $S \geq 0$, so it has a square root: $\sqrt{S}$ (recall that a function of a hermitian operator can be defined by the spectral representation: if $S = \sum_d d|d\rangle\langle d|$, then $\sqrt{S} = \sum_d \sqrt{d}|d\rangle\langle d|$). I claim that not only is $S \geq 0$, but $S > 0$ – all of its eigenvalues are positive, $d > 0$. (If it were otherwise, there would be a vector v with $0 = Sv$, which means

$$0 = v^\dagger S v = \sum_{g\in\mathsf{G}} \parallel D(g)v \parallel^2 \quad \Longrightarrow \quad D(g)v = 0, \forall g \ .$$

But this would violate the requirement that $D(e) = \mathbb{1}$.) Therefore, we can define

$$D'(g) \equiv \sqrt{S} D(g) \sqrt{S}^{-1}$$

which are unitary:

$$D'(g)^\dagger D'(g) = \sqrt{S}^{-1} D(g)^\dagger \sqrt{S}\sqrt{S} D(g) \sqrt{S}^{-1} = \cdots = \mathbb{1}.$$

I won't spoil the fun by writing out the steps here. ∎

For a compact but non-discrete group, the same argument goes through with $\sum_g D(g)^\dagger D(g)$ replaced by $\int_g D(g)^\dagger D(g)$. This ability to take averages over a compact group has many consequences and is a stratagem we will use all the time.

To understand reducibility, a big help is **Schur's lemma**: Suppose we are given collections of linear operators $A_\alpha : U \to U, B_\alpha : V \to V$ each of which act irreducibly (think of α as labelling group elements), and

$$\Lambda : U \to V \qquad \text{such that } \Lambda A_\alpha = B_\alpha \Lambda, \ \forall \alpha. \tag{2.5}$$

(Such a Λ is called an *intertwining operator.*) Then either (a) $\Lambda = 0$ or (b) Λ is 1-1 and onto (invertible), $\dim U = \dim V$ and $A_\alpha = \Lambda^{-1} B_\alpha \Lambda$.

This very useful statement is true for a simple reason. The condition (2.5) says that $\ker(\Lambda) \subset U$ and $\mathrm{Im}(\Lambda) \subset V$ are invariant subspaces of $\{A_\alpha\}$ and $\{B_\alpha\}$ respectively[13] So by the irreducibility assumption, either $\ker \Lambda = U, \mathrm{Im}\,\Lambda = 0$ or $\ker \Lambda = 0, \mathrm{Im}\,\Lambda = V$. ∎

[13] A little more explicitly: if $|a\rangle \in \ker \Lambda$, $\Lambda|a\rangle = 0$. Then $\Lambda A_\alpha |a\rangle = B_\alpha \Lambda |a\rangle = 0$, so $A_\alpha |a\rangle \in \ker \Lambda$, too.

A corollary is: if $\{A_\alpha\}$ act irreducibly on V and

$$\Lambda A_\alpha = A_\alpha \Lambda \tag{2.6}$$

then $\Lambda = \lambda 1\!\!1_V$ (λ could be zero). Here's why: (2.6) implies $(\Lambda - x1\!\!1)A_\alpha = A_\alpha(\Lambda - x1\!\!1)$. But $\det(\Lambda - x1\!\!1)$ is a polynomial in x of degree $\dim V$, which therefore has at least one root, say at $x = \lambda$. But this means that $\Lambda - \lambda 1\!\!1$ is not invertible, and by Schur's lemma must vanish. ■

A consequence of Schur's lemma is that each irrep can be put in a canonical form. A basis change on an irrep that preserves the representation matrices satisfies $S^{-1}D(g)S = D(g), \forall g \in \mathsf{G}$. But then Schur's lemma says that $S = \lambda 1\!\!1$. So we can (and will from now on) assume that whenever we speak about a particular irrep, we choose the same list of unitary representation matrices.

Orthogonality of matrix elements. Our most important application of Schur's lemma is the following. Let $D_a(g) : V_a \to V_a$ be an irrep of dimension $d_a = \dim V_a$. The following equation is sometimes called the Grand Orthogonality Theorem:

$$\frac{1}{|\mathsf{G}|}\sum_{g\in\mathsf{G}} \left(D^a(g^{-1})\right)_{ij} \left(D^b(g)\right)_{kl} = \frac{1}{d_a}\delta_{jk}\delta_{il}\delta^{ab} \tag{2.7}$$

for any two irreps. For unitary reps, we can replace the first factor by $\left(D^a(g^{-1})\right)_{ij} = \left(D^a(g)\right)^\dagger_{ij}$, so

$$\frac{1}{|\mathsf{G}|}\sum_{g\in\mathsf{G}} \left(D^a(g)\right)^\dagger_{ij} \left(D^b(g)\right)_{kl} = \frac{1}{d_a}\delta_{jk}\delta_{il}\delta^{ab} \tag{2.8}$$

Proof: For any operator $M : V_b \to V_a$, let $\Lambda^M \equiv \sum_g D^a(g^{-1})MD^b(g)$. Then $D^a(g)\Lambda^M = \Lambda D^b(g)$, $\forall g, M$, by relabelling the summation variable. Schur's lemma applies so[14]

$$\Lambda^M_{il} = \sum_g \left(D^a(g^{-1})\right)_{ij} M_{jk} \left(D^b(g)\right)_{kl} = \lambda(M)\delta_{il}\delta^{ab}.$$

Since this is true for every such M, take M to be 0 everywhere and 1 in the jk entry, so

$$\sum_g \left(D^a(g^{-1})\right)_{ij} \left(D^b(g)\right)_{kl} = \lambda_{jk}\delta_{il}\delta^{ab}.$$

To determine the prefactor λ_{jk}, set $a = b$ and *contract* $i = l$ (by *contract* the two indices i and l I mean multiply by δ_{il} and sum over both) to find $\delta_{jk}|\mathsf{G}| = \lambda_{jk}d_a$. ■

One more quick application of Schur's lemma: Each representation operator $D(g)$ commutes with all the others $D(g)D(h) = D(h)D(g)$, and therefore on an irrep must

[14]More explicitly: Schur implies either $\Lambda^M = 0$ or $d_a = d_b$ and $D^a(g) = \Lambda D^b(g)\Lambda^{-1}$ which means the irreps are the same, which means $\Lambda^M = \lambda 1\!\!1$.

be proportional to the identity. And for an abelian group, every subspace is an invariant subspace. This means every irreducible representation of an abelian group is 1-dimensional.

[End of Lecture 5]

2.2 Characters

The character of a representation R is a function that eats group elements and spits out numbers:

$$\chi_R(g) \equiv \mathrm{tr} D_R(g) = \sum_i (D_R(g))_{ii} \,.$$

These objects have many virtues. In particular we will see that a representation is specified by its character, up to trivial relabellings.

- Characters are basis independent, *i.e.* unchanged by a similarity transformation, $\mathrm{tr} S^{-1} D S = \mathrm{tr} D$ by cyclicity of the trace.
- Characters are *class functions* – $\chi(g)$ only depends on the conjugacy class of g:

  $$\chi(g^{-1}hg) = \mathrm{tr} D(g^{-1}hg) = \mathrm{tr} D(g^{-1}) D(h) D(g) = \mathrm{tr} D(h) = \chi(h)$$

  using the group law and cyclicity of the trace.
- $\chi_R(e) = \dim R$.
- Characters play nice with the operations $\otimes$ and $\oplus$:

  $$\chi_{D_1 \otimes D_2}(g) = \chi_{D_1}(g) \chi_{D_2}(g), \quad \chi_{D_1 \oplus D_2}(g) = \chi_{D_1}(g) + \chi_{D_2}(g) \,. \tag{2.9}$$

- Characters of different irreducible representations are orthonormal, in the sense that

  $$\frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \chi_{R_a}(g^{-1}) \chi_{R_b}(g) = \delta_{ab},$$

  and hence they are different for different representations. For unitary representations, this is the same as

  $$\frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \chi_{R_a}(g)^\star \chi_{R_b}(g) = \delta_{ab}. \tag{2.10}$$

This last crucial fact follows from the Grand Orthogonality Fact (2.7) just by contracting indices $i = j$ and $k = l$. Notice that by the class function property, the relation can be rewritten as

$$\frac{1}{|\mathsf{G}|} \sum_{\text{conjugacy classes},\alpha} n_\alpha \chi_{R_a}(\alpha)^\star \chi_{R_b}(\alpha) = \delta_{ab} \tag{2.11}$$

where n_α is the number of elements in the conjugacy class labelled α. So characters are orthogonal with respect to the inner product on conjugacy classes defined by

$$\langle \chi_1, \chi_2 \rangle \equiv \frac{1}{|\mathsf{G}|} \sum_\alpha n_\alpha \chi_1(\alpha)^\star \chi_2(\alpha). \tag{2.12}$$

The character table is square. By the *character table* of a group, I just mean $\chi^a_\alpha \equiv \chi_{R_a}(g \in C_\alpha)$ regarded as a matrix. The boldface statement is:

The number of irreps of G is equal to its number of conjugacy classes.

Proof: We already know from (2.11) that the number of of conjugacy classes n_C is at least as big as the number of irreps, n_R: the LHS is a matrix of rank n_R, but the RHS is a matrix of rank (at most) n_C.

The goal of the following argument is to show that the number of conjugacy classes can't be bigger than the number of irreps. [This an adaptation of Prop. 2.30 of Fulton and Harris.] If there were a class function f which did not come from the character of a representation (such as a function which was 1 on an extra conjugacy class), by (2.10), it would have $0 = \langle f, \chi_a \rangle$ for all irreps R_a. But then consider the object $S = \sum_{g \in G} f(g) D^a(g)$. It commutes with all of the representation matrices $D^a(h)S = SD^a(h), \forall h \in G$. Here's why:

$$D^a(h)S = \sum_g f(g) D^a(hg) = \sum_{g'=h^{-1}gh} f(hg'h^{-1}) D^a(g'h) = \sum_{g'} f(g) D^a(g') D^a(h) = SD^a(h).$$

But then Schur's lemma says $S = \lambda 1\!\!1_{V^a}$, and taking the trace of both sides (and using cyclicity of the trace and integration by parts),

$$\lambda d_a = \text{tr} S = \sum_g f(g) \chi^a(g) = \langle f, \chi_{\bar{a}} \rangle^\star = 0.$$

(At the last step I used the fact that $\chi_R(g)^\star = \chi_{\bar{R}}(g)$ is the character of the representation $\bar{R}$ with operators $D(g)^\star$.) So $\lambda = 0$ and so $S = 0$. Now we defined $S = S^a$ above for a particular irrep labelled a, but the conclusion that $S^a = 0$ is true for every irrep. In fact, we could define $S^R = \sum_{g \in G} f(g) D_R(g)$ for *any* representation and the conclusion would be the same (since any representation is a direct sum of irreps). But the matrix elements of the unitary irreps provide an orthonormal basis for the set of all functions on the group – this is the content of the Grand Orthogonality Theorem. From this we can conclude that $f = 0$. ■

This result means that the characters of the irreps are a *basis* of class functions on G. Using (2.10), then, an arbitrary class function can be decomposed as

$$F(g) = \sum_{\text{irreps},a} \chi_a(g) c_a^F, \quad c_a^F = \frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \chi_{R_a}(g^{-1}) F(g). \tag{2.13}$$

Cyclic groups and Fourier series. You may notice that this maneuver (2.13) is just like Fourier decomposition. This is not a coincidence. Consider the case $\mathsf{G} = \mathbb{Z}_n$. Since it is abelian, the conjugacy classes each contain one element. The irreps are $D_k\left(g^\ell\right) = \omega^{k\ell}, k = 0..n-1, \ell = 0..n-1$ ($\omega \equiv e^{2\pi \mathbf{i}/n}$), and the characters are $\chi_k\left(g^\ell\right) = \omega^{k\ell}$, too. Then in this special case (2.13) *is* Fourier decomposition:

$$F(\ell) = \sum_{k=0}^{n-1} \omega^{k\ell} F_k, \quad F_k = \frac{1}{n} \sum_{\ell=0}^{n-1} \omega^{-k\ell} F(\ell).$$

Characters are orthonormal in another sense as well:

$$\sum_{\text{irreps},a} \chi_{R_a}(g_\alpha)^\star \chi_{R_a}(g_\beta) = \frac{|\mathsf{G}|}{n_\alpha} \delta_{\alpha\beta} \tag{2.14}$$

where again n_α is the number of elements in the conjugacy class labelled α. Once we know the character table is square, (2.14) follows from (2.11): the latter says that $S_{\alpha a} \equiv \sqrt{\frac{n_\alpha}{|\mathsf{G}|}} \chi_{R_a}(g_\alpha)$ is a unitary matrix, $S^\dagger S = \mathbb{1}$[15].

An important practical consequence of complete reducibility is that any unitary representation is

$$R = \bigoplus_{\text{irreps},a} \Big(\underbrace{R_a \oplus \cdots \oplus R_a}_{m_a^R \text{ times}} \Big) = \underbrace{R_1 \oplus \cdots \oplus R_1}_{m_1^R \text{ times}} \oplus \underbrace{R_2 \oplus \cdots \oplus R_2}_{m_2^R \text{ times}} \oplus \cdots \equiv \bigoplus_{\text{irreps},a} R_a^{\oplus m_a^R} = \bigoplus_{\text{irreps},a} V_a^R \otimes R_a \tag{2.15}$$

where $\dim V_a^R = m_a^R$. The only data here are the m_a^R, the number of times irrep a appears in R. Using (2.9), (2.15) implies that the character of R is

$$\chi_R(g) = m_1^R \chi_1(g) + m_2^R \chi_2(g) + \cdots . \tag{2.16}$$

[15] S is square (it is the character table with some rescaling) so $SS^\dagger = \mathbb{1}$ as well. Here is a proof: $S \in \mathsf{GL}(N, \mathbb{C})$ and $S^\dagger S = \mathbb{1}$ says $V^\dagger \in \mathsf{GL}(N, \mathbb{C})$ is its left inverse in this group. But in a group left-inverse is the same as right-inverse as you showed on the homework.

And therefore we can pick off the coefficients by taking the overlaps

$$m_a^R = \langle \chi^{R_a}, \chi^R \rangle \equiv \frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \chi_{R_a}(g)^\star \chi_R(g).$$

Consider for example the regular representation R_{reg} of any group G. From the definition, $\chi_{R_{\text{reg}}}(g) = \delta_{g,e}|\mathsf{G}|$ – by the sudoku rule, any element other than e must move every other element. Therefore

$$m_a^{R_{\text{reg}}} = \frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \chi_{R_a}(g)^\star \chi_{R_{\text{reg}}}(g) = \chi_{R_a}(e) = d_a.$$

Recalling that $\dim(R_{\text{reg}}) = |\mathsf{G}|$, this means that

$$|\mathsf{G}| = \sum_{\text{irreps},a} d_a^2,$$

an important constraint on the dimensions of the irreps of any group.

Now we must see some examples of character tables. One way to construct them is to know the representation matrices for every irrep, and just take the traces for one element of each conjugacy class. We can do this for $\mathbb{Z}_3 = \langle g | g^3 = e \rangle$:

$\mathbb{Z}_3$	n_C	1	ω	ω^2
e	1	1	1	1
g	1	1	ω	ω^2
g^2	1	1	ω^2	ω

I am using Zee's convention for character tables: each column corresponds to an irrep, each row to a conjugacy class. Rows are orthogonal with respect to the ordinary (complex) dot product. Columns are orthogonal with respect to the inner product (2.12), so I include a column with n_C, the number of elements of each conjugacy class. Notice that the characters of the identity conjugacy class (the first row) give the dimensions of the irreps: $d_a = \chi_a(e)$. Check that $\sum_a d_a^2 = |\mathsf{G}|$. Notice that for $\mathbb{Z}_3$ the characters are complex numbers – we'll learn that this means that these representations are not real representations.

The more common situation, though, is we're figuring out what the irreps are at the same time. For example, for S_3, we know there is a trivial rep, and we know there's the alternating rep $(-1)^\pi$, which I'll call $\mathbf{1}'$. For 1d reps, the character *is* the representation. Since there are three conjugacy classes, this leaves only one more rep, which must satisfy $6 = 1 + 1 + d^2$, so it's 2-dimensional. From here we can figure out the character table just by demanding the orthogonality conditions:

	S_3	n_C	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{2}$
$(1) = \yng(3)$	$\cdot$	1	1	1	2
$(12) = \yng(2,1)$	$\mathbb{Z}_2$	3	1	-1	x
$(123) = \yng(1,1,1)$	$\mathbb{Z}_3$	2	1	1	y

$\longrightarrow$

	S_3	n_C	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{2}$
$(1) = \yng(3)$	$\cdot$	1	1	1	2
$(12) = \yng(2,1)$	$\mathbb{Z}_2$	3	1	-1	0
$(123) = \yng(1,1,1)$	$\mathbb{Z}_3$	2	1	1	-1

We can determine x and y by demanding that the first two rows and the first and last row are orthogonal. In this table, I've added a new column. For each conjugacy class, I put the subgroup generated by its elements. Having worked hard to build this character table, let's use it in some examples.

Collider physics of representation theory. Earlier we referred to irreps as the elementary particles of representation theory. Here is the analog of a collider experiment: given two irreps R_a and R_b, we can make a new, generally reducible rep by taking their tensor product:

$$R_a \otimes R_b = \oplus_c R_c^{\oplus m_c^{ab}}$$

which in turn has a decomposition into irreps. To find the multiplicity m_c^{ab} of a given irrep c in this product, we use the orthonormality of the characters:

$$m_c^{ab} = \langle \chi_c, \chi_{a\otimes b} \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_c(g)^\star \chi_{a\otimes b}(g) = \frac{1}{|G|} \sum_{g\in G} \chi_c(g)^\star \chi_a(g) \chi_b(g).$$

This gives a sort of product law on the irreps of any group – an *algebra* of irreps. It's a product that is harder to visualize than a group product, since superpositions are allowed.

For example, for S_3 we have the following. For any rep, $a \otimes \mathbf{1} = a$. In this case we can just element-wise multiply the columns of the character table and compare:

$$\chi_{\mathbf{1}'}\chi_{\mathbf{1}'} = \chi_{\mathbf{1}}, \quad \chi_{\mathbf{2}} \otimes \chi_{\mathbf{1}'} = \chi_{\mathbf{2}}, \quad \chi_{\mathbf{2}} \otimes \chi_{\mathbf{2}} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \chi_{\mathbf{1}} + \chi_{\mathbf{1}'} + \chi_{\mathbf{2}}.$$

Simple application. Suppose we have a tight-binding model on an equilateral triangle. What are the degeneracies of the levels?

What is the symmetry group of the equilateral triangle, *i.e.* the group of transformations that maps it to itself? Besides the identity, we can rotate by $2\pi/3$ or $4\pi/3$ about the center (this is a $\mathbb{Z}_3$ subgroup), or we can make a reflection across the three medians (each of these is a $\mathbb{Z}_2$ subgroup). These operations permute the three vertices: the $\mathbb{Z}_3$ subgroup are the permutations (123) and (132), while the reflections are (12),

(23) and (31). The symmetry group of the regular n-sided polygon is called D_n, so we've just learned that $D_3 = S_3$.

The vertices transform in the defining, reducible **3** representation of S_3. To answer the question about degeneracies, we need to know how does this compose into irreps. So let's compute its character:

$$\chi_{\mathbf{3}} \begin{pmatrix} (1) \\ (12) \\ (123) \end{pmatrix} \equiv \begin{pmatrix} \chi_{\mathbf{3}}\left((1)\right) \\ \chi_{\mathbf{3}}\left((12)\right) \\ \chi_{\mathbf{3}}\left((123)\right) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

(In the first expression I've 'vectorized' the character function, partly to avoid the awful notation with nested parens in the middle expression.) Where did the numbers come from? The key point is that the character is a *trace* – a sum over diagonal elements, elements which are mapped to themselves by the representation matrix, fixed points. Because it is a permutation representation (that is, it is also a group action), the character for the **3** of a given operation is just the number of vertices that it fixes. More generally, the character for the defining representation of S_n is $\chi_{\mathbf{n}}(\alpha) =$ the number of one-cycles in the conjugacy class α.

So we know that the **3** of S_3 has a decomposition of the form (2.15), and that its character has a decomposition of the form (2.16) and we want to know the m_a. So we need to solve the equation

$$\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} m_{\mathbf{1}} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} m_{\mathbf{1}'} + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} m_{\mathbf{2}} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} m_{\mathbf{1}} \\ m_{\mathbf{1}'} \\ m_{\mathbf{2}} \end{pmatrix},$$

with integer ms. In this case it's easy to do by inspection, but there's a better general way. Rewrite it in the form of a matrix equation

$$\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \chi \begin{pmatrix} m_{\mathbf{1}} \\ m_{\mathbf{1}'} \\ m_{\mathbf{2}} \end{pmatrix} \equiv \chi m,$$

or, with indices, $\chi^{\mathbf{3}}_\alpha = \chi_\alpha{}^a m^{\mathbf{3}}_a$. Here χ is the character table regarded as a (here) 3×3 matrix, $\chi_\alpha{}^a$. Then the solution is just

$$m = \chi^{-1} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{that is,} \quad m_a = \left(\chi^{-1}\right)_{a\alpha} \chi_{3'}(\alpha)$$

(sum on α implied). In this case, $\chi^{-1} = \frac{1}{3!} \begin{pmatrix} 1 & 3 & 2 \\ 1 & -3 & 2 \\ 2 & 0 & -2 \end{pmatrix}$, and therefore $\begin{pmatrix} m_{\mathbf{1}} \\ m_{\mathbf{1}'} \\ m_{\mathbf{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, *i.e.* $\mathbf{3} = \mathbf{1} \oplus \mathbf{2}$. (We knew already that $(1,1,1)$ was an invariant subspace, so it's not surprising to find a trivial rep, $\mathbf{1}$, in there.)

Notice that it's basically impossible to get the wrong answer here[16]. If you make a mistake, like switching the order in which the reps appear in midstream, you'll almost certainly get non-integer answers for the degeneracies. Since the entries of the matrix χ^{-1} are not integers, it can seem rather like a miracle, and definitely is a huge constraint, that $(\chi^{-1})_{a\alpha}\chi_R(\alpha)$ must be an integer for *any* representation R and each irrep a.

Therefore, there is a twofold degenerate level of the tight-binding model on the equilateral triangle. How do I know this? The statement that D_3 is a symmetry of this Hamiltonian means $[H, D_{\mathbf{3}}(g)] = 0$ for all $g \in G$. Thus the Hamiltonian is an intertwiner for this representation! So, by Schur's lemma, on each irrep it must be proportional to the identity matrix. This means that it has degeneracies of the size of each irrep that appears in the decomposition of the Hilbert space.

Let's be bold and generalize to the tight-binding model on the regular n-gon, with symmetry $D_n = \langle a, b | a^n = e, b^2 = e, bab = a^{-1}\rangle$. This symmetry group has a $\mathbb{Z}_n = \langle a | a^n = e\rangle$ subgroup. Just using that we can get quite far in this case, since the adjacency (hopping) matrix is just $H = (D(a) + D(a)^\dagger)$. What are the eigenvalues of $D(a)$? This is the same as reducing it into irreps, which we've seen for the cyclic group is the same as Fourier decomposition: $D(a) = \sum_k \omega^k |k\rangle\langle k|$, with $|k\rangle = \frac{1}{n}\sum_{\ell=1}^n \omega^{-\ell k} |\ell\rangle, k = 1..n$. Therefore the eigenvalues of H are $2\cos 2\pi k/n, k = 1..n$.

To understand a bit more about this answer let's ask how $D(b)$ acts on these states. (Notice that we could have used the character table for the full D_n instead, but chose not to.) Let's take $D(b)$ to act as the reflection

$$D(b)|i\rangle = |-i\rangle = |n-i\rangle .$$

Therefore

$$D(b)|k\rangle = \frac{1}{n}\sum_{\ell=1}^n \omega^{-\ell k}|n-\ell\rangle = \frac{1}{n}\sum_{\ell=1}^n \omega^{\ell k}|\ell\rangle = |-k\rangle .$$

The fact that $[H, D(b)] = 0$ guarantees that the states $|\pm k\rangle$ have the same energy, it is a parity symmetry.

[End of Lecture 6]

Building an irrep from the character table. So we've shown that S_3 must have a 2d rep with characters $\chi_{\mathbf{2}}\begin{pmatrix}(1)\\(12)\\(123)\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}$. But what are the representation

[16] Actually, I must confess that I did have an error in my notes here for a little while. I noticed it only when I added the collider example above.

matrices? Let's try to build them in the basis where (123) and (321) (which commute with each other since they are inverse) are both diagonal. Since they each generate a $\mathbb{Z}_3$ subgroup, their eigenvalues must be cube roots of unity $\{\omega, \omega^2\}$ (other than 1). So what could they be but

$$D(123) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, D(321) = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}.$$

Switching the two would work, but that's just a basis transformation. The characters are $\chi_{\mathbf{2}}(123) = \text{tr}D(123) = \omega + \omega^2 = -1$ which is right. For $D(12)$ we need a traceless 2×2 matrix which under conjugation interchanges the two matrices above (since $(12)(123)(12) = (321)$). It must be $D(12) = \sigma^x$. This is enough information to specify the rest of the matrices, since (12) and (123) generate the whole $S_3 = D_3$. If you are curious, $D(23) = D(123)D(12)D(321) = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}$, also traceless and unitary. You can guess $D(13)$.

To appreciate how highly constrained the representation matrices are, consider the Grand Orthogonality fact

$$\sum_g D^a(g^{-1})_{ij} D^b(g)_{kl} = \frac{|G|}{d_a} \delta^{ab} \delta_{il} \delta_{kj}$$

with $a = \mathbf{2}$ and $b =$ the trivial representation. This says that for all ij

$$\sum_g D^a(g)_{ij} = 0.$$

Furthermore, with $b =$ the sign representation $\mathbf{1}'$, we have another set of (four) constraints:

$$\sum_g (-1)^g D^a(g)_{ij} = 0.$$

Building character tables without knowing the irreps. I also want to emphasize how strongly constrained is the character table of a group. If I call $N \equiv$ the number of irreps of $G =$ the number of conjugacy classes of G, there are actually $2N^2$ constraints on these N^2 numbers: For each pair of rows (conjugacy classes, α, β, including $\alpha = \beta$) we have

$$\sum_{a=1}^{N} \chi_a(\alpha)^\star \chi_a(\beta) = \delta_{\alpha\beta} |G| / n_\alpha \tag{2.17}$$

and for each pair of columns (irreps, a, b, including $a = b$) we have

$$\sum_{\alpha=1}^{N} n_\alpha \chi_a(\alpha)^\star \chi_b(\alpha) = \delta_{ab} |G|. \tag{2.18}$$

Similarly, we can work out the character table for S_4. I give two solutions. One is completely systematic, but uses some prior information about the group. The other attempts to use only subgroup structure and the row and column orthogonality, but uses some educated guesses.

S_4 character table, method one.

1. The first step is to work out the conjugacy classes and their sizes. For S_n these are Young diagrams with n boxes, and the number of elements in conjugacy class $n_\alpha = \frac{n!}{\prod_j j^{k_j} k_j!}$ where the diagram has k_j j-cycles ($\equiv$ columns of height j). Don't forget to include the 1-cycles in this formula.

2. Next, we decompose the order of the group into a sum of squares, the dims of the irreps. For $|S_4| = 24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$, there is only one way to do this with five numbers. At this point the information we have is:

S_4		n_C	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3'}$
$(1) =$	$\cdot$	1	1	1	2	3	3
$(12) =$	$\mathbb{Z}_2$	6	1				
$(12)(34) =$	$\mathbb{Z}_2$	3	1				
$(123) =$	$\mathbb{Z}_3$	8	1				
$(1234) =$	$\mathbb{Z}_4$	6	1				

3. The characters of the sign representation $\mathbf{1}$ are easy: it's -1 for the odd permutations:

S_4		n_C	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3'}$
$(1) =$	$\cdot$	1	1	1	2	3	3
$(12) =$	$\mathbb{Z}_2$	6	1	-1			
$(12)(34) =$	$\mathbb{Z}_2$	3	1	1			
$(123) =$	$\mathbb{Z}_3$	8	1	1			
$(1234) =$	$\mathbb{Z}_4$	6	1	-1			

4. A representation of S_4 that we know is the defining $\mathbf{4}$ dimensional representation. We've seen that it's reducible since the uniform state (the ones vector) is invariant. Its character is

$$\chi_{\mathbf{4}} \begin{pmatrix} e \\ (12) \\ (12)(34) \\ (123) \\ (1234) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}.$$

Since this is a permutation representation, the first equation follows by counting fixed points. We'll see the importance of the second equation in a moment. Its norm is therefore

$$\langle \chi_{\mathbf{4}}, \chi_{\mathbf{4}} \rangle = \frac{1}{24}\left(1 \cdot 4^2 + 6 \cdot 2^2 + 3 \cdot 0^2 + 8 \cdot 1^2 + 6 \cdot 0^2\right) = \frac{16 + 24 + 8}{24} = 2.$$

This means that it contains two irreps. That is, we've confirmed that there is a 3-dimensional irrep (which we knew from the diophantine equation). Moreover, since $\mathbf{4} = \mathbf{1} + \mathbf{3}$ we have $\chi_{\mathbf{4}} = \chi_{\mathbf{1}} + \chi_{\mathbf{3}}$. Thus we know

S_4		n_C	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3'}$
$(1) =$	$\cdot$	1	1	1	2	3	3
$(12) =$	$\mathbb{Z}_2$	6	1	-1		1	
$(12)(34) =$	$\mathbb{Z}_2$	3	1	1		-1	
$(123) =$	$\mathbb{Z}_3$	8	1	1		0	
$(1234) =$	$\mathbb{Z}_4$	6	1	-1		-1	

Notice that our answer for $\chi_{\mathbf{3}}$ that we found by this trick has norm 1 and is orthogonal to the other already-known characters.

5. One more trick. The tensor product $\mathbf{1'} \otimes \mathbf{3}$ gives a 3-dimensional representation with character $\chi_{\mathbf{1'}\otimes\mathbf{3}} = \chi_{\mathbf{1'}}\chi_{\mathbf{3}}$. Its norm is 1 so it is an irrep, and it is orthogonal to $\chi_{\mathbf{3}}$, so this must be $\mathbf{3'} = \mathbf{1'} \otimes \mathbf{3}$. So we have

S_4		n_C	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3'}$
$(1) =$	$\cdot$	1	1	1	2	3	3
$(12) =$	$\mathbb{Z}_2$	6	1	-1	w	1	-1
$(12)(34) =$	$\mathbb{Z}_2$	3	1	1	x	-1	-1
$(123) =$	$\mathbb{Z}_3$	8	1	1	y	0	0
$(1234) =$	$\mathbb{Z}_4$	6	1	-1	z	-1	1

Notice that $\chi_{\mathbf{3'}} = \chi_{\mathbf{3}\otimes\mathbf{1'}}$ is orthogonal to $\chi_{\mathbf{3}}$ (and the other columns we know already).

6. From here, row orthonormality of each row with itself gives

$$4 + |w|^2 = 24/6 = 4, 4 + |x|^2 = 24/3 = 8, 2 + |y|^2 = 24/8 = 3, 4 + |z|^2 = 24/6 = 4$$

so we learn $w = 0, |x| = 2, |y| = 1, |z| = 0$.

One more generally-useful piece of information: Since every permutation is conjugate in S_n to its inverse (it has the same cycle structure), $\chi(g) = \chi(g^{-1}) = \chi(g)^\star$, the characters are all real.

So we have $x = \pm 2, y = \pm 1$:

S_4		n_C	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3'}$
$(1) =$	$\cdot$	1	1	1	2	3	3
$(12) =$	$\mathbb{Z}_2$	6	1	-1	0	1	-1
$(12)(34) =$	$\mathbb{Z}_2$	3	1	1	± 2	-1	-1
$(123) =$	$\mathbb{Z}_3$	8	1	1	± 1	0	0
$(1234) =$	$\mathbb{Z}_4$	6	1	-1	0	-1	1

Orthogonality between rows 1 and 3 gives $1 + 1 + 2x - 3 - 3 = 0$ so $x = 2$. Orthogonality between rows 1 and 4 gives $1 + 1 + 2y = 0$ so $y = -1$.

S_4 **character table, second method.** Here is another series of (mostly deductive) steps that solves this puzzle without using prior knowledge of some irreps.

1. The first step is to work out the conjugacy classes and their sizes. For S_n these are Young diagrams with n boxes, and the number of elements in conjugacy class $n_\alpha = \frac{n!}{\prod_j j^{k_j} k_j!}$ where the diagram has k_j j-cycles ($\equiv$ columns of height j). Don't forget to include the 1-cycles in this formula.

2. Next, we decompose the order of the group into a sum of squares, the dims of the irreps. For $|S_4| = 24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$, there is only one way to do this. At this point the information we have is:

S_4		n_C	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3'}$
$(1) =$	$\cdot$	1	1	1	2	3	3
$(12) =$	$\mathbb{Z}_2$	6	1				
$(12)(34) =$	$\mathbb{Z}_2$	3	1				
$(123) =$	$\mathbb{Z}_3$	8	1	x			
$(1234) =$	$\mathbb{Z}_4$	6	1	y			

3. In this enlarged table, I've also indicated the subgroup generated by elements of each conjugacy class. This is valuable information because it means, for example, that the characters in 1d reps of an element generating a $\mathbb{Z}_n$ must be nth roots of unity. More generally, if $g^n = 1$, then $D_R(g)^n = 𝟙$, so all of the eigenvalues of $D_R(g)$ are nth roots of unity, so $\chi_R(g) = \mathrm{tr} D_R(g)$ is the sum of these nth roots of unity[17]. In the special case where R is one-dimensional, there is just one term in the sum.

[17] Incidentally, this shows that every entry in the character table for a finite group is a sum of (finite) roots of unity and hence an algebraic integer.

So $x \in \{1, \omega, \omega^2\}$ But column orthogonality with the trivial representation forbids $x = \omega$ or ω^2, since none of the other conjugacy classes can cancel this contribution. So $x = 1$.

The same argument says $y \in \{\pm 1, \pm \mathbf{i}\}$, but by column orthogonality with the first column, y can only be ± 1. Therefore

S_4		n_C	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3'}$
$(1) =$	$\cdot$	1	1	1	2	3	3
$(12) =$	$\mathbb{Z}_2$	6	1	a			
$(12)(34) =$	$\mathbb{Z}_2$	3	1	b			
$(123) =$	$\mathbb{Z}_3$	8	1	1			
$(1234) =$	$\mathbb{Z}_4$	6	1	± 1			

4. Column orthonormality of $1'$ with itself then says $1^2 + 6|a|^2 + 3|b|^2 + 8 + 6 = 24$, from which we conclude $|a| = |b| = 1$. We already knew $a, b \in \{\pm 1\}$ since those conjugacy class elements generate $\mathbb{Z}_2$ subgroups. Column orthogonality between the first two columns gives

$$1 + 6a + 3b + 8 \pm 6 = 0$$

which is only solved by $b = 1, a = -1, y = -1$. Therefore

S_4		n_C	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3'}$
$(1) =$	$\cdot$	1	1	1	2	3	3
$(12) =$	$\mathbb{Z}_2$	6	1	-1	a	b	c
$(12)(34) =$	$\mathbb{Z}_2$	3	1	1			
$(123) =$	$\mathbb{Z}_3$	8	1	1			
$(1234) =$	$\mathbb{Z}_4$	6	1	-1	d	e	f

5. Row orthonormality for the second row says $1^2 + (-1)^2 + |a|^2 + |b|^2 + |c|^2 = 24/6 = 4$, so $|a|^2 + |b|^2 + |c|^2 = 2$. Row orthogonality between the first and second rows says $2a + 3b + 3c = 0$. The same arguments for the last row say $|d|^2 + |e|^2 + |f|^2 = 0$ and $2d + 3e + 3f = 0$ and in addition $2 + a^\star d + b^\star e + c^\star f = 0$. Hang on a second: why am I bothering to write complex conjugates here? Since every permutation is conjugate in S_n to its inverse (it has the same cycle structure), the characters are all real.

   At this point, let's *guess* that these objects are not only real but integers[18]. Then solution of these equations is $(a, b, c) = (0, 1, -1)$ and $(d, e, f) = (0, -1, 1)$. (We

[18]This guess will turn out to work here, but the characters need not be integers. We already have seen examples where they are nth roots of unity. More generally they are always *algebraic integers* – roots of polynomials with integer coefficients. We will give a proof of this statement around equation (2.26).

could have switched the two but that just amounts to a relabelling of **3** and **3′**, which we haven't defined yet in any other way.) There might be others, but let's try it and see what happens.

S_4		n_C	**1**	**1′**	**2**	**3**	**3′**
$(1) = \ydiagram{4}$	$\cdot$	1	1	1	2	3	3
$(12) = \ydiagram{3,1}$	$\mathbb{Z}_2$	6	1	−1	0	1	−1
$(12)(34) = \ydiagram{2,2}$	$\mathbb{Z}_2$	3	1	1	l	m	n
$(123) = \ydiagram{2,1,1}$	$\mathbb{Z}_3$	8	1	1	g	h	i
$(1234) = \ydiagram{1,1,1,1}$	$\mathbb{Z}_4$	6	1	−1	0	−1	1

6. Row orthogonality between the unknown rows and the last row gives $1-1-h+i = 0, 1-1-m+n = 0$, so $h = i, m = n$

S_4		n_C	**1**	**1′**	**2**	**3**	**3′**
$(1) = \ydiagram{4}$	$\cdot$	1	1	1	2	3	3
$(12) = \ydiagram{3,1}$	$\mathbb{Z}_2$	6	1	−1	0	1	−1
$(12)(34) = \ydiagram{2,2}$	$\mathbb{Z}_2$	3	1	1	l	m	m
$(123) = \ydiagram{2,1,1}$	$\mathbb{Z}_3$	8	1	1	g	h	h
$(1234) = \ydiagram{1,1,1,1}$	$\mathbb{Z}_4$	6	1	−1	0	−1	1

7. Orthonormality for the 4th row says $1^2+1^2+g^2+2h^2 = 24/8 = 3$, which requires $g^2+2h^2 = 1$. The only integer solution is $h = 0, g = \pm 1$, so let's try that. $r_1 \perp r_4$ then requires $2+2g = 0$ so $g = -1$. $r_3 \perp r_4$ then requires $2+gl = 0$ so $l = 2$. $r_1 \perp r_3$ then requires $2+4+6m = 0$ so $m = -1$.

The final result is:

S_4		n_C	**1**	**1′**	**2**	**3**	**3′**
$(1) = \ydiagram{4}$	$\cdot$	1	1	1	2	3	3
$(12) = \ydiagram{3,1}$	$\mathbb{Z}_2$	6	1	−1	0	1	−1
$(12)(34) = \ydiagram{2,2}$	$\mathbb{Z}_2$	3	1	1	2	−1	−1
$(123) = \ydiagram{2,1,1}$	$\mathbb{Z}_3$	8	1	1	−1	0	0
$(1234) = \ydiagram{1,1,1,1}$	$\mathbb{Z}_4$	6	1	−1	0	−1	1

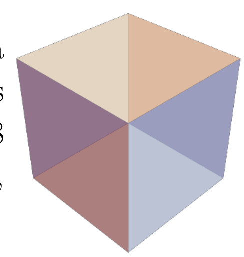
In the discussion above, we imagine we are given a group (say a presentation of it, or some other definition) and from that we can determine its character table. An interesting question is whether we can figure out all possible character tables without starting from the information of the list of all possible groups! Such a method is called 'bootstrap': the idea is that we impose just essential consistency conditions on the thing we want to find and try to carve out the space of allowed values of the thing. This strategy has been very successful in various areas of physics, such as the study of scattering amplitudes, of conformal field theory, of topological phases of matter. Here, for each N, we would like to find $N + N^2$ numbers ($n_\alpha \in \mathbb{Z}_+$ and $\chi^\alpha_a \in \mathbb{C}$, with $a = 1..N, \alpha = 1..N$) satisfying the $2N^2$ conditions (2.17) and (2.18). Some questions: Is the fact that $n_\alpha||G|$ for each α automatic from this input, or should we impose that separately? Is the fact that χ^α_a is an algebraic integer automatic from this input?

Cube example. Finally we return to the motivating example at the beginning of the notes – the tight-binding model on the vertices of the cube. The group of rotations that map the cube to itself is called O (for octahedron). Why octahedron? Put a vertex in the center of each of the six faces of the cube, and draw edges between vertices in pairs of faces that share an edge. Put a face for every vertex of the cube. Voila, an octahedron. The cube and the octahedron are dual under this operation that exchanges p-cells of one with $(D - p)$-cells of the other. (It is an avatar of Poincaré duality.) For our purposes here, the important point is that the cube and the octahedron share the same symmetry group.

But what is the group, O? Here is an inventory of the elements, besides the identity:

- Draw a line through the centers of two opposite faces. There is a $\mathbb{Z}_4$ subgroup of $\pi/2$ rotations fixing this line. Since there are 3 choices of pairs of opposite faces, this gives $(4 - 1) \cdots 3 = 9$ elements. There are 6 order-4 elements and 3 order-2 elements. Each of these fix 2 faces.

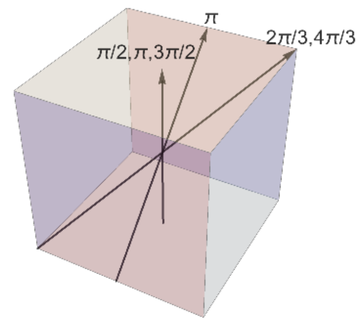
- Draw a line from a vertex to the farthest vertex (this is called a 'large diagonal'). There is a $\mathbb{Z}_3$ subgroup of $2\pi/3$ rotations fixing this line. There are 4 pairs of opposite vertices, so this gives $(3-1)\cdot 4 = 8$ elements. These fix no faces. To visualize better the order-3 elements, consider the cube viewed from a large diagonal, as at right.



- Draw a line from the midpoint of an edge to the midpoint of the farthest parallel edge. There is a $\mathbb{Z}_2$ subgroup of π rotations fixing this line. There are 2 pairs of opposite edges in each of 3 directions, so this gives $(2 - 1) \cdot 6 = 6$ elements. These fix no faces.

Altogether we find $1 + 9 + 8 + 6 = 24$ elements. Different choices of faces, vertex and edge in the above list produce elements that are conjugate to each other.

Since $24 = 4!$ we might guess that this group is isomorphic to S_4. What are the four things being permuted? The large diagonals of the cube. The groups of elements above correspond respectively to the conjugacy classes (1234) and (12)(34) together (order 4 and order 2), (123) (order 3) and (12) (order 2). (To verify that the π rotation about the line between the midpoints of opposite edges indeed exchanges two of the four long diagonals I had to play with a 6-sided die for a while.)



$\mathsf{O} = S_4$		n_C	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3}'$
$(1) = \yng(4)$	$\cdot$	1	1	1	2	3	3
$(12) = \yng(3,1)$	$\mathbb{Z}_2$	6	1	-1	0	1	-1
$(12)(34) = \yng(2,2)$	$\mathbb{Z}_2$	3	1	1	2	-1	-1
$(123) = \yng(2,1,1)$	$\mathbb{Z}_3$	8	1	1	-1	0	0
$(1234) = \yng(1,1,1,1)$	$\mathbb{Z}_4$	6	1	-1	0	-1	1

Given the above inventory of the elements, it is not hard to compute the characters of this 6-dimensional representation coming from the action on the faces of the cube. Again we count fixed points:

$$\chi_{\Gamma_F}\begin{pmatrix} (1) \\ (12) \\ (12)(34) \\ (123) \\ (1234) \end{pmatrix} \equiv \begin{pmatrix} \chi_{\Gamma_F}((1)) \\ \chi_{\Gamma_F}((12)) \\ \chi_{\Gamma_F}((12)(34)) \\ \chi_{\Gamma_F}((123)) \\ \chi_{\Gamma_F}((1234)) \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} \overset{!}{=} Cm.$$

[End of Lecture 7]

The last equation specifies the multiplicities m_a^Γ of each irrep in this reducible representation, and C is the character table regarded as a 5×5 matrix. We conclude

$$m = C^{-1}\begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{24}\begin{pmatrix} 1 & 6 & 3 & 8 & 6 \\ 1 & -6 & 3 & 8 & -6 \\ 2 & 0 & 6 & -8 & 0 \\ 3 & 6 & -3 & 0 & -6 \\ 3 & -6 & -3 & 0 & 6 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

that is, $\Gamma_F = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3}'$. Notice that the dimensions add up correctly. A less elegant way to think about the step where we took the inverse of the character table is that we solved the equation

$$\begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} = m_{\mathbf{1}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + m_{\mathbf{1'}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + m_{\mathbf{2}} \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} + m_{\mathbf{3}} \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} + m_{\mathbf{3'}} \begin{pmatrix} 3 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} .$$

On the homework, you can do the case where we act on the faces of the octagon (or vertices of the cube).

This does not quite answer the problem stated at the beginning of the class. To answer that question (*e.g.* in the incarnation: starting with an initial set of numbers on the faces of the cube, if at each time-step we replace each temperature by the average of its four neighbors, what is the temperature after t steps and what is the error in the estimate?), we must know not only the degeneracies but also the eigenvalues.

We know one eigenvector: which functions on the faces transform in the trivial representation? Ones that are uniform. This state $|u\rangle = \frac{1}{\sqrt{6}} \sum_j |j\rangle$ has eigenvalue 1 under $H = \frac{1}{4} \sum_{\langle ij \rangle} |i\rangle\langle j|$. (I normalize H in this way so that its operation corresponds to replacing the value on a face with the average of its neighbors.)

Since H is hermitian, all its other eigenvectors $\sum_j \psi_j |j\rangle$ have to be orthogonal to this one. This means that they have to satisfy $\sum_j \psi_j = 0$. With a small further geometrical insight, we can now solve the whole problem: consider functions that are the same on opposite faces. There is a 2-dimensional space of such functions that are orthogonal to the uniform function (there are 3 sets of opposite faces, and we require $a + b + c = 0$), this is the **2**. Finally, there is a 3-dimensional space of functions where opposite faces have opposite values – this is the **3**.

What happens if we act with H on a state in the **3**? A face with value b has neighbors with values $a, -a, c, -c$, so the final value is $a - a + c - c = 0$. H annihilates a state in the **3**.

Acting on a state in the **2**, a face labelled b turns into $\frac{a+a+c+c}{4} = \frac{1}{2}(a + c) = -\frac{1}{2}b$, where I used the $a + b + c = 0$ constraint.

Thus: the eigenvalues are: $1, 0, 0, 0, -\frac{1}{2}, -\frac{1}{2}$. So now we can answer the question: what is the final value? First write the H operator in its spectral representation $H = \sum_\lambda \lambda P_\lambda$ where P_λ is the projector onto the subspace with eigenvalue λ. Then repeated action of H on an initial state $|f_0\rangle = \sum_j f_0(j) |j\rangle$ gives

$$|f_n\rangle = H^n |f_0\rangle = \sum_\lambda \lambda^n P_\lambda |f_0\rangle .$$

Since $1^n \gg 0^n, \left(-\frac{1}{2}\right)^n$, this rapidly approaches $P_1 |f_0\rangle = |u\rangle\langle u|f_0\rangle = \frac{1}{6}\sum_k f_0(k) \sum_j |j\rangle$. The numbers all approach the mean of the initial numbers. What about the rate of approach to the final configuration? The **3** doesn't contribute at all after the initial step, and after t steps, the contribution of the **2** is suppressed by a factor of 2^{-t}.

You might not be entirely satisfied by the use of an extra piece of geometric insight here. It is actually possible to answer the question completely systematically with a little bit more work. In particular, using only the characters and the reducible representation matrices, we can construct *projection operators* onto the irreps (actually onto the subspace composed of all copies of a given irrep).

Reflections. Actually, the symmetry group of an abstract, undecorated cube is larger than O. O just contains the rotations. If we also allow reflections, we add one more generator I'll call P, which we can describe in terms of its action on the coordinates of the space in which the cube is embedded with its center at the origin as $D(P) : (x, y, z) \to (-x, -y, -z)$. This is not a rotation because $\det D(P) = -1$. (P stands for 'parity'.)

If we allow such reflections, the symmetry group is then $\mathsf{O} \times \mathbb{Z}_2$. This is a common situation in crystallography, and so it is worth recording the character table for $G \times \mathbb{Z}_2 = G \times \langle P | P^2 = e\rangle$. Each conjugacy class of G produces two conjugacy classes, C and CP, and each irrep of G produces two irreps: one which is even under P and one which is odd. [19]

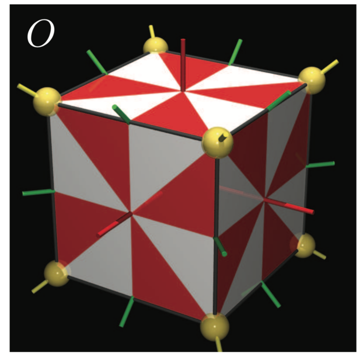
$G \times \mathbb{Z}_2$	R_a^+	R_a^-
C_α	χ_α^a	χ_α^a
PC_α	χ_α^a	$-\chi_\alpha^a$

(2.19)

[19]**Digression and diatribe on names and crystallography.** In the context of crystallography, instead of $+$ and $-$ these are labelled with subscripts g and u, which are the first letters of the German words 'gerade' and 'ungerade' (for 'even' and 'odd'). What a waste of two perfectly good letters. It is almost as bad as s, p, d, f... in spectroscopy. In that context, $\mathsf{O} \times \mathbb{Z}_2$ is called O_h.

While I'm at it, I have to complain some more about the crystallographers in the guise of teaching you some of their notation. I've said that the symmetry group of the regular n-gon was called D_n, the dihedral group. There is a large group of people who will instead call it C_{nv}. These people (some of them are even my friends) also use C_n to denote $\mathbb{Z}_n$.

What is an object that has O symmetry but not $\mathsf{O} \times \mathbb{Z}_2$ symmetry? At right is a picture of such an object which I got from Dan Arovas, who got it from here. I guess it can be called a 'chiral cube,' since it has a handedness.



2.3 Projection operators

The characters give a means to find not only how many of a given irrep appear in a representation, but actual projectors into those irreps. Here it is: given a reducible representation with representation operators D,

$$P_a = \frac{d_a}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \chi_a(g)^\star D(g). \tag{2.20}$$

This operator projects onto $\underbrace{R_a \oplus R_a \oplus \cdots}_{m_a \text{ times}}$. Note that $P_a = P_a^\dagger$. These projectors are orthogonal $P_a P_b = \delta_{ab} P_a$ by virtue of the GOT (see the homework). This formula has a strong air of plausibility about it, but next we discuss some technology that makes it inevitable.

Group algebra. The regular representation actually has more structure. Let's denote the basis vectors by $\mathbf{g}_i$, so a general element of the vector space is $\mathbf{x} = \sum_{i=1}^{|\mathsf{G}|} x_i \mathbf{g}_i$, $x_i \in \mathbb{C}$. Define a multiplication rule on this space by

$$\mathbf{g}\mathbf{x} = \sum_{i=1}^{|\mathsf{G}|} x_i \mathbf{g}\mathbf{g}_i,$$

where $\mathbf{g}\mathbf{g}_i$ is another basis element. This is called the *group algebra*, $\mathbb{C}[G]$. Algebra just means a group where you can add as well as multiply (and multiply by scalars), or a vector space where two vectors can also be multiplied to give another vector.

Here's a reason this is useful. Any representation $D(g)$ of the group is also a representation of the group algebra. (A representation of an algebra just means an algebra homomorphism – a map which plays nicely with all the structure of the algebra.)

$$D(x_1 \mathbf{g}_1 + x_2 \mathbf{g}_2 + \cdots) = x_1 D(g_1) + x_2 D(g_2) + \cdots . \tag{2.21}$$

Now consider the following elements of the group algebra:

$$\mathbf{e}^a_{ij} = \frac{d_a}{|\mathsf{G}|}\sum_{g\in\mathsf{G}} \left(D^a_{ij}(g)\right)^\star \mathbf{g},$$

where $D^a_{ij}(g)$ are the matrix elements for irrep a. What happens when we multiply by another element? I claim that

$$\mathbf{h}\mathbf{e}^a_{ij} = \sum_{k=1}^{d_a} \left(D^a(h)^T\right)_{ik} \mathbf{e}^a_{kj} \tag{2.22}$$

where $\left(M^T\right)_{ik} \equiv M_{ki}$ is the transpose operation. The steps involved are: relabel summation variables $g' \equiv hg$, just as in the character-table-is-square proof (in fact this construction is also a proof of that fact) and use the fact that $D^a(h^{-1}g) = D^a(h^{-1})D^a(g)$ is a representation. Then (2.22) implies (using the Grand Orthogonality Fact) that

$$\mathbf{e}^a_{ij}\mathbf{e}^b_{kl} = \delta^{ab}\delta_{jk}\mathbf{e}^a_{il}. \tag{2.23}$$

This is the multiplication rule for the set of matrices $e_{\alpha\beta}$ with zeros everywhere except for a 1 in the $\alpha\beta$th entry. They form a basis for the algebra (since there are $\sum_{\text{irreps},a} \dim R_a^2 = |\mathsf{G}|$ of them and they are orthonormal and hence linearly independent), so an arbitrary element is $\mathbf{g} = \sum_{aij}\left(D^a(g)^T\right)_{ij}\mathbf{e}^a_{ij}$ (as you can check by expanding with arbitrary coefficients $\mathbf{g} = \sum_{aij} c_{aij}(g)\mathbf{e}^a_{ij}$, multiplying the BHS by $\mathbf{e}^b_{kl}$ and solving for the coefficients $c_{aij}(g)$ using (2.23) and (2.22)). To see the projection property, consider an arbitrary vector $|v\rangle$ in the carrier space, and construct

$$|v, ika\rangle \equiv \hat{D}(\mathbf{e}^a_{ik})\,|v\rangle$$

(I am using (2.21) to get a linear operator on V from a group algebra element). In the given rep, this transforms into

$$\hat{D}(g)\,|v, ika\rangle = \hat{D}(g)\hat{D}(\mathbf{e}^a_{ik})\,|v\rangle = \hat{D}(g\mathbf{e}^a_{ik})\,|v\rangle \overset{(2.22)}{=} \hat{D}(\mathbf{e}^a_{jk})\,|v\rangle\, D^a(g)_{ji} = |v, jka\rangle\, D^a(g)_{ji}.$$

So if it isn't zero, $|v, ika\rangle$ transforms in the rep R^a.

Finally, the projectors onto irreps are

$$\mathbf{P}^a = \sum_i \mathbf{e}^a_{ii} = \frac{d_a}{|\mathsf{G}|}\sum_{g\in\mathsf{G}} \chi^a(g)^\star\mathbf{g}.$$

They satisfy $\mathbf{P}^a\mathbf{P}^b = \delta^{ab}\mathbf{P}^a$ (using (2.23)), and $\sum_a \mathbf{P}^a = \mathbf{e}$, the identity element of the group algebra $\mathbb{C}[\mathsf{G}]$ (straight from the definition using row orthogonality). Acting in a given reducible representation using (2.21), this becomes the projector we wrote above. Notice that this object can be constructed just from the character table, without knowing explicitly the matrices for the irreps (unlike the $\mathbf{e}^a_{ij}$) – we only need to know the matrices for the representation we're decomposing.

Example. So in the example of the equilateral triangle, using the matrices in (2.2),

$$P_{\mathbf{1}} = \frac{1}{6}\left(\sum_g D(g)\right) = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$P_{\mathbf{1}'} = \frac{1}{6}\left(D(e) + (D(123) + D(321)) - (D(12) + D(23) + D(31))\right) = 0$$

$$P_{\mathbf{2}} = \frac{2}{6}\left(2D(e) - (D(123) + D(321))\right) = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} .$$

So you can see that $P_{\mathbf{1}} = |u\rangle\langle u|$, the projector onto the ones vector. These projectors are orthogonal $P_a P_b = \delta_{ab} P_a$. Once we have the projectors, it's easy to figure out the eigenvalues of a matrix that commutes with the symmetry (as long as the multiplicities m_a are not too large). The tight-binding hamiltonian on the equilateral triangle is

$$H = -t \sum_{i=1}^{3} \left(|i\rangle\langle i+1| + |i+1\rangle\langle i|\right)$$

where the arguments of the kets are understood modulo 3. In fact this operator has a lot in common with $P_{\mathbf{2}}$:

$$H = tP_{\mathbf{2}} - 2tP_{\mathbf{1}}.$$

Its eigenvectors are therefore the two basis elements of the **2** plus an orthonormal vector, which is the ones vector. The eigenvalues and eigenvectors are therefore:

$$\mathbf{2} \text{ has energy } t, \quad \mathbf{1} \text{ has energy } -2t.$$

We conclude that the first excited state is doubly degenerate, and the gap above the groundstate is $3t$. No matrices were (explicitly) diagonalized in the course of this calculation. [End of Lecture 8]

Classical mechanics examples. [Zee III.2, Arovas §2.7, Georgi, 2d edition, §1.16, 1.17] Representation theory is more obviously useful in quantum mechanics than in classical mechanics, since the former is basically just linear algebra (!), whereas the latter is a horrible monstrosity that can involve lots of non-linear things. This last statement is not true if we restrict our attention to small oscillations about equilibrium. For a collection of N particles (say equal mass for simplicity) in d dimensions, with x^{ia} is the deviation from equilibrium of the ath particle in the ith direction, Newton's equation is

$$\ddot{x}^{ia} = -\sum_{bj} H^{ia,jb} x^{jb} \tag{2.24}$$

where H is a real symmetric matrix since it comes from the second derivatives of the potential. We can regard $A \equiv ia$ as a multi-index. Plugging in $x^A(t) = x^A e^{\mathbf{i}\omega t}$ (2.24) is $H^{AB}x^B = \omega^2 x^A$, and we want to find the normal modes x^A and their spectrum – the eigenvectors and eigenvalues of the H^{AB}.

If the equilibrium configuration and H have enough symmetry, G, we can do this without even writing down the matrix H^{AB}, never mind diagonalizing it. x^A lives in a dN-dimensional representation R of G. Decomposing $R = \oplus_a V_a^R \otimes R_a$ into irreps that occur $m_a^R = \dim V_a^R$ times reduces this to a problem of diagonalizing a collection of $m_a \times m_a$ matrices.

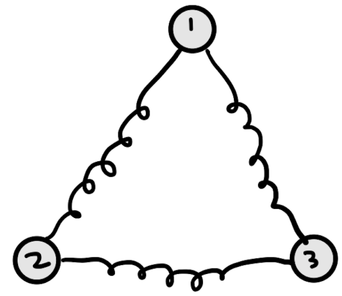
Warmup: $d = 1, N = 2$. 

Here $G = S_2$, no matter where the equilibrium is. The representation on (x_1, x_2) is $D(e) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, D(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The character is $\chi\begin{pmatrix} e \\ (12) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, since the swap fixes no one. The character table of $S_2 = \mathbb{Z}_2$ is

S_2		n_C	$\mathbf{1}$	$\mathbf{1'}$
(1)	$\cdot$	1	1	1
(12)	$\mathbb{Z}_2$	1	1	-1

. Therefore $R = \mathbf{1} \oplus \mathbf{1'}$. If the potential is just a function of the particle separation, one eigenvector is obvious: the uniform vector is a singlet, with eigenvalue 0, since it doesn't stretch the springs – this is called a *zeromode*. The other eigenvector must be orthogonal to this, so it is $\begin{pmatrix} x \\ -x \end{pmatrix}$, with eigenvalue 2.

Let's do $d = 2, N = 3$ with the equilibrium configuration an equilateral triangle, like a symmetrical triatomic molecule. This preserves $G = D_3 = S_3$. It is a $dN = 6$-dimensional representation with character



$$\chi_{\mathbf{6}} \begin{pmatrix} e \\ (12) \\ (123) \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}.$$

This is not quite a permutation representation, but it is still possible to compute the trace of the operators by counting. The identity fixes all 6 coordinates, hence the 6. The rotation operation which maps the picture back to itself involves a rotation by $2\pi/3$ combined with a relabelling ($3 \to 1 \to 2$). This fixes noone. Similarly, the reflection operation also involves a relabelling and fixes no one.

There is another useful way to think about this 6d rep: it is $\mathbf{6} = \mathbf{2} \otimes \mathbf{3}$, where $\mathbf{3}$ is the fundamental (reducible) rep of S_3, and $\mathbf{2}$ is the irrep. Why is this? Well, this picture actually provides a construction of the $\mathbf{2}$ of S_3: it comes from the action of S_3 on a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by $2\pi/3$ rotations and reflections of the plane. The 6d rep above is the product of this with the $\mathbf{3}$ because of the relabelling required to put the particles back where they started[20].

You can see from the character that it's actually the regular rep of S_3. Using the inverse of the character table, we extract $(\chi^{-1})_a^{\alpha} \chi_{\mathbf{6}}(\alpha) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}_a$, so $\mathbf{6} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2} \oplus \mathbf{2}$. (A rep of dim d_a appears d_a times in the regular rep.) So in this example, in principle we actually have to diagonalize a 2×2 matrix acting on the two $\mathbf{2}$s.

Actually, we don't. If the potential is really just a function of the particle sepa-

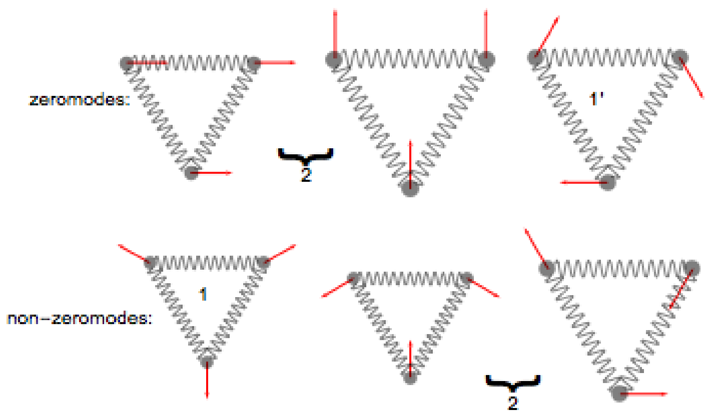
[20] I have to admit that I find this confusing. There are two other possibilities for the nature of the 6d rep here. One possibility is that we don't act on the labels at all, in which case the rep is $(\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}) \otimes \mathbf{2} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{2}$ with character $3\chi_{\mathbf{2}} = \begin{pmatrix} 6 \\ 0 \\ -3 \end{pmatrix}$.

The third possibility is the one discussed in Tony Zee's book, which gives the character $\begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} = \chi_{S_3}^{-1} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$. I believe this answer arises because he takes the reflection $D(23)$ to act as $(x_1, y_1, x_2, y_2, x_3, y_3) \mapsto (x_1, y_1, x_3, y_3, x_2, y_2)$ rather than $(x_1, y_1, x_2, y_2, x_3, y_3) \mapsto (-x_1, y_1, -x_3, y_3, -x_2, y_2)$, the latter of which is indeed traceless. Notice that the latter operation fixes the *equilibrium* position of particle 1, but not the general configuration which has $x_1 \neq 0$.

These answers are different! I believe the one in the text above is the correct one.

rations, there are some zeromodes: translating everyone in x or y doesn't stretch the springs. This is $\begin{pmatrix}1\\0\\1\\0\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\\1\\0\\1\end{pmatrix}$. Similarly, if we rotate all three particles about their center of mass, the springs don't stretch. These three modes account respectively for one of the **2**s and for one of the singlets.

The other singlet is not a zeromode, but is quite understandable: it is the breathing mode, where all the particles move radially away from their center of mass. Since H is ral and symmetric, the final **2** is determined by the fact that it has to be orthonormal to these other 4 modes.



The little arrows I drew in this figure were all calculated just using group theory, using (2.20). We need the explicit matrix representation of the irreps. Fortunately, using $\mathbf{6} = \mathbf{2} \otimes \mathbf{3}$, we have $(D_{\mathbf{6}}(g))_{im,jn} = (D_{\mathbf{3}}(g))_{i,j}\,(D_{\mathbf{2}}(g))_{m,n}$. The matrices for the **3** are in (2.2). The matrices for the **2** are just $2\pi/3$ rotations and reflections[21].

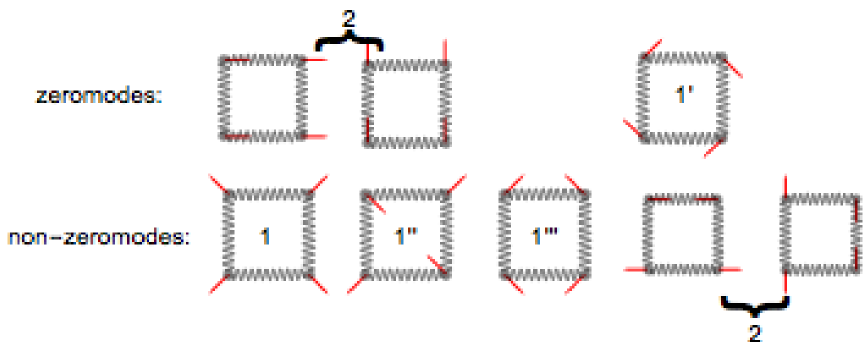
Putting this together (I recommend mathematica), we find, just by adding, that $P^{\mathbf{1}}$ is a rank-one matrix whose image is spanned by the vector $(1/2, -\sqrt{3}/6, -1/2, -\sqrt{3}/6, 0, 1/\sqrt{3})$ (the image of the projector can be found by acting on an arbitrary vector). (Note that the rank of a projector is just its trace, since its eigenvalues are all zero or one.) These three pairs of numbers are the directions of the three arrows in the bottom left mode labelled **1**.

Similarly, $P^{\mathbf{1}'}$ is a rank one projection matrix whose image describes the picture labelled $\mathbf{1}'$. $P^{\mathbf{2}}$ is a rank-4 projector. The projector for the multiplet of zeromodes is

[21] If you insist: the generators are represented by $D_{\mathbf{2}}(23) = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ and $D_{\mathbf{2}}(123) = \begin{pmatrix}c & s\\ -s & c\end{pmatrix}$, with $c = \cos(2\pi/3) = -1/2, s = \sin(2\pi/3) = \sqrt{3}/2$.

$T_x + T_y = \frac{1}{3}|u\rangle\langle u| \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + |u\rangle\langle u| \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |u\rangle\langle u| \otimes 1\!\!1_2$ where $|u\rangle$ is the (normalized) ones vector. The projector on the nontrivial bit is then $\mathbf{P^2} - T_x - T_y$, whose image is two dimensional and spanned by the bottom right pictures (pick a random vector, act on it with $\mathbf{P^2} - T_x - T_y$, and normalize it; then pick another one and do Gram-Schmidt to get a second orthogonal basis vector).

Here are the normal modes for the case with 4 atoms in a square, made using the same code (basically just changing some 3's to 4's):



Comment on accidental degeneracy. Back to QM on a Hilbert space $\mathcal{H}$, carrying a representation D of G. If $[H, D(g)] = 0, \forall g \in G$, the spectrum of H decomposes into *multiplets* of G (this just means irreps), $\mathcal{H} = \text{span}\{|a, \mu, \ell\rangle\}$. Here there are three kinds of indices: a labels an irrep of G, $\mu = 1..d_a$ labels states within that irrep, and finally $\ell = 1..m_a$ labels different invariant subspaces transforming under the same rep R_a. In this basis

$$\langle a, \mu, \ell| H |a', \mu', \ell'\rangle = \delta_{aa'}\delta_{\mu\mu'} H^a_{\ell\ell'}.$$

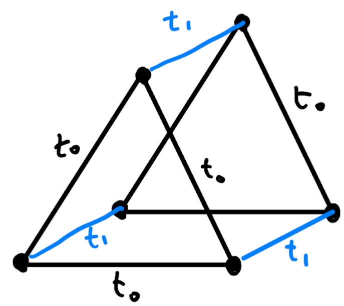
(This is a version of the Wigner-Eckhart theorem.) The matrices H^a are $m_a \times m_a$ and contain any information about H that goes beyond group theory of G (*e.g.* if G is trivial, it is all of H). Note that nothing about group theory guarantees that the basis $|\ell\rangle$ is orthonormal – that's up to you.

Now: when can there be degeneracies between multiplets? Collisions between levels of a given H^a – between copies of the same irrep – require tuning 3 parameters in $H^a_{\ell\ell'}$ and therefore basically doesn't happen. This is just ordinary level-repulsion. Here's why: focus on the two levels that might collide and forget about the rest. The general hermitian 2×2 matrix is then $h_2 = d_0 1\!\!1 + \vec{d} \cdot \vec{\sigma}$ with spectrum $d_0 \pm \sqrt{|\vec{d}|^2}$. The two levels only collide if $0 = d_x = d_y = d_z$ – three real conditions on the parameters.

However, if for some reason someone forbids the off-diagonal terms in h_2, then fewer conditions are required. How could this happen? Suppose there is a symmetry, say we know $[h_2, \sigma^z] = 0$. Then this requires $d_y = d_z = 0$, and we only need to tune one parameter d_x to get a collision.

Back to the general case, a collision of levels of H^a and $H^{b\neq a}$ can happen at codimension 1 in the space of hamiltonians. G-symmetry forbids the off-diagonal terms that would mix the levels.

For example, consider a tight-binding model on the vertices of a triangular prism, whose cross-section is an equilateral triangle. This has D_3 symmetry (and actually a bit more because there's a also a reflection exchanging the two triangles – this group is called $D_{3h} = D_3 \times \mathbb{Z}_2$ by the crystallographers). The Hilbert space transforms in the $\mathbf{3} \oplus \mathbf{3} = \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1}$ of D_3. The character for this 6d representation of D_{3h} is a 6 and five 0s – the first three entries are just the character of the $\mathbf{3}$ of D_3, and then the elements that permute the two triangles have no fixed points. You can see from (2.19) that this means that the decomposition into irreps of D_{3h} is $\mathbf{2}_+ \oplus \mathbf{1}_+ \oplus \mathbf{2}_- \oplus \mathbf{1}_-$ (using the notation of (2.19) for representations of $D_3 \times \mathbb{Z}_2$). Call t_1 the hopping along the triangles and t_0 the hopping between the triangles. I claim that by tuning one parameter t_1/t_0 to $\frac{3}{2}$ then a $\mathbf{1}_+$ collides with a $\mathbf{2}_-$ to form what is sometimes called a 'supermultiplet' (this name should be avoided – everything in physics is super-something). (For more detail of this example, see Dan Arovas' notes page 83.)



Algebra of classes. Here is another application of the group algebra, which I'll denote $\mathbb{C}[G]$ here. Consider the following object in the group algebra, associated with a conjugacy class C_α of size n_α:

$$\mathbf{C}_\alpha \equiv \frac{1}{n_\alpha} \sum_{g \in C_\alpha} \mathbf{g}.$$

This object commutes with all the elements of the group algebra:

$$\mathbf{g}^{-1} \mathbf{C}_\alpha \mathbf{g} = \mathbf{C}_\alpha$$

since conjugation by g takes each element of C_α to another one. The set of elements of the group algebra that commute with all the others is called the *center* of the group algebra, $Z(\mathbb{C}[G])$. It is a subalgebra and a subgroup. An arbitrary element in the center of the group algebra can be written[22] in terms of the $\mathbf{C}_\alpha$:

$$\mathbf{x} = \sum_\alpha x_\alpha C_\alpha.$$

But consider what happens if we multiply two of these objects

$$\mathbf{C}_\alpha \mathbf{C}_\beta = \mathbf{C}_\beta \mathbf{C}_\alpha \in Z(\mathbb{C}[G])$$

– we get another element of the center of the group algebra, which can itself be expanded as

$$\mathbf{C}_\alpha \mathbf{C}_\beta = \sum_\gamma c^\gamma_{\alpha\beta} \mathbf{C}_\gamma.$$

$c^\gamma_{\alpha\beta} = c^\gamma_{\beta\alpha}$. This product on the conjugacy classes is called the algebra of classes. It is a remarkable thing.

Notice that earlier we found a product on the irreps (by taking tensor products and decomposing) and now we've found a product on the conjugacy classes. There are some very strong relations between irreps and conjugacy classes. In particular, the character table is an invertible matrix taking one space to the other. Might these products be related?

Since the linear operators $\mathbf{C}_\alpha$ all commute, the matrices $(\mathbf{C}_\alpha)^\gamma{}_\beta = c^\gamma_{\alpha\beta}$ all commute with each other and can be simultaneously diagonalized. Can you guess who their eigenvalues and eigenvectors are?

In fact,

$$\mathbf{C}_\alpha \mathbf{P}^a = \lambda^a_\alpha \mathbf{P}^a \tag{2.26}$$

where $\mathbf{P}^a$ is the projector onto irrep a, $\mathbf{P}^a = \frac{d_a}{|G|} \sum_{g\in G} (\chi^a(g))^\star \, \mathbf{g}$. And the eigenvalues are proportional to the characters: $\lambda^a_\alpha = \frac{\chi^a_\alpha}{\chi^a_e}$ where $\chi^a_e = d_a$ is the character of the identity element. [End of Lecture 9]

If you know the algebra of classes $c^\gamma_{\alpha\beta}$, you can use this fact (2.26) to determine the character table, just by diagonalizing the matrices, and using $\sum_\alpha n_\alpha |\chi^a_\alpha|^2 = |G|$ and

[22] To see why, consider such an arbitrary element of the center of the group algebra. It must satisfy

$$\mathbf{x} = \sum_i x_i \mathbf{g}_i \overset{!}{=} \mathbf{h}^{-1}\mathbf{x}\mathbf{h} = \sum_i x_i \mathbf{h}^{-1}\mathbf{g}_i\mathbf{h}, \quad \forall h \in G. \tag{2.25}$$

The point is that in order for $\mathbf{x}$ to be in the center, the x_i must be constant on conjugacy classes. By definition: $h^{-1}g_i h$ is some other element g_j of the same conjugacy class with g_i and we must have $x_i = x_j$ in order to satisfy (2.25). And every element within each conjugacy class can be obtained from a given element by conjugating with some h. Thanks to Brian Tran and Ahmed Akhtar for providing the proof.

$\chi^a_e > 0$ to fix the normalization. (Starting from scratch, this may not be the best way to go.)

Characters are algebraic integers. The equation (2.26) implies that any character of a finite group element is an algebraic integer: the entries of the matrices $(\mathbf{C}_\alpha)^\gamma{}_\beta$ are rational numbers (actually the denominators only come from the factor of $1/n_\alpha$ in the definition of the $\mathbf{C}_\alpha$), and (after rationalizing denominators) the characteristic polynomial for such a matrix has integer coefficients. The eigenvalues $\frac{\chi^a_\alpha}{\chi^a_e}$ are the roots of this polynomial and hence are algebraic integers. Since $\chi^a_e = d_a$ is an integer, this means χ^a_α is an algebraic integer.

Fusion of classes and fusion of irreps. So, is the algebra of classes the same as the (semi-)ring[23] of irreps? The key to answering this question is called the *Verlinde formula* (in a slightly broader context). Here's the idea. In terms of the characters, the fusion rules for irreps are:

$$\chi_a \chi_b = m^c_{ab} \chi_c. \tag{2.27}$$

Since $\chi_a\chi_b = \chi_b\chi_a$ the matrices $(m_a)_b{}^c$ all commute $m_a m_b = m_b m_a$ and can be simultaneously diagonalized:

$$m_a = S\Lambda_a S^{-1}, \text{or with indices, } (m_a)_b{}^c = S^\alpha_b \left(\Lambda_a\right)^\beta_\alpha \left(S^{-1}\right)^c_\beta, \text{ with } \left(\Lambda_a\right)^\beta_\alpha = \lambda^\beta_a \delta^\beta_\alpha. \tag{2.28}$$

But now we use the fact that $m_0 = \mathbb{1}$, where 0 denotes the trivial representation. This says $S = m_0 S$, or

$$S^\alpha_a = m^c_{a0} S^\alpha_c \overset{(2.28)}{=} \sum_\beta S^\beta_0 \lambda^\beta_a \underbrace{(S^{-1})^c_\beta S^\alpha_c}_{=\delta^\alpha_\beta} = S^\alpha_0 \lambda^\alpha_a$$

from which we conclude a relation between the eigenvalues and eigenvectors of m_a:

$$\lambda^\alpha_a = \frac{S^\alpha_a}{S^\alpha_0}. \tag{2.29}$$

Plugging this back into (2.28) gives

$$m^c_{ab} = \sum_\alpha \frac{S^\alpha_a S^\alpha_b (S^{-1})^c_\alpha}{S^\alpha_0}.$$

Now who is the matrix that diagonalizes the fusion coefficients? Look at (2.27) again, and rewrite it as

$$(m_a)_b{}^c \chi_c(\alpha) = \chi_a(\alpha)\chi_b(\alpha).$$

[23] Some words about terminology that you should ignore: What's the difference between a ring and an algebra? (This sounds like the setup to a bad joke.) The latter allows us to multiply by elements of an arbitrary field (here $\mathbb{C}$). Why do I say 'semi-ring'? A ring is supposed to have inverses. But the additive inverse of χ_a is $-\chi_a$, which is not the character of any actual representation. It is sometimes called the character of a 'virtual representation'.

This says that the eigenvalues of the m_a are the characters χ_a^α – and so are the eigenvectors! Here the conjugacy class α is the label on the eigenvector and eigenvalue. (Personally, I find this equation so simple as to be confusing.) So we have proved that the fusion coefficients are:

$$m_{ab}^c = \sum_\alpha \frac{\chi_a^\alpha \chi_b^\alpha (\chi^{-1})_\alpha^c}{\chi_0^\alpha}. \tag{2.30}$$

Here $\chi_0^\alpha = \text{tr}_{\text{trivial rep}} D(\alpha) = 1$. This is a special case of the Verlinde formula, which is usually regarded as a statement about conformal field theory. These manipulations work for any compact group. For more on this point of view, see here.

By the same sequence of steps, a similar formula to (2.30) holds for $c_{\alpha\beta}^\gamma$. [24] Moreover, we've proved that the structure constants for the algebra of classes $c_{\alpha\beta}^\gamma$ and for the ring of irreps m_{ab}^c have the same eigenvalues (and hence, by (2.29), eigenvectors). So at least in this sense, they are the same.

---

[24]Here is a more direct proof of that formula. Consider the object $\Lambda_a^\alpha \equiv \frac{1}{n_\alpha}\sum_{g\in C_\alpha} D^a(g)$, where $D^a(g)$ is an irrep and C_α is a conjugacy class. Just like in our argument that the character table is square, this object commutes with all of the representation matrices $D^a(h)\Lambda_a^\alpha = \Lambda_a^\alpha D^a(h), \forall h \in G$. Here's why: Λ_a^α is just the object I called S in that argument made from the class function $f_\alpha(g)$ that is only nonzero on the character α: $f_\alpha(g) = \begin{cases} 1, & g \in C_\alpha, \\ 0, & \text{else}\end{cases}$. Or, explicitly:

$$D^a(h)\Lambda_a^\alpha = \frac{1}{n_\alpha}\sum_{g\in C_\alpha} D^a(hg) = \frac{1}{n_\alpha}\sum_{g'=h^{-1}gh} f(hg'h^{-1})D^a(g'h) = \frac{1}{n_\alpha}\sum_{g'} f(g)D^a(g')D^a(h) = \Lambda_a^\alpha D^a(h).$$

So by Schur's lemma, it is

$$\Lambda_a^\alpha = \lambda_a^\alpha \mathbb{1}_a$$

where $\mathbb{1}_a$ is the identity operator on the carrier space of irrep a. Taking the trace of the BHS, $\lambda_a^\alpha = \frac{\chi_a^\alpha}{d_a}$, so

$$\Lambda_a^\alpha = \frac{\chi_a^\alpha}{d_a}\mathbb{1}_a. \tag{2.31}$$

Now the class algebra $\mathbf{C}_\alpha\mathbf{C}_\beta = c_{\alpha\beta}^\gamma \mathbf{C}_\gamma$ implies that for each a, $\Lambda_a^\alpha\Lambda_a^\beta = c_{\alpha\beta}^\gamma\Lambda_a^\gamma$. But then (2.31) implies $\lambda_a^\alpha\lambda_a^\beta = c_{\alpha\beta}^\gamma\lambda_a^\gamma$. Therefore for all a, $\chi_a^\alpha\chi_a^\beta/d_a = c_{\alpha\beta}^\gamma\chi_a^\gamma$.

Now we multiply both sides by $\bar{\chi}_a^\delta$ and sum over a, using the fact (row orthogonality) that $\sum_a \chi_a^\alpha\bar{\chi}_a^\beta = \delta^{\alpha\beta}\frac{|G|}{n_\alpha}$ to get

$$c_{\alpha\beta}^\gamma = \sum_a \frac{n_\gamma}{d_a|G|}\chi_a^\alpha\chi_a^\beta\bar{\chi}_a^\gamma.$$

By the same logic, a more direct proof of the formula for m_{ab}^c follows by starting with $m_{ab}^c\chi_c^\alpha = \chi_a^\alpha\chi_b^\alpha$ multiplying both sides by $n_\alpha\bar{\chi}_d^\alpha$ (where $\bar{\chi}_a \equiv \chi_{\bar{a}} = \chi_a^\star$), and summing over α using column orthogonality $\sum_\alpha n_\alpha\bar{\chi}_d^\alpha\chi_c^\alpha = |G|\delta_{dc}$ to get

$$m_{ab}^c = \sum_\alpha \frac{n_\alpha}{|G|}\chi_a^\alpha\chi_b^\alpha\bar{\chi}_c^\alpha.$$

This also avoids the need to introduce χ^{-1}. But they are related by the fact that $\sqrt{\frac{n_\alpha}{|G|}}\chi_a^\alpha$ is unitary.

2.4 Real versus complex representations

[Zee §II.4, Stone-Goldbart problem 14.24] Given a representation R of G with linear operators $D(g)$, let $\bar{R}$ be the representation made from $D(g)^\star$. You can check that this is also a representation[25]. Its characters are $\chi_{\bar{R}}(g) = \chi_R(g)^\star$. In particle physics, if we have a particle in representation R, then $\bar{R}$ is the representation in which the antiparticle transforms.

A representation R is *not complex* if $D^\star \sim D$ (that is, if $\exists$ a similarity transformation S such that $D(g)^\star = SD(g)S^{-1}$ for all $g \in G$). If, in addition, there exists a basis where $D(g)_{ij} \in \mathbb{R}, \forall g \in G$, then the representation D is called *real*. A representation which is not complex but not real is called *pseudo-real*. For example (with apologies for the sudden appearance of a continuous group), consider the two dimensional (defining) representation of $\mathsf{SU}(2)$, in which an arbitrary element is represented by $U = \begin{pmatrix} a & -b^\star \\ b & a^\star \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$. Without the condition on the norms, $UU^\dagger = \mathbb{1}$ is unitary (in $\mathsf{U}(2)$), and $\det U = |a|^2 + |b|^2 = 1$ is the condition to be in $\mathsf{SU}(2)$ ("special unitary"). Notice that there are three real parameters specifying an element of $\mathsf{SU}(2)$. The conjugate representation of the $\mathbf{2}$ then has representation matrices

$$U^\star = \begin{pmatrix} a^\star & -b \\ b^\star & a \end{pmatrix} = \epsilon^{-1} \begin{pmatrix} a & -b^\star \\ b & a^\star \end{pmatrix} \epsilon, \quad \epsilon \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which is a similarity transformation showing that $\mathbf{2} \sim \overline{\mathbf{2}}$. But: there is no basis with $U_{ij} \in \mathbb{R}$, because a 2×2 real unitary matrix with determinant 1 has only one parameter (for real a, b, $a^2 + b^2 = 1$ describes a circle). (For an example of a finite group with a pseudoreal rep, consider the subgroup of π-rotations in $\mathsf{SU}(2)$. The 2d rep is still 2d.)

If $D^\star \sim D$ then $\chi_{\bar{R}}(g) = \chi_R(g)^\star = \chi_R(g)$, the characters are real. So if any character of R is a complex number, then R is a complex representation[26]. However, a representation with $\chi_R = \chi_{\bar{R}}$ could still be not real, *i.e.* it could be pseudoreal.

How to tell? For everything that follows, we assume R is a unitary rep. Suppose we have two objects x, y in the representation R (think of them as column vectors), $x \mapsto D(g)x, y \mapsto D(g)y$.

Claim: $\exists$ a G-invariant bilinear $y^T S x \Leftrightarrow R$ is not complex.

[25] Notice that $D(g)^\dagger$ would not make a representation, since $D(g)^\dagger D(h)^\dagger = (D(h)D(g))^\dagger = D(hg)^\dagger$ – they multiply in the wrong order. Thanks to Ahmed Akhtar for reminding me of this.

[26] The converse is also true: if $\chi_{\bar{R}}(g) = \chi_R(g) \forall g \in G$ then $R \sim \bar{R}$, *i.e.* R is pseudoreal. This is because the number of times the rep R appears in the decomposition of $\bar{R}$ is $\langle \chi_{\bar{R}}, \chi_R \rangle = \langle \chi_R, \chi_R \rangle$. So it is the same as R.

$\boxed{\Leftarrow}$: If $D^\star = SD(g)S^{-1}$ then $D(g)^T = SD(g)^\dagger S^{-1}$, so $y^T Sx \mapsto y^T D(g)^T SD(g)x = y^T SD(g)^\dagger S^{-1}SD(g)x = y^T Sx$.
$\boxed{\Rightarrow}$: If $\exists S$ such that $y^T Sx \mapsto y^T D(g)^T SD(g)x \stackrel{!}{=} y^T Sx,\ \forall x, y$ then $D(g)^T SD(g) = S \Leftrightarrow SD(g)^{-1}S^{-1} = (D(g)^T)^{-1} = D(g)^\star$, so the representation is not complex.

Another way to state the claim is: If the trivial rep appears in the decomposition of $R \otimes R$, then R is not complex.

This claim also explains one reason why we might care. Suppose we are in the situation described in the introduction, where we have a field theory of a certain collection of fields ϕ with a certain collection of symmetries G (assume discrete, for simplicity). A central question in this situation is: are the fields *massive* or *massless*? In the context of particle physics this means the corresponding particles are massive or massless and you can see why we care about that. In a condensed matter context, this determines whether the correlation length is finite. Such terms are constrained by the fact that the fields ϕ transform in some representation R of G.

Suppose we are in the situation where the fields describe particles which are their own antiparticles, where $R = \bar{R}$. This could be the case for neutrinos. A sufficient condition for such fields to be massive (in the absence of some tuning) is then if the symmetries allow us to add a term of the form $\phi^T\phi$ to the Lagrangian density (*i.e.* add $\int d^D x \phi^T(x)\phi(x)$ to the action) – the existence of an invariant bilinear S in $R \otimes R$.

To understand more about the nature of our friend S, suppose an irrep $D \sim D^\star$ is not complex. Then $SD(g)S^{-1} = D^\star(g)$. Taking transpose of the BHS gives $D(g^{-1}) \stackrel{\text{unitary}}{=} D(g)^\dagger = D(g)^{\star T} = (S^{-1})^T D(g)^T S^T$. Therefore $D(g^{-1}) = (S^{-1})^T D(g)^T S^T$ for all g, including g^{-1}, so

$$D(g) = (S^{-1})^T D(g^{-1})^T S^T = (S^{-1})^T SD(g)S^{-1}S^T = (S^{-1}S^T)^{-1}D(g)S^{-1}S^T.$$

The key point here is that taking inverse twice is doing nothing. Since this is true for all g, Schur's lemma implies $S^{-1}S^T = \eta\mathbb{1}$, *i.e.* $S^T = \eta S$. But $S = (S^T)^T = (\eta S)^T = \eta^2 S$. Therefore $\eta = \pm 1$, that is, S is either totally symmetric or totally antisymmetric. I claim that these two cases are real and pseudoreal, as defined above. That is, if $\eta = 1$, then there is a basis where $D(g)$ is real for all g. Note that an invertible AS matrix is even dimensional, so the latter case only occurs for even-dimensional reps.

First $S \propto$ a unitary matrix, *i.e.* $S^\dagger S = \mathbb{1}$. Here's why: $\forall g, S = D(g)^T SD(g)$ and $S^\dagger = D(g)^\dagger S^\dagger D(g)^\star$. Therefore

$$S^\dagger S = D(g)^\dagger S^\dagger \underbrace{D(g)^\star D(g)^T}_{=\mathbb{1}} SD(g),\ \forall g,$$

so Schur's lemma implies $S^\dagger S \propto \mathbb{1}$. So rescale S so that it's unitary from now on.

Now, If $\eta = +1$ I claim that $W \equiv \sqrt{S}$ is also unitary and symmetric[27]. From this it follows that $W^{-1} = W^\dagger = W^\star$ which implies

$$W^2 D(g) W^{-2} = D(g)^\star \implies W D(g) W^{-1} = W^{-1} D(g)^\star W = W^\star D(g)^\star (W^{-1})^\star = (W D(g) W^{-1})^\star$$

is real!

$D \sim D^\star$ says

$$S D(g) S^{-1} = D^\star(g) = (D^T(g))^{-1} \quad \Leftrightarrow \quad D^T(g) S D(g) = S, \ \forall g \in G$$

So, if we were feeling fancy, we could regard the similarity transformation S as a matrix representation of a G-invariant (invertible) quadratic form. Note that for an irrep, Schur's lemma says there can only be one such form, up to scaling: If there were two, $D^T(g) S_{1,2} D(g) = S_{1,2}$, then for any $x \in \mathbb{C}$, $(S_1 - xS_2) D(g) = D^\star(g)(S_1 - xS_2)$, so $S_1 - xS_2$ is an intertwiner between R and $R^\star$. But $\det(S_1 - xS_2)$ has a root, λ, so can't be an isomorphism and we conclude $S_1 = \lambda S_2$.

Frobenius-Schur indicator. For any matrix X of the correct shape and G compact, let $S_X \equiv \frac{1}{|G|} \sum_{g\in G} D^T(g) X D(g)$. This is G-invariant in the sense that

$$S_X \mapsto D^T(h) S_X D(h) = \frac{1}{|G|} \sum_{g \in G} D(h)^T D(g)^T X D(g) D(h) = S_X, \quad \forall h \in G.$$

So $y^T S_X x$ is a G-invariant bilinear! According to the claim above, this means $S_X = 0$ if R is complex.

Now choose $X = e_{il}$ where $(e_{il})_{jk} = \delta_{ij}\delta_{kl}$, the matrix with a 1 in the il entry and 0 everywhere else. Then

$$(S_{e_{il}})_{jk} = \frac{1}{|G|} \sum_g (D^T e_{il} D(g))_{jk} = \sum_g (D(g)^T)_{ji} (D(g))_{lk} = \sum_g D(g)_{ij} D(g)_{lk}$$

This equation is true for all $ijkl$ (the indices il are implicit on the LHS). Now contract $j = l$:

$$\sum_{g \in G} D(g)_{ij} D(g)_{jk} = \sum_{g\in G} D(g^2)_{ik}.$$

[27] Here's why: Take the log: $S = e^{\mathbf{i}H}$. S is unitary means $H = H^\dagger$ is hermitian. S is symmetric means $S = S^T = e^{\mathbf{i}H^T}$ means that $H^T = H + 2\pi n \mathbb{1}, n \in \mathbb{Z}$. But taking the transpose doesn't change the diagonal elements, so n must be zero. (Alternatively, take transpose of both sides, to get $H = H^T + 2\pi n \mathbb{1}$; combining this with the original equation, we conclude that n must be zero [Thanks to Hongrui Li for this argument].) Then $W = e^{\mathbf{i}H/2}$ is unitary since H is hermitian, and $W^T = e^{\mathbf{i}H^T/2} = W$.

In the case with $\eta = -1$, this doesn't work: we require $S = e^{\mathbf{i}G}$ with $G^T \stackrel{?}{=} G + \mathbf{i}\pi\mathbb{1}$, which would give $\sqrt{S}^T = e^{\mathbf{i}\pi/2}\sqrt{S}$ (if it were even possible to find such a G, which it is not, by the argument above).

This is 0 if R is complex. We can make a simpler indicator by taking the trace, which is $\sum_{g\in G}\chi(g^2)$. [End of Lecture 10]

If instead, R is not complex, we must have $S_X^T = \eta S_X$:

$$S^T = \frac{1}{|G|}\sum_{g\in G} D(g)^T X^T D(g) = \eta \sum_{g\in G} D(g)^T X D(g).$$

For our friend $X = e_{il}$ above, this gives

$$(S_{e_{il}})_{jk} = \sum_{g\in G} \underbrace{(D(g)^T)_{jl}}_{=D(g)_{lj}} D(g)_{ik} = \eta \sum_{g} \underbrace{(D(g)^T)_{ji}}_{=D(g)_{ij}} D(g)_{lk}.$$

This equation is a huge set of constraints. Contract $i = j$ to get

$$\sum_{g\in G} D(g^2)_{lk} = \eta \sum_{g\in G} \chi(g) D(g)_{lk}$$

and now when we contract $l = k$ we get

$$\sum_{g\in G} \chi(g^2) = \eta \sum_{g\in G} \chi(g)^2.$$

But using the fact that R is not complex, $\chi_R = \chi_R^\star$, and character orthogonality says the RHS is $|G|$ since R is an irrep.

Therefore (drumroll), the Frobenius-Schur indicator for an irrep R is

$$\eta_R \equiv \frac{1}{|G|}\sum_{g\in G} \chi_R(g^2) = \begin{cases} 0, & R \text{ complex} \\ 1, & R \text{ real} \\ -1, & R \text{ pseudoreal} \end{cases}.$$

Two checks that this makes sense: If $[g_1] = [g_2]$ (g_1 and g_2 are in the same conjugacy class) then $[g_1^2] = [g_2^2]$. For the trivial rep, $\eta_R = 1$.

The same techniques allow us to answer questions like: How many square roots does a given group element have? In how many ways can a group element be written as a product of two squares? How many homomorphisms are there from the trefoil knot group (the fundamental group of the complement of the trefoil knot, $\langle a, b | a^2 = b^3\rangle$) to G?

To see the idea of how to do these things, rewrite the FS indicator as

$$\eta_a = \frac{1}{G}\sum_{g\in G} \chi_a(g^2) = \frac{1}{|G|}\sum_{h\in G} \sigma(h)\chi_a(h)$$

where $\sigma(h) \equiv$ the number of elements $g \in G$ satisfying $g^2 = h$. Now multiply the BHS by $\chi_a^\star(h')$, sum over a and use character orthogonality to isolate the desired $\sigma(h)$.

2.5 Induced representations

Given a rep $D^W(h) : W \to W$ of $H \subset G$, we can make a (reducible, in general) rep of G. The carrier space is $W \times V_{G/H}$, where $V_{G/H} \equiv \text{span}\{|x\rangle, x \in G/H\}$, so the dimension of this *induced* representation is $\dim W \cdot |G/H|$. Here's the idea: recall (from the homework) that G acts on G/H by $x = \{g_1, g_2, \cdots\} \mapsto \{gg_1, gg_2, \cdots\}$. Select a representative $a_x \in G$ of each $x \in G/H$. Then the action of G on G/H can be written as $ga_x = a_{gx}h$, with $h \in H$, where a_{gx} is the representative of gx. Here's the definition of the rep of G: For $h \in H$, let

$$D(h)|n, 0\rangle \equiv |m, 0\rangle D^W_{mn}(h). \tag{2.32}$$

Here 0 means the coset with the identity. This is just the ordinary rep of H, acting trivially on the other index. For the particular representatives we've chosen, let

$$D(a_x)|n, 0\rangle \equiv |n, x\rangle. \tag{2.33}$$

Now determine the rest by requiring it to be a rep of G. In practice you can just impose this demand as needed, but it's actually possible to find a formula for the action on a general element:

$$D(g)|n, x\rangle = D(g)D(a_x)|n, 0\rangle = D(ga_x)|n, 0\rangle = D(a_{gx}h)|n, 0\rangle \tag{2.34}$$

$$= D(a_{gx})D(h)|n, 0\rangle = D(a_{gx})|m, 0\rangle D^W(h)_{mn} = |m, gx\rangle D^W(h)_{mn}. \tag{2.35}$$

You can check that it's a representation $D(g_1g_2)|n, x\rangle \stackrel{!}{=} D(g_1)D(g_2)|n, x\rangle$.

A good way to write the action is to describe its action on a general state $|f\rangle \equiv \sum_{nx} f_n(x)|n, x\rangle$ as $f_n(x) \mapsto D^W_{nm}(h) f_m(g^{-1}x)$, where h is defined by $ga_x = a_{gx}h$.

Here's the simplest possible example: take $H = \mathbb{Z}_2 = \langle\sigma|\sigma^2 = e\rangle \subset G = \mathbb{Z}_4 = \langle\tau|\tau^4 = e\rangle$ with the inclusion $\sigma = \tau^2$, and consider the nontrivial rep of $\mathbb{Z}_2$, with $D(\sigma) = -1$. The set $\mathbb{Z}_4/\mathbb{Z}_2$ has two elements, which we can label by e, τ, so the induced rep is two-dimensional, with carrier space $\text{span}\{|e\rangle, |\tau\rangle\}$ (we don't need a label for the 1d rep of H). Then $D(\tau)|e\rangle = |\tau\rangle$ from (2.33). And then $D(\tau)|\tau\rangle = D(\tau)D(\tau)|e\rangle = D(\tau^2)|e\rangle = D(\sigma)|e\rangle = -|e\rangle$. So in this basis

$$D(e) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, D(\tau) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, D(\tau^3) = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, D(\tau) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The character is $\begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} = \chi \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ where χ is the character table for $\mathbb{Z}_4$. So this is $\mathbf{1}_1 \oplus \mathbf{1}_3$, where I've labelled the irreps of $\mathbb{Z}_4$ as $D_k(g^\ell) = \omega^{kl}$.

A less trivial example: the (reducible) **8** of S_4 acting on the vertices of the cube is $\text{Ind}_{\mathbb{Z}_3}^{S_4}$(trivial rep of $\mathbb{Z}_3$), induced from the trivial rep of a $\mathbb{Z}_3$ subgroup of $2\pi/3$ rotations. Notice that there are eight cosets, $|S_4/\mathbb{Z}_3| = 8$, so this gives a rep of the right dimension. Another hint is that the $\mathbb{Z}_3$ subgroup fixes a pair of vertices. Similarly, the $\mathbb{Z}_2$ subgroup that fixes a pair of edges induces the 12-dimensional rep on the edges, and the $\mathbb{Z}_4$ subgroup that fixes a pair of faces induces the 6-dimensional rep on the faces. The general idea, for example in the case of vertices, is to think about the world from the point of view of a single vertex: the subgroup of S_4 that fixes the vertex is $\mathbb{Z}_3$ (which acts trivially on it), so this is the subgroup to consider. In the language of Wigner, $\mathbb{Z}_3$ is the 'little group' of the vertex.

Frobenius reciprocity. So this gives a way to make a reducible rep of G from irreps of H. It is easy to make a reducible rep of H from irreps of G: an irrep of G is also a rep of $H \subset G$, which is in general reducible. Interestingly, there is a close relation between these two operations – they are adjoints of each other in a certain sense, which I want to explain next.

If $f(h)$ is a function on H, we can extend it to a function on G just by setting $f(g) = 0$ if $g \notin H$. To make a class function on G, just average over orbits by conjugation: $\text{Ind}_H^G[f](s) \equiv \frac{1}{|H|}\sum_{g\in G} f(g^{-1}sg)$. Now apply this to $\chi_W(h) \equiv \text{tr}_W D^W(h)$ the character of the rep of H. I claim that the result is just the character of the induced representation $\text{Ind}_H^G[W]$:

$$\chi_{\text{Ind}_H^G[W]}(g) = \text{Ind}_H^G[\chi_W](g).$$

Finally, consider a rep $D^V(g): V \to V$ of G. Let's call $\text{Res}_H^G(V)$ the rep of $H \subset G$ that this produces, just by restricting to elements of H, $D^V(h): V \to V$. Given two functions on (any compact) G, there is an inner product

$$\langle \phi_1, \phi_2 \rangle_G \equiv \frac{1}{|G|}\sum_{g\in G} \phi_1(g^{-1})\phi_2(g).$$

(This is the same as the inner product for characters.)

Now, given ψ, ϕ class functions on H and G respectively, I claim that

$$\left\langle \psi, \text{Res}_H^G[\phi] \right\rangle_H = \left\langle \text{Ind}_H^G[\psi], \phi \right\rangle_G.$$

That is, the two maps Res and Ind are adjoints with respect to this inner product. (This fact is called *Frobenius reciprocity.*) This is a cool thing: taking ϕ and ψ to be the characters of irreps V_b of G and W_a of H, it says that the number of times irrep W_a of H appears in the decomposition of irrep V_b of G (when regarded as a rep of H) is the same as the number of times irrep V_b of G appears in the decomposition of the rep of G induced from the irrep W_a of H.

Promise of physics applications. The operation of making induced representations (the fancy word for it is 'functor') is especially useful for constructing unitary representations of non-compact groups from reps of their compact subgroups. Examples relevant to physics are the symmetry group of euclidean space $E(d)$ (translations and rotations) and the Poincaré group (translations and rotations and boosts). Perhaps more later on this.

Here is a brief preview: The symmetry group of $\mathbb{R}^2$ is sometimes called $E(2)$ (for Euclidean) and includes translations and rotations. It has a subgroup $E(2) \supset \mathsf{SO}(2)$. The irreps of $\mathsf{SO}(2)$ are labelled by the (integer) angular momentum m. The rep of $E(2)$ induced by the inclusion $\mathsf{SO}(2) \subset E(2)$ is labelled by $|p, m\rangle$, where $p \in E(2)/\mathsf{SO}(2) = \mathbb{R}_+$. This is the magnitude of the momentum.

Similarly, the Poincaré group (the Lorentz group and translations in $\mathbb{R}^{3,1}$) contains an $\mathsf{SO}(3)$ subgroup. The states of a massive spin-J particle arise as the induced representation from the spin-J representation of $\mathsf{SO}(3) = \text{span}\{|\sigma\rangle, \sigma \in \{-J, -J + 1, \cdots J-1, J\}\}$. The resulting states are labelled by an element $\vec{\beta} \in \mathsf{SO}(1,3)/\mathsf{SO}(3)$ (an element of $\mathsf{SO}(1,3)$ can be labelled by a rotation vector $\vec{\theta}$ and a boost vector $\vec{\beta}$; the quotient by $\mathsf{SO}(3)$ precisely gets rid of the rotation part). One way to think about the role of the $\mathsf{SO}(3)$ subgroup that we started with is that it is the subgroup of Poincaré that preserves the 4-momentum of the massive particle (in the rest frame it is $k_0^\mu = (m, 0, 0, 0)^\mu$, but by conjugating by a boost the subgroup preserving any timelike 4-vector is the same). The $\vec{\beta}$ labels the boost that takes k_0^μ to an arbitrary timelike 4-vector: $k^\mu(\beta) = \Lambda(\beta)^\mu_\nu k_0^\nu$.

For a massless particle, instead the momentum has a form conjugate to $p^\mu = (E, 0, 0, E)^\mu$, a timelike vector. The subgroup of the Lorentz group preserving this is only the $\mathsf{SO}(2)$ of rotations about $\hat{z}$. Each (unitary) rep of $\mathsf{SO}(2)$ (labelled by the integer angular momentum) induces a corresponding (unitary) rep of Lorentz and Poincaré.

2.6 Representations of S_n and Young tableaux

[Georgi, 2d edition §1.24] Recall that for any group, the number of irreps is the number of conjugacy classes. For S_n we saw that conjugacy classes are in 1-1 correspondence with Young diagrams with n boxes. This suggests, and it's true, that the irreps can also be labelled by Young diagrams. Let's construct an irrep of S_n for each Young diagram with n boxes.

There are $n!$ ways to put the numbers $1, 2, \cdots n$ into the boxes of a Young diagram, like $\begin{array}{|c|c|c|c|}\hline 6&5&3&2\\\hline 1&7\\\cline{1-2} 4\\\cline{1-1}\end{array}$. We can associate (in an arbitrary way) each such arrangement with a permutation (so if I arbitrarily number the boxes from left to right and top to bottom, then $\begin{pmatrix}1&2&3&4&5&6&7\\6&5&3&2&1&7&4\end{pmatrix}$ is the permutation in the example above), which in turn specifies a basis state of the regular representation, $\left|\begin{pmatrix}1&2&3&4&5&6&7\\6&5&3&2&1&7&4\end{pmatrix}\right\rangle$. Recall that all the irreps fit into the regular representation, with each rep R_a appearing d_a times.

Important notation warning: For legibility below, I will denote *e.g.* $\left|\begin{pmatrix}1&2&3&4&5&6&7\\6&5&3&2&1&7&4\end{pmatrix}\right\rangle \equiv |6532174\rangle$. This is not cycle notation.

Now to find the invariant subspaces of the regular rep $D(\pi)\,|i_1 i_2 \cdots\rangle = |\pi_{i_1}\pi_{i_2}\cdots\rangle$, just make states where we

- *symmetrize* in the rows, so for example $\left|\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right\rangle \equiv (|12\rangle + |21\rangle)/\sqrt{2}$, (where here $|12\rangle$ denotes the basis vector of the regular rep associated with the identity element of S_2) and
- *antisymmetrize* in the columns, so for example $\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right\rangle \equiv (|12\rangle - |21\rangle)/\sqrt{2}$.

You can see that these are invariant subspaces. A more complicated example is

$$\left|\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right\rangle \equiv (|123\rangle + |213\rangle - |321\rangle - |231\rangle)/\sqrt{4}.$$

Claim: states made from the same Young diagram span an invariant subspace under the action of S_n in the regular representation. This is because

$$D_{R_{\text{reg}}}(\pi)\left|\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right\rangle = \left|\begin{array}{|c|c|}\hline \pi_1&\pi_2\\\hline \pi_3\\\cline{1-1}\end{array}\right\rangle,$$

so the action of S_n preserves the symmetrization structure.

For example, the states $|\ydiagram{2}\rangle$ and $|\ydiagram{1,1}\rangle$ specify 1d representations of S_2, the trivial and sign representation, respectively.

For S_3, $|\ydiagram{3}\rangle \propto |123\rangle + 5$ terms is the 1d trivial representation, while $\left|\ydiagram{1,1,1}\right\rangle \propto |123\rangle - |231\rangle + \cdots = \sum_{\pi \in S_n} (-1)^\pi |\pi\rangle$ is the sign representation. The interesting case is the Young diagram $\ydiagram{2,1}$. Writing out the definitions of each of the 6 states gives:

$$\begin{pmatrix} \left|\ytableaushort{12,3}\right\rangle \\ \left|\ytableaushort{32,1}\right\rangle \\ \left|\ytableaushort{23,1}\right\rangle \\ \left|\ytableaushort{13,2}\right\rangle \\ \left|\ytableaushort{31,2}\right\rangle \\ \left|\ytableaushort{21,3}\right\rangle \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} |123\rangle \\ |213\rangle \\ |321\rangle \\ |231\rangle \\ |132\rangle \\ |312\rangle \end{pmatrix} \tag{2.36}$$

The matrix here has rank two: rows 12, 34 and 56 are just minus each other (related by interchanging labels in the same column of the tableaux), and there is one more relation which says the sum over the orbits of the cyclic permutation (rows 135 and 246) gives zero. This leaves a two dimensional representation of S_3.

Why did this happen? The independent vectors in the list (2.36) correspond to ways of placing the numbers $1, 2, \ldots, n$ in the boxes of the Young diagram such that in each row and column the numbers are ordered – the point is that labellings where they are out of order are obtained by the action of permutations on this set. Such a thing is called a *Young tableau*. So we have (ignore the $\oplus$ signs if you wish, really I am just listing the basis elements of the irreps that we get from this construction)

$$S_2 : \ytableaushort{12} \oplus \ytableaushort{1,2} = \mathbf{1} \oplus \mathbf{1}'$$

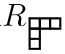
$$S_3 : \ytableaushort{123} \oplus \left(\ytableaushort{12,3}, \ytableaushort{13,2} \right) \oplus \ytableaushort{1,2,3} = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1}'$$

$$S_4 : \ytableaushort{1234} \oplus \left(\ytableaushort{123,4}, \ytableaushort{134,2}, \ytableaushort{124,3} \right) \oplus \left(\ytableaushort{12,34}, \ytableaushort{13,24} \right) \oplus \left(\ytableaushort{12,3,4}, \ytableaushort{13,2,4}, \ytableaushort{14,2,3} \right) \oplus \ytableaushort{1,2,3,4}$$
$$= \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{2} \oplus \mathbf{3}' \oplus \mathbf{1}'. \tag{2.37}$$

Notice that $\sum_a d_a^2 = |\mathsf{G}|$. It's not obvious from what I said (but it's true) that each irrep appears exactly *once* from this construction, even though it appears d_a times in the regular rep.

So what is the dimension of the irrep of S_n associated with a given Young diagram, λ? $\dim R_\lambda = \#$ of Young tableaux for the given diagram. To count the tableaux, we use the following *hook rule*: In each box of the Young diagram put its *hook length*, the number of boxes below and to the right, including the box itself. (The name comes from the fact that the hook length is the number of boxes through which a *hook* passes, where a hook starts at the bottom of the column of the box, makes a right turn at the box, and passes out of the diagram to the right.) For example, for , we have

5	2	1
2		
1		

[WARNING: this is not a Young tableau]. The dimension of the associated representation is $\dim R_\lambda = \frac{n!}{\prod_{\text{boxes}} \text{hook lengths}}$. In this example, $\dim R$ $= \frac{5!}{5 \cdot 2^2} = 6$.

Warning: as we will see, there is also a relationship between Young diagrams and irreps of various Lie groups. The formula for the dimension is not the same!

[End of Lecture 11]

2.7 Projective representations

[Dan Arovas's notes §2.1.5] In quantum mechanics a state is often only defined up to a multiple of a phase $e^{\mathbf{i}\phi}$. So you might think that we can relax a bit the definition of a representation to allow

$$D(g)D(h) = \omega(g,h)D(gh) \tag{2.38}$$

where $\omega(g,h) \in \mathsf{U}(1)$ is a phase $|\omega(g,h)| = 1$. We must still demand associativity:

$$\begin{aligned} D(g)D(h)D(k) &= (D(g)D(h))D(k) = \omega(g,h)D(gh)D(k) = \omega(g,h)\omega(gh,k)D(ghk) \\ &= D(g)(D(h)D(k)) = D(g)\omega(h,k)D(hk) = \omega(g,hk)\omega(h,k)D(ghk) \end{aligned}$$

from which we conclude

$$1 = \frac{\omega(g,h)\omega(gh,k)}{\omega(h,k)\omega(g,hk)}. \tag{2.39}$$

A function $\omega : G \times G \to \mathsf{U}(1)$ satisfying this associativity condition is called a *cocycle*. A set of $D(g)$s with nontrivial cocycle is called a *projective representation*.

When are two such things equivalent and what does trivial mean? By rephasing the generators $D(g) \mapsto \gamma(g)D(g)$, with $\gamma : G \to \mathsf{U}(1)$, (2.38) becomes $\gamma(g)D(g)\gamma(h)D(h) = \omega(g,h)\gamma(gh)D(gh)$, so

$$\omega(g,h) \mapsto \omega(g,h)\frac{\gamma(gh)}{\gamma(g)\gamma(h)}.$$

So if $\omega(g,h) = \frac{\gamma(g)\gamma(h)}{\gamma(gh)}$ for some function $\gamma : G \to \mathsf{U}(1)$, then this is actually equivalent to an ordinary (linear) representation of G.

To crystallize what we've just learned, define $\Omega^p \equiv \Omega^p(G,\mathsf{U}(1)) \equiv$ maps from $\underbrace{G \times G \times \cdots G}_{p \text{ times}} \to \mathsf{U}(1)$. An element of Ω^p is called a p-cochain. We can construct a (co)chain *complex*, which is a sequence of maps with the property that the image of one is contained in the kernel of the next:

$$\Omega^1 \xrightarrow{\delta_1} \Omega^2 \xrightarrow{\delta_2} \Omega^3 \tag{2.40}$$

$$\gamma \mapsto \delta_1\gamma(g,h) = \frac{\gamma(h)\gamma(g)}{\gamma(gh)} \tag{2.41}$$

$$\omega \mapsto \delta_2\omega(g,h,k) = \frac{\omega(g,h)\omega(gh,k)}{\omega(h,k)\omega(g,hk)}.$$

You can check that $\operatorname{Im}\delta_1 \subset \ker\,\delta_2$. So $\omega \in \ker\,\delta_2$ is a cocycle, defining a projective representation. The equivalence relation is the map δ_1. Therefore, inequivalent projective reps correspond to elements of the quotient

$$\frac{\ker\,\delta_2 \subset \Omega^2}{\operatorname{Im}\delta_1 \subset \Omega^2} \equiv H^2(G,\mathsf{U}(1)).$$

This object is called the 2nd group cohomology of G. It is a group under multiplication.

Time for some examples.

Example 1: $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \sigma, \tau | \sigma^2 = e, \tau^2 = e, \sigma\tau = \tau\sigma \rangle$, and $D(\sigma) = \mathbf{i}X, D(\tau) = \mathbf{i}Y, D(\sigma\tau) = \mathbf{i}Z$. Notice that, even though this is an abelian group, we don't need to have $\omega(g, h) = \omega(h, g)$. In fact $D(\sigma)D(\tau) = -D(\tau)D(\sigma)$. This is a projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by π rotations in the **2**-dimensional rep (a spinor) of $\mathsf{SU}(2)$. In fact it is an ordinary representation of a much smaller group with only eight elements, namely Q_8.

Something I should mention for cultural background. In general, a projective representation of G is an ordinary rep of a larger 'covering' group $\tilde{G}$. Such a situation arises when we have an *extension* $\tilde{G}$ of G by an abelian group A, which is defined by an exact sequence (meaning a complex with trivial cohomology)

$$1 \longrightarrow A \xrightarrow{\psi} \tilde{G} \xrightarrow{\pi} G \longrightarrow 1.$$

The complex being exact means that there is trivial cohomology: $\text{Im}\,\psi = \ker\,\pi$, and similarly for the other steps. This means that $\text{Im}\,\psi \subset \tilde{G}$ is a normal subgroup, and $\tilde{G}/A = G$ is a group (namely G). (So for example S_n is an extension of A_n by $\mathbb{Z}_2$.) An extension is *central* if $\text{Im}\,\psi \subset Z(\tilde{G})$, the center of $\tilde{G}$, when this is the case, representations of $\tilde{G}$ give projective reps of G. Each element of G is associated with $|A|$ elements of $\tilde{G}$; this describes the possible ambiguities in the phase of the product of two elements of G.

Here it is in detail for the example above (on its side):

$$\begin{array}{cccccccccc}
1 & 1 & & & & & & & & \\
\downarrow & \downarrow & & & & & & & & \\
\mathbb{Z}_2 & 1 & -1 & & & & & & & \\
\downarrow & \downarrow & \downarrow & & & & & & & \\
Q_8 & 1 & -1 & i & j & k & -i & -j & -k \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbb{Z}_2 \times \mathbb{Z}_2 & 1 & 1 & \tau & \sigma & \tau\sigma & \tau & \sigma & \tau\sigma \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Even, fixing G and A, $\tilde{G}$ is not unique. An equivalent (up to rephasing) projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$, with $D(\sigma) = X, D(\tau) = Z$ can be lifted to a linear representation of D_4.

Example 2: $G = \mathsf{U}(1) = \{e^{\mathbf{i}\theta}, \theta \in [0, 2\pi)\}$. Representations of $\mathsf{U}(1)$ are (all one dimensional) of the form $U_q(\theta) = e^{\mathbf{i}q\theta}$ with $q \in \mathbb{Z}$. Why do we need $q \in \mathbb{Z}$? Consider what happens if $q \in \mathbb{Z} + \frac{1}{2}$. Then it's still true that $U(\theta)U(\theta') = U(\theta + \theta')$ as phases, but $\theta + \theta'$ need not remain in the fundamental domain $[0, 2\pi)$. Putting it back in that range gives

$$U(\theta)U(\theta') = U((\theta + \theta')_{2\pi})\omega(\theta, \theta'), \quad \omega(\theta, \theta') = \begin{cases} 1, & \theta + \theta' \in [0, 2\pi) \\ -1, & \theta + \theta' \in [2\pi, 4\pi) \end{cases}.$$

Example 3: $G = \mathsf{SO}(3)$. A spinor representation, that is, one with half-integral angular momentum, is a projective representation of $\mathsf{SO}(3)$. For example for the **2**, the representation matrices are

$$U(\theta, \hat{n}) = e^{-\mathbf{i}\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}} \overset{\text{Taylor}}{=} \cos\theta/2 - \mathbf{i}\hat{n}\cdot\vec{\sigma}\sin\theta/2$$

where $\vec{\sigma}$ are the Paulis and we used $(\hat{n}\cdot\vec{\sigma})^2 = \mathbb{1}$. To see that this is a projective rep of $\mathsf{SO}(3)$, notice that $U(2\pi, \hat{n}) = -\mathbb{1}$. Moreover, $U\sigma^a U^\dagger = R_{ab}\sigma^b$, where R_{ab} is the representation matrix for the same rotation in the **3** (vector rep). Since U and $-U$ map to the same element, you can see that it is double-valued. So these groups fit in to the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathsf{SU}(2) \longrightarrow \mathsf{SO}(3) \longrightarrow 1$$

where the $\mathbb{Z}_2$ is generated by the matrix $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$. So $\mathsf{SO}(3) = \mathsf{SU}(2)/\mathbb{Z}_2$.

Example 4: Magnetic translations. Ordinary translations are realized on say the Hilbert space of a free particle in d dimensions by the linear operators $\hat{T}(\vec{x}) = e^{\mathbf{i}\vec{p}\cdot x/\hbar} = e^{\vec{x}\cdot\vec{\nabla}}$. So for example $\hat{T}(\vec{x})\psi(\vec{r}) = \psi(\vec{r} + \vec{x})$ (this is Taylor's theorem) and $\hat{T}(\vec{x})\hat{T}(\vec{x}') = \hat{T}(\vec{x} + \vec{x}')$. The group is $G = \mathbb{R}^d$ under ordinary vector addition. It was important here that $[p_i, p_j] = 0$.

Take $d = 3$ and turn on a uniform magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. The canonical momentum is replaced by $\vec{p} \to \vec{\pi} = \vec{p} + \frac{e}{c}\vec{A}$. These satisfy $[\pi^i, \pi^j] = -\mathbf{i}\frac{\hbar e}{c}\epsilon_{ijk}B^k$. Operators that commute with the free hamiltonian $H = \frac{\pi^2}{2m}$ are the 'guiding center momenta' $\vec{\kappa} \equiv \vec{\pi} - \frac{e}{c}\vec{B} \times \vec{r}$ which have commutation relations $[\kappa^i, \kappa^j] = +\mathbf{i}\frac{\hbar e}{c}\epsilon_{ijk}B^k$. The *magnetic translation operators*, which represent the symmetry transformation by a finite amount, are $\hat{T}_B(\vec{x}) \equiv e^{\mathbf{i}\vec{\kappa}\cdot\vec{x}/\hbar}$ and satisfy

$$\hat{T}_B(\vec{x})\hat{T}_B(\vec{x}') = e^{-\mathbf{i}\pi\vec{B}\cdot\vec{x}\times\vec{x}'/\phi_0}\hat{T}_B(\vec{x} + \vec{x}'),$$

where $\phi_0 = \frac{\hbar c}{e}$ is the Dirac quantum of magnetic flux. So this is a projective representation of translations.

Example 5: 1d SPT states. This is the most ambitious example. A 1d SPT state (SPT stands for 'symmetry-protected topological') is a symmetric gapped groundstate of a local many-body hamiltonian H which has a symmetry G[28]. 'Gapped' means that the first excited state has an energy larger than the groundstate by an amount (the gap) that stays finite in the thermodynamic limit. We assume that the Hilbert space is $\mathcal{H} = \otimes_x \mathcal{H}_x$, a product of local Hilbert spaces of dimension $d < \infty$ (say $d = 2$), and we assume that the symmetry is represented on $\mathcal{H}$ as $U = \prod_x u_x$ (this is called an 'on-site' representation, as opposed to something that acts on many sites at once).

A *trivial* SPT state is one where H can be deformed – through G-symmetric hamiltonians – to one like $H_0 = -\sum_x X_x$ (X_x is the pauli matrix acting on the qubit at site x) without closing the gap. (Two states that are related in this way are said to be in the same *phase* (with symmetry G).) The crucial property of H_0 is that its groundstate is a product of symmetric states, $|\text{trivial}\rangle = \otimes_x |+\rangle_x$ ($X_x |+\rangle_x = |+\rangle_x$).

SPT states are featureless in the bulk – they are *paramagnets*, *i.e.* don't break any of the G symmetry (and they don't have topological order, *i.e.* anyonic excitations); the signature of a nontrivial SPT is something happening at the boundary of the system, specifically a projective representation of G there. An example: take a chain of qubits, with $H = -\sum_i Z_{i-1}X_iZ_{i+1}$. This has $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with generators $U_e = \prod_{i \text{ even}} X_i$, $U_o = \prod_{i \text{ odd}} X_i$. The groundstate is called a *cluster state*. We can understand this model completely from the following fact:

$$H = SH_0S^\dagger, \quad S = \prod_i \mathsf{CZ}_{i,i+1} \tag{2.42}$$

where $\mathsf{CZ}_{ij} \equiv e^{\mathbf{i}\frac{\pi}{4}(1-Z_i)(1-Z_j)} = \mathsf{CZ}_{ji}$ is the control-Z operation: in the Z basis ($Z|s\rangle = (-1)^s |s\rangle, s = 0,1$), it acts as Z on the second bit only if the first bit is 1. Or, most simply,

$$\mathsf{CZ}\,|s_1s_2\rangle = (-1)^{s_1s_2}|s_1s_2\rangle\,.$$

Notice that $\mathsf{CZ}_{ij} = \mathsf{CZ}_{ji}$. The key algebraic fact for purposes of (2.42) is

$$X_i\mathsf{CZ}_{ij}X_i = \mathsf{CZ}_{ij}Z_j \tag{2.43}$$

conjugating by X_i spits out a Z_j. So the groundstate of H is $|\Psi\rangle \equiv S\,|\text{trivial}\rangle$. The trick is that on a circle (with periodic boundary conditions), $[U_{e,o}, S] = 0$, so this state is symmetric. But the individual unitaries $\mathsf{CZ}_{i,i+1}$ making up S do not commute with the symmetries, as you can see from (2.43). Now think about what happens if there is

[28]In $d > 1$ I would have to add the further restriction that an SPT state has short-range entanglement. This would take us too far afield.

a boundary: In the middle of the chain, these extra Z_js all cancel. But at the two ends, there are leftover Zs. This means that the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the groundstate(s) of the open chain is essentially the projective representation described above:

$$U_o |\Psi\rangle \equiv X_1 X_3 \cdots \mathsf{CZ}_{12}\mathsf{CZ}_{23} \cdots |\text{trivial}\rangle = \mathsf{CZ}_{12} Z_2 Z_2 \mathsf{CZ}_{23} \cdots |\text{trivial}\rangle = |\Psi\rangle .$$

$$U_e |\Psi\rangle \equiv X_2 X_4 \cdots \mathsf{CZ}_{12}\mathsf{CZ}_{23} \cdots |\text{trivial}\rangle = Z_1 \mathsf{CZ}_{12} \mathsf{CZ}_{23} Z_3 Z_3 \cdots |\text{trivial}\rangle = Z_1 |\Psi\rangle .$$

$$U_o U_e |\Psi\rangle = X_1 X_3 \cdots Z_1 |\Psi\rangle = -Z_1 U_o |\Psi\rangle = -Z_1 |\Psi\rangle .$$

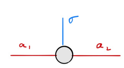
$$U_e U_o |\Psi\rangle = Z_1 |\Psi\rangle .$$

So U_o and U_e don't commute when acting on the groundstate of the edge![29] Instead, they form a Heisenberg algebra (like $XZ = -ZX$), which has no 1d representations, only a 2d irrep, here with basis $\{|\Psi\rangle, Z_1|\Psi\rangle\}$.

The physical consequence of this is that there are degenerate doublets at the ends of the chain. A very similar (but harder to show analytically) phenomenon is realized in spin-1 Heisenberg chains. Such a system has a symmetry group $\mathsf{SO}(3)$, but has a nontrivial SPT phase (called the Haldane phase) where the ends of the chain carry degenerate doublets – the **2** is a projective representation of $\mathsf{SO}(3)$, as described above. Evidence for these degenerate doublets have been observed, *e.g.*, in $Y_2BaNi_{1-x}Mg_xO_5$.

More generally, we can invoke a fact about gapped ground states in 1d: such a state can be written as a *matrix product state*:

$$= \sum_{a_{1,2\ldots}=1}^{\chi} \cdots M^{\sigma_1}_{a_1 a_2} M^{\sigma_2}_{a_2 a_3} \cdots |\cdots \sigma_1, \sigma_2 \cdots\rangle$$



$$\equiv M^{\sigma}_{a_1 a_2} \qquad (2.44)$$

$\sigma = 1..d$, $a = 1..\chi$. χ, the range of the auxiliary index, is called the *bond dimension*. This encodes the groundstate (a vector with d^L components,) in terms of χ^2 numbers (times L if the Ms for different sites are different). In such a state, each site is manifestly entangled with the rest of the system only through its neighbors. The statement that *any* gapped groundstate of a 1d local Hamiltonian can be written this way is a result of Hastings.

We want this state to be invariant under the action of $U = \prod_x u$ (so that it is a paramagnet). Each u acts on a single site: $u_x |\sigma_x\rangle = u_{\sigma_x \sigma'_x} |\sigma'_x\rangle$.

$$U \sum_{a_{1,2\ldots}=1}^{\chi} \cdots M^{\sigma_1}_{a_1 a_2} M^{\sigma_2}_{a_2 a_3} \cdots |\cdots \sigma_1, \sigma_2 \cdots\rangle = \sum_{a_{1,2\ldots}=1}^{\chi} \cdots M^{\sigma_1}_{a_1 a_2} u_{\sigma_1 \sigma'_1} M^{\sigma_2}_{a_2 a_3} u_{\sigma_2 \sigma'_2} \cdots |\cdots \sigma_{1'}, \sigma_{2'} \cdots\rangle$$

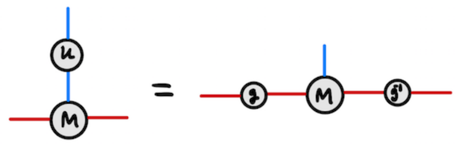
[29]You may be puzzled about how it can be that two operators both of the form $\prod X$ can fail to commute. The trick is that there is another end of the chain. U_e and U_o actually do commute on the groundstate of the whole chain, but because of a cancellation of signs coming from the two endpoints, which can be on opposite sides of the galaxy.

Clearly it would be invariant if at each site

$$M^{\sigma_1}_{a_1 a_2} u_{\sigma_1 \sigma_1'} \stackrel{?}{=} M^{\sigma_1'}_{a_1 a_2}.$$

But this is more than we need. Suppose instead that at each site we can factorize the effects of u as

$$M^{\sigma_1}_{a_1 a_2} u_{\sigma_1 \sigma_1'} = g_{a_1 a_1'} M^{\sigma_1'}_{a_1' a_2'} \left(g^{-1}\right)_{a_2' a_2} .$$



The gs may form a projective representation of G, since the phases will cancel in the previous expression – it's like the action of u is *fractionalized*. Then for a closed chain, the effects of the transformation would cancel between each pair of Ms, by  But if there were a boundary, there would be dangling gs. This leads to a classification of 1d SPTs in terms of $H^2(G, \mathsf{U}(1))$. For more on this I recommend these notes. Under some assumptions, SPTs in d dimensions are classified by $H^{d+1}(G, \mathsf{U}(1))$.

In this context, the group operation on the cohomology is realized just by *stacking* the representative systems on top of each other, and adding generic G-invariant couplings between them.

[End of Lecture 12]

3 Lie groups and Lie algebras

3.1 Lie algebra and structure constants

[Zee, §I.3; Georgi §2,3,6,7,8; Stone-Goldbart §15.3] Recall that a Lie group is a group where we can do calculus. You already know the key idea for understanding the representation theory of Lie groups from your study of quantum mechanics. In that context, you know that given a hermitian operator H, you can make a unitary operator $U(t) = e^{\mathbf{i}tH}$, actually a one-parameter family of unitary operators. This is a solution of the Schrödinger equation for time evolution $-\mathbf{i}\partial_t U = HU$ (I've set $\hbar = 1$ because why wouldn't I?) with the initial condition $U(0) = \mathbb{1}$. This produces a finite transformation $U(t)$ from the information about the infinitesimal transformation $U(\epsilon) = \mathbb{1} + \mathbf{i}\epsilon H$. The story becomes more interesting if we have multiple such hermitian operators (we'll learn to call them generators of the Lie algebra) which do not commute.

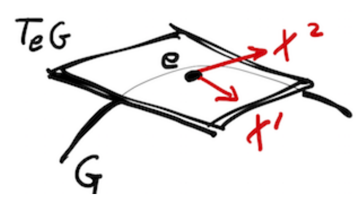
Suppose $g(\vec{s}) \in G$ depend smoothly on a set of $d_G \equiv \dim G$ real parameters. 'Smoothly' means we can do all the calculus we want. We'll set the useful convention for our parameters that $g(0) = e$, the identity in G. In any representation R of G, then, $D(g(\vec{s}))|_{s=0} = D(e) = \mathbb{1}$. *Near* the identity, the smoothness assumption says we can Taylor expand:

$$D(\vec{\epsilon}) \overset{\text{Taylor}}{=} \mathbb{1} + \mathbf{i}\epsilon^A X^A + \mathcal{O}(\epsilon^2), \quad X^A = -\mathbf{i}\partial_{s^A} D(\vec{s})|_{s=0}.$$

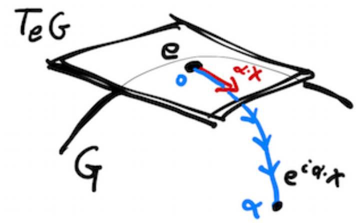
The factor of $\mathbf{i}$ is a convention so that for a unitary rep, $D^\dagger D = \mathbb{1} \implies X^A = (X^A)^\dagger$ the X^A are hermitian (rather than anti-hermitian). If the coordinates aren't redundant, the X^A will be linearly independent. Note that the X^A are $\dim R \times \dim R$ matrices.

As you can see, we can add X^As with complex coefficients – they form a vector space. Actually there is some more structure – there is also a product on this vector space, so it forms an algebra, the Lie algebra $\mathfrak{g}$ associated with G.

Geometrical aside: Here we have defined the objects X^A in a representation, so they are just matrices whose size is the dimension of the carrier space. It is possible to define the Lie algebra generators more abstractly, without a choice of representation, as a basis of tangent vectors to the group manifold at the identity element. If these words don't mean anything to you, please just ignore them. We don't lose much by focussing right away on a representation, since that's where most of the physics is, anyway.



So the key step, Lie's central idea, is to think of a finite transformation as a succession of infinitely many infinitesimal transformations. The identity



$$\lim_{N\to\infty}\left(\mathbb{1}+\frac{☼}{N}\right)^N = e^{☼}$$

is still true if ☼ is a matrix, $☼ = \mathbf{i}s^A X^A$. So

$$D(\vec{s}) = \lim_{N\to\infty}\left(\mathbb{1}+\frac{\mathbf{i}s^A X^A}{N}\right)^N = e^{\mathbf{i}s^A X^A}.$$

Multiplying elements along the same direction $\vec{s}$ is easy, $D(\lambda_1\vec{s})D(\lambda_2\vec{s}) = D((\lambda_1+\lambda_2)\vec{s}) = D(\lambda_2\vec{s})D(\lambda_1\vec{s})$, since that's just like the $e^{\mathbf{i}tH}$ example above. Any choice of $\vec{s}\cdot\vec{X}$ generates a 1-parameter abelian subgroup.

But in general, $D(\vec{s})D(\vec{t}) \neq D(\vec{s}+\vec{t})$. To see what happens instead, I claim that

$$D(\vec{s})D(\vec{t}) = e^{\mathbf{i}s^A X^A} e^{\mathbf{i}t^A X^A} = e^{\mathbf{i}r^A X^A} \tag{3.1}$$

for some choice of r^A. This is because these exponentials form a representation of $G_0 \equiv \{g \in G, \text{ connected by a path to } e\}$, the component of G connected to the identity, which is a subgroup of G. (Note that (3.1) is *not* true of the exponential map on a general manifold.) By the smoothness assumption, we can use calculus to find $\vec{r}$. A useful trick is to rescale $\vec{s},\vec{t}$ by the same small parameter ϵ; then terms in the Taylor expansion at the same order in ϵ must match on the BHS. At the end we can set $\epsilon = 1$.

$$\mathbf{i}\vec{r}\cdot\vec{X} = \ln(1+K) = K - \frac{1}{2}K^2 + \mathcal{O}(\epsilon^3) \quad \text{with}$$

$$K \equiv e^{\mathbf{i}\vec{s}\cdot\vec{X}}e^{\mathbf{i}\vec{t}\cdot\vec{X}} - 1 = \mathbf{i}\vec{s}\cdot\vec{X} + \mathbf{i}\vec{t}\cdot\vec{X} - \frac{1}{2}(\vec{s}\cdot\vec{X})^2 - \frac{1}{2}(\vec{t}\cdot\vec{X})^2 - \vec{s}\cdot\vec{X}\vec{t}\cdot\vec{X} + \mathcal{O}(\epsilon^3).$$

This would be just linear $\mathbf{i}\vec{r}\cdot X \stackrel{?}{=} \mathbf{i}\vec{s}\cdot\vec{X} + \mathbf{i}\vec{t}\cdot\vec{X}$ if $[X^A, X^B] = 0$. In general, this is not the case, and instead

$$\mathbf{i}\vec{r}\cdot X = \mathbf{i}\vec{s}\cdot\vec{X} + \mathbf{i}\vec{t}\cdot\vec{X} - \frac{1}{2}[\vec{s}\cdot X, \vec{t}\cdot\vec{X}] + \mathcal{O}(\epsilon^3)$$

and therefore

$$[\vec{s}\cdot X, \vec{t}\cdot\vec{X}] = -2\mathbf{i}\left(\vec{r}-\vec{s}-\vec{t}\right)\cdot\vec{X} \equiv \mathbf{i}\vec{u}\cdot\vec{X}.$$

In order for this to be true for all s, t, since the $\{X^A\}$ span the vector space, we require $u^A = s^A t^B f_{ABC}$ where

$$[X^A, X^B] = \mathbf{i}f_{ABC}X^C$$

– the generators must form an algebra under commutators. That is, the commutator of two generators is a linear combination of generators. This is the product operation on $\mathfrak{g}$ I referred to above. Notice that the ordinary matrix product $X^A X^B$ is not necessarily such a linear combination of Xs. I claim that this one relation, obtained at $\mathcal{O}(\epsilon^2)$, is enough to guarantee that the group law is obeyed to all orders in ϵ. The idea is that we know the *only* deviation from linearity ($\vec{r} \stackrel{?}{=} \vec{s} + \vec{t}$) comes from commutators, and if the commutator is again an element of the algebra, then so are any repeated commutators.

Comments on BCH formulae. To see this more explicitly, we can derive an explicit formula for $e^{-A}e^{A+B}$, which would be e^B if $[A, B] = 0$. This is a version of the Baker-Cambell-Hausdorff formula.

Given a matrix A, define the matrix operator ad_A, which implements the adjoint action of A, by

$$\text{ad}_A B \equiv [A, B].$$

Note that this operator is a *derivation*, in the sense that it satisfies the product rule

$$\text{ad}_A(BC) = (\text{ad}_A B)C + B(\text{ad}_A C) \tag{3.2}$$

and, because the trace of a commutator vanishes (for finite-dimensional matrices, at least),

$$\text{tr} \circ \text{ad} = 0$$

one can "integrate by parts" under the trace. So ad_X is very much like a matrix version of a derivative "in the X direction".

First note that $\exp \text{ad} = \text{Ad}_{\exp}$. By this I mean

$$e^{tA} B e^{-tA} = e^{t\text{ad}_A} B \equiv F(t). \tag{3.3}$$

The proof is just: the BHS satisfies $\dot{F}(t) = [A, F(t)], F(0) = B$, which has a unique solution. The left hand side is the group version of the adjoint action, which we might write as $\text{Ad}_g B \equiv gBg^{-1}$, so that (3.3) can be written (WLOG at $t = 1$, and on the right as an operator equation, since it's true for all B) as

$$\exp\left(\text{ad}_A\right) B = \text{Ad}_{\exp(A)} B \quad \text{or} \quad \exp\left(\text{ad}_A\right) = \text{Ad}_{\exp(A)} \quad . \tag{3.4}$$

Next we are going to derive a BCH-like formula

$$e^{A+B} = e^A G \tag{3.5}$$

where G would be e^B if $[A, B] = 0$, and we will see explicitly that G depends only on $\text{ad}_A^n B$, repeated commutators. To do this, let

$$G(s) \equiv e^{-sA} e^{s(A+B)}.$$

Then

$$\partial_s G(s) = -AG(s) + e^{-sA}(A + B)e^{sA}e^{-sA}e^{s(A+B)} = B(s)G(s)$$

with

$$B(s) \equiv e^{-sA} B e^{sA} = e^{-s\text{ad}_A} B.$$

The equation $\partial_s G(s) = B(s)G(s)$ is familiar from perturbation theory in quantum mechanics; to solve it, observe that

$$G(s) = G(0) + \int_0^s dt B(t)G(t) = G(0) + \int_0^s dt_1 B(t_1)G(t_1) + \int_0^s dt_1 \int_0^{t_1} dt_2 B(t_1)B(t_2)G(t_2) \cdots$$

and therefore the solution is

$$G(s) = \sum_{n=0}^{\infty} \int_0^s dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n B(t_1)B(t_2) \cdots B(t_n).$$

By introducing the notion of time-ordering $\mathcal{T} B(t_1)B(t_2) \equiv \begin{cases} B(t_1)B(t_2), & t_1 \geq t_2 \\ B(t_2)B(t_1), & t_2 > t_1 \end{cases}$, this can be repackaged as

$$G(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^s dt_1 \int_0^s dt_2 \cdots \int_0^s \mathcal{T} B(t_1)B(t_2) \cdots B(t_n) = \mathcal{T} e^{\int_0^s dt B(t)}.$$

Therefore

$$G(s) = \mathcal{T} e^{\int_0^s dt B(t)} G(0) \ .$$

Since $G(0) = 1$ we get

$$e^{-A} e^{A+B} = G(1) = \mathcal{T} e^{\int_0^1 dt e^{-t\text{ad}_A} B}$$

which is the form we were after. This is maybe not the most useful expression. But you can see explicitly that only objects of the form $\text{ad}_A^n B$ appear, so if A, B are generators of a Lie algebra, the whole thing is determined by the commutator algebra.

A more common and perhaps slightly more useful form of the CBH formula is

$$\log\left(e^X e^Y\right) = X + \int_0^1 dt g\left(e^{\text{ad}_X} e^{t\text{ad}_Y}\right)(Y), \quad g(z) \equiv \frac{\log z}{1-1/z}, \tag{3.6}$$

and $g(A)$ for a linear operator A is defined by its Taylor series about $z = 1$. The proof involves some more-generally-useful ingredients. First,

$$\partial_t e^{X+tY}|_{t=0} = e^X \frac{1-e^{-\text{ad}_X}}{\text{ad}_X}(Y).$$

This follows by writing $e^{X+tY} = \left(e^{X/m+tY/m}\right)^m$ (for any m) and differentiating using the product rule[30]:

$$\partial_t e^{X+tY}|_{t=0} = \sum_{k=0}^{m-1} \left(e^{X/m}\right)^{m-k-1} \partial_t e^{X/m+tY/m}|_{t=0} \left(e^{X/m}\right)^k \tag{3.7}$$

$$= e^{\frac{m-1}{m}X} \sum_{k=0}^{m-1} \frac{1}{m} e^{-\frac{k}{m}\text{ad}_X} Y \tag{3.8}$$

$$\overset{m\to\infty}{\to} e^X \int_0^1 ds e^{-s\text{ad}_X} Y = e^X \frac{1-e^{-\text{ad}_X}}{\text{ad}_X} Y. \tag{3.9}$$

From this it follows by the chain rule that

$$e^{-Z(t)} \partial_t e^{Z(t)} = \frac{1-e^{-\text{ad}_{Z(t)}}}{\text{ad}_{Z(t)}} \partial_t Z(t). \tag{3.10}$$

Now let $Z(t) \equiv \log e^X e^{tY}$. Then we have $e^{Z(t)} = e^X e^{tY}$ and

$$e^{-Z(t)} \partial_t e^{Z(t)} = \left(e^X e^{tY}\right)^{-1} e^X e^{tY} Y = Y.$$

Comparing this to (3.10), we have

$$Y = \frac{1-e^{-\text{ad}_{Z(t)}}}{\text{ad}_{Z(t)}} \partial_t Z(t).$$

This implies

$$\partial_t Z(t) = \left(\frac{1-e^{-\text{ad}_{Z(t)}}}{\text{ad}_{Z(t)}}\right)^{-1} Y.$$

[30] As Georgi emphasizes, the following form of this identity (the same as the first expression in (3.9)) is quite appealing:

$$\partial_t e^{X+tY}|_{t=0} = \int_0^1 ds e^{(1-s)X} Y e^{sX}.$$

Since the derivative doesn't commute with the exponential, we don't know on which side to put it. So we just average over all possibilities! What else could it be?

But now $g(z) = \left(\frac{1-z^{-1}}{\log(z)}\right)^{-1}$, so (3.6) follows by integrating up the BHS over t. Note that we need to use (3.4) to write the argument of $g(z)$ as $z = e^{-\mathrm{ad}_{Z(t)}} = e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y}$.

For more details, see the book by Hall, §5.3-5.5.

So a Lie algebra looks like $[X_A, X_B] = \mathbf{i} f_{ABC} X_C$. The f_{ABC} are called the *structure constants* of the Lie algebra. Under some assumptions, they encode the whole structure. They have the following properties.

- Since $[A, B] = -[B, A]$, $f_{ABC} = -f_{BAC}$.
- For unitary reps (any rep for a compact group), $[X_A, X_B]^\dagger = -\mathbf{i} f^\star_{ABC} X_C = -\mathbf{i} f_{ABC} X_C$ so $f_{ABC} = f^\star_{ABC}$, the structure constants are real.
- The *Jacobi identity* is

  $$0 = [X_A, [X_B, X_C]] + 2 \text{ cyclic permutations.}$$

  It follows just by writing out all 12 terms and cancelling them in pairs. (It is true for any three matrices, not just for the Lie algebra generators.) In terms of the structure constants it says

  $$f_{BCD} f_{ADE} + f_{ABD} f_{CDE} + f_{CAD} f_{BDE} = 0. \tag{3.11}$$

  In terms of the adjoint action, it has the clearest statement:

  $$[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]}. \tag{3.12}$$

The crucial point is that representations of the Lie algebra $\mathfrak{g}$ are representations of G_0 (and the difference between G and G_0 is some finite-group problem which we've already solved). And the former are much easier to understand. To be clear, by a representation of a Lie algebra, I mean a set of matrices $D(X_A)$ whose commutators satisfy the Lie algebra[31] – a homomorphism from the Lie algebra to $\mathsf{GL}(n)$ for some n, the dimension of the representation. To get from this a representation of the Lie group, use the same carrier space, and just exponentiate arbitrary linear combinations of these matrices: $D(g(\theta_A)) = e^{\mathbf{i}\theta_A D(X_A)}$.

Here, then, is an example of a representation of any Lie group, the adjoint representation. This is just the map $X \to \mathrm{ad}_X$, which is a representation of $\mathfrak{g}$ because of

[31] As you may have noticed by now I will often just write X_A for the matrices themselves, when I really mean $D(X_A)$ in some representation.

(3.12). More prosaically, consider T^A with matrix elements $(T_A)_{BC} \equiv -\mathbf{i} f_{ABC}$. The Jacobi identity in the form (3.11) can be rewritten as

$$[T_A, T_B] = \mathbf{i} f_{ABC} T_C$$

so these matrices satisfy the Lie algebra. The dimension of this rep is the number of generators of the Lie algebra, which is d_G, the dimension of the group. [End of Lecture 13]

A good way to think about the adjoint rep is a bit like the regular representation of a finite group: it has a basis $|X_A\rangle$ labelled by generators – the carrier space is $\text{span}\{|X_A\rangle\}$ – and the action of a generator is

$$D_{\text{adj}}(X_A) |X_B\rangle = |[X_A, X_B]\rangle . \tag{3.13}$$

(Often you'll see X_A in place of $D(X_A)$ here.) Other ways to write this are

$$D_{\text{adj}}(X_A) |X_B\rangle = |[X_A, X_B]\rangle = |\mathbf{i} f_{ABC} X_C\rangle = \mathbf{i} f_{ABC} |X_C\rangle = (T^A)_B{}^C \left|X^C\right\rangle ,$$

so you see it agrees with the definition above.

For any (unitary) rep we can define an inner product on the Lie algebra by

$$\text{tr}_R X^A X^B = \kappa^{AB}.$$

Since this is a symmetric matrix, we can diagonalize it by a similarity transformation, by taking linear combinations $(X^A)' = L_{AB} X^B$. (Nothing specified a basis for the Lie algebra so far.) In terms of the new X's, the inner product is $k^A \delta^{AB}$. If the k^A are nonzero, we can rescale the generators to set $|k^A| = \lambda$ for all A (for some constant λ to be chosen later). But we cannot change the signs of the k^A. I claim that $k^A > 0$ for all A for a *compact* Lie group with no $\mathsf{U}(1)$ factors, explanation later. The short version is that if $k^A > 0$, then the Lie algebra is a subalgebra of $\mathsf{so}(n)$ for some n, and we agree that $\mathsf{SO}(n)$ is a compact group. In this case, the range of the parameters s^A has some restriction, since there will exist some $s_\star$ for which $e^{\mathbf{i} s_\star \cdot X} = 1\!\!1$.

Now, if $\text{tr} X^A X^B = \lambda \delta^{AB}$, then multiplying the BHS of the Lie algebra by X^D and taking trace gives

$$f_{ABC} = -\frac{\mathbf{i}}{\lambda} \text{tr}[X^A, X^B] X_C = f_{BCA} = -f_{BAC} = -f_{ACB} = -f_{CBA}$$

– the structure constants in such a basis are totally antisymmetric in all three indices. (If you like, we used the $\text{tr} \circ \text{ad} = 0$ (IBP) identity.) Furthermore, the adjoint rep is unitary in this basis – the T_A are imaginary and antisymmetric, hence hermitian. Note that the inner product in the adjoint rep is

$$\langle X_A | X_B \rangle \propto \text{tr}_{\text{adj}} T^A T^B.$$

Concrete example: $\mathsf{SO}(n)$. Recall that $\mathsf{O}(n) = \{n \times n \text{ real matrices } R \text{ with } R^T R = 1\!\!1\}$. The component connected to the identity, "$\mathsf{O}(n)_0$" in the notation above, is $\mathsf{SO}(n) = \{R \in \mathsf{O}(n), \det R = 1\}$.

Following the strategy advocated above, now consider an infinitesimal rotation: $R(\theta) = 1\!\!1 + A + \cdots$.

$$1\!\!1 \overset{!}{=} R^T R = (1\!\!1 + A^T)(1\!\!1 + A) + \cdots = 1\!\!1 + A^T + A + \cdots$$

which says $A = -A^T$ is antisymmetric.

For $n = 2$, there is only one antisymmetric matrix, up to scalar multiples: $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so $A = \theta \mathcal{J}$ (with some foresight in the naming of variables). So $R = 1\!\!1 + \theta \mathcal{J} + \mathcal{O}(\theta^2) = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} + \mathcal{O}(\theta^2)$. For finite θ, this is

$$R(\theta) \overset{\text{Lie}}{=} \lim_{N \to \infty} \left(1 + \frac{\theta \mathcal{J}}{N}\right)^N = e^{\theta \mathcal{J}} \overset{\text{Taylor}}{=} 1 + \theta \mathcal{J} + \frac{\theta^2}{2} \mathcal{J}^2 + \cdots = \sum_{k=0}^{\infty} \frac{\theta^k \mathcal{J}^k}{k!}$$

Since $\mathcal{J}^2 = -1\!\!1$, we have $\mathcal{J}^{2\ell} = 1\!\!1(-1)^\ell, \mathcal{J}^{2\ell+1} = (-1)^\ell \mathcal{J}$, so

$$R(\theta) = \sum_{\ell=0}^{\infty} \frac{\theta^{2\ell}(-1)^\ell}{(2\ell)!} 1\!\!1 + \sum_{\ell=0}^{\infty} \frac{\theta^{2\ell+1}(-1)^\ell}{(2\ell+1)!} \mathcal{J} = \cos\theta 1\!\!1 + \sin\theta \mathcal{J} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \equiv e^{\mathbf{i}\theta J}$$

where $J \equiv -\mathbf{i}\mathcal{J}$ is hermitian. This is the familiar form of a 2d rotation matrix in terms of the angle of rotation θ. Since there is only one generator of the Lie algebra, $\mathsf{SO}(2)$ is an abelian group. Notice that this maneuver fails to reach all of $\mathsf{O}(n)$, such as $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which has $\det Z = -1$.

For $n = 3$, any antisymmetric matrix is a linear combination of these three:

$$\mathcal{J}^1 = \begin{pmatrix} 0 & & \\ & & 1 \\ & -1 & \end{pmatrix}, \quad \mathcal{J}^2 = \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix}, \quad \mathcal{J}^3 = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 0 \end{pmatrix}$$

(where I only write the nonzero entries), $A = \theta^A \mathcal{J}^A \equiv \mathbf{i}\theta^A J^A$. More compactly, $(\mathcal{J}_i)^j{}_k = \epsilon_{ijk}$, the Levi-Civita symbol – totally antisymmetric and equal to 1 for $ijk = 123$. So $R(\theta^A) = e^{\theta^A \mathcal{J}^A} = e^{\mathbf{i}\theta^A J^A} \in \mathsf{SO}(3)$. If you like, compare this expression with the definition of Euler angles; do not ask me to do it.

Since we have more than one generator for $n > 2$, we can study the commutator algebra. Consider $R = 1\!\!1 + A + \cdots$. If R' is another rotation, then

$$RR'R^{-1} = (1\!\!1 + A)R'(1\!\!1 - A) + \cdots = R' + [A, R'] + \cdots$$

If R' is also close to the identity, $R' = \mathbb{1} + A' + \cdots$, then $RR'R^{-1} = 1 + A' + [A, A'] + \cdots$. For $n = 3$, $A = \theta \cdot \mathcal{J}$, $A' = \theta' \cdot \mathcal{J}$. Since $([A, A'])^T = -[A, A']$, this extra term is *also* an antisymmetric 3×3 matrix, and hence can be expanded in the $\mathcal{J}$s: $[A, A'] = \theta'' \cdot \mathcal{J}$. Let $J_i \equiv -\mathbf{i}\mathcal{J}_i$ as above. By explicit calculation, $[J_1, J_2] = \mathbf{i}J_3 = -[J_2, J_1]$, plus its cyclic images[32], which can be summarized nicely as

$$[J_i, J_j] = \mathbf{i}\epsilon_{ijk}J_k \tag{3.14}$$

where ϵ_{ijk} is the Levi-Civita symbol again. This is the Lie algebra $\mathsf{so}(3) = \mathsf{su}(2)$. Notice that the real matrices satisfy $[\mathcal{J}_i, \mathcal{J}_j] = \epsilon_{ijk}\mathcal{J}_k$ with no i. The *commutator* of two real antisymmetric matrices is real antisymmetric (though the ordinary product is not!).

A confession and a definition: Actually the matrices R that we've been writing are the representation matrices $D(\vec{\theta})_{ij}$ for the $\mathbf{n}$, the n-dimensional *fundamental* (or *defining*) representation of $\mathsf{SO}(n)$. An object that transforms in this rep is called a *vector* of $\mathsf{SO}(n)$. I emphasize that the algebra (3.14), however, is satisfied by the generators in *any* representation.

For general n: an arbitrary real antisymmetric matrix is a linear combination $A = \sum_{r<s} A_{rs}\mathcal{J}_{(rs)}$ of these

$$\left(\mathcal{J}_{(rs)}\right)^{ij} \equiv \begin{cases} 0, & \text{everywhere except} \\ 1, & \text{in the } rs \text{ entry} \\ -1, & \text{in the } sr \text{ entry to make it AS} \end{cases} = \delta^{ri}\delta^{sj} - \delta^{sj}\delta^{ri}.$$

This is zero for $r = s$, and $\mathcal{J}_{(rs)} = -\mathcal{J}_{(sr)}$ so there are $n(n-1)/2 = \dim \mathsf{SO}(n)$ of these. To make hermitian generators, let $J_{(rs)} = -\mathbf{i}\mathcal{J}_{(rs)}$. So an arbitrary element of $\mathsf{SO}(n)$ (rather, its representative in the $\mathbf{n}$) is

$$R(\vec{\theta}) = \exp\left(\mathbf{i}\sum_{r<s}\theta_{(rs)}J_{(rs)}\right).$$

Notice that $J_{(rs)}$ generates a rotation in the rs plane. In 3d, there is this weird accident that specifying a plane is the same as specifying which direction is normal to it, so we can label the generators $J_{(ij)} = J_k\epsilon_{ijk}$.

The $\mathsf{SO}(n)$ Lie algebra looks more complicated than it is. Think about $n = 4$. Then $[J_{(12)}, J_{(34)}] = 0$ since they act on orthogonal subspaces. And we already know $[J_{(12)}, J_{(23)}] = \mathbf{i}J_{(31)}$ since this is part of $\mathsf{so}(3)$. More generally, there are three cases, depending on how many labels (0,1, or 2) the two victims have in common. If there are

[32] One way to see that the algebra must be invariant under cyclic permutations $x \to y \to z$ is that this transformation is accomplished by $R(\hat{y}, \pi/2) \circ R(\hat{z}, \pi/2)$, an element of $\mathsf{SO}(3)$.

0 or 2, the commutator vanishes. In the remaining case, the shared index gets erased: $[J_{(rs)}, J_{(rq)}] = \mathbf{i}J_{(sq)}$ (no sum on r). Other cases are related by the antisymmetry of the commutator or the labels on J. This information can be (rather inelegantly) summarized by the equation

$$[J_{(rs)}, J_{(pq)}] = \mathbf{i}\left(\delta_{rp}J_{(sq)} + \delta_{sq}J_{(rp)} - \delta_{sp}J_{(rq)} - \delta_{rq}J_{(sp)}\right). \tag{3.15}$$

The structure on the RHS follows from the antisymmetry under either $r \leftrightarrow s$ or $p \leftrightarrow q$ or $(rs) \leftrightarrow (pq)$.

Let's think more about the case of $\mathsf{so}(4)$. Let $J_i \equiv \frac{1}{2}J_{(jk)}\epsilon_{ijk}$, $K_i \equiv J_{i4}$. These satisfy the algebra

$$\begin{aligned}
[J_i, J_j] &= \mathbf{i}\epsilon_{ijk}J_k, \quad \mathsf{so}(3) \subset \mathsf{so}(4)\\
[K_i, K_j] &= \mathbf{i}\epsilon_{ijk}J_k\\
[J_i, K_j] &= \mathbf{i}\epsilon_{ijk}K_k, \quad K \text{ is a vector of } \mathsf{SO}(3)
\end{aligned}$$

The first equation is not a surprise. The second equation follows, up to a constant, from the fact that under $(\vec{x}, x_4) \mapsto (\vec{x}, -x_4)$, $\vec{J} \mapsto \vec{J}, \vec{K} \mapsto -K$, so the RHS can't have any K. The third equation is the statement that K_k transforms as a vector of $\mathsf{SO}(3)$ under conjugation. What is the finite transformation? Under a rotation in $\mathsf{SO}(3)$,

$$K^j \mapsto e^{\mathbf{i}\theta^i \mathrm{ad}_{J^i}}K^j \simeq K^j + \mathbf{i}\theta^i[J^i, K^j] + \mathcal{O}(\theta^2) \stackrel{\star}{=} R(\theta)K^jR(\theta)^{-1}, \quad R(\theta) \equiv e^{\mathbf{i}\theta\cdot J}. \tag{3.16}$$

To see the equality with the $\star$, we are using $\exp \mathrm{ad} = \mathrm{Ad}\exp$. Or more directly, observe that (after rescaling $\theta^i \to s\theta^i$) the BHS satisfies the ordinary differential equation

$$\partial_s F(s)^j = \mathbf{i}[s\theta^i J^i, K^j], \quad \text{with initial condition } F(s=0)^j = K^j,$$

which has a unique solution. For example, $e^{-\mathbf{i}\varphi J^3}K_1 e^{\mathbf{i}\varphi J^3} = \cos\varphi K_1 + \sin\varphi K_2$.

One more step about $\mathsf{SO}(4)$: let $\vec{J}_\pm \equiv \frac{1}{2}(\vec{J} \pm \vec{K})$. These each satisfy the $\mathsf{SO}(3)$ algebra, and they commute $[J_+^i, J_-^j] = 0$. Therefore $\mathsf{so}(4) = \mathsf{so}(3) \oplus \mathsf{so}(3)$, meaning that as vector spaces, we take the direct sum, and also that the generators of the different summands commute. (Note that people (including me will sometimes write $\mathsf{so}(3) \times \mathsf{so}(3)$ for this relation instead. This would be more accurate as a statement about the groups obtained by exponentiating these algebras.)

A very similar analysis applies to the (non-compact) Lorentz groups $\mathsf{SO}(p, q)$ for various p, q. The analog of $R^TR = 1\!\!1$ is

$$\eta_{\mu\sigma} = \Lambda_\mu{}^\nu \eta_{\nu\rho}(\Lambda^T)^\rho{}_\sigma = (1\!\!1 + A)_\mu{}^\nu \eta_{\nu\rho}(1\!\!1 + A^T)^\rho{}_\sigma + \cdots$$

which says $A_\mu{}^\nu\eta_{\nu\sigma} + \eta_{\mu\rho}(A^T)^\rho{}_\sigma = 0$. Generators like $\mu\nu = ij$ with both indices along positive directions are antisymmetric, while generators labelled $\mu\nu = i0$ with one negative direction are symmetric. These are rotations and boosts, respectively. Notice that hermiticity requires the symmetric generators to be *real*, so the exponentiated version looks like $e^{\eta B}$ with $\eta \in \mathbb{R}$, rather than $e^{\mathbf{i}\theta J}$. This is the origin of the non-compactness.

Notice something very weird about $\mathsf{SO}(3)$: this is the *adjoint* representation of $\mathsf{SO}(3)$ (in fact the ad in (3.16) is the reason for the name), but it is also the fundamental **3**-dimensional representation.

Simplicity and semi-simplicity. An *invariant subalgebra* of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ that is closed under commutators with elements of $\mathfrak{g}$: $\mathfrak{h} = \text{span}\{X \in \mathfrak{g} | [X, Y] \in \mathfrak{h}, \forall Y \in \mathfrak{g}\}$. The point of this definition is that exponentiating the elements of an invariant subalgebra $\exp(\mathfrak{h}) = H \subset G$ produces an invariant subgroup: If $h = e^{\mathbf{i}X}, g = e^{\mathbf{i}Y}$ for $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, then $g^{-1}hg = e^{\mathbf{i}X'} \in H$ with

$$\mathfrak{h} \ni X' = e^{-\mathbf{i}Y} X e^{\mathbf{i}Y} = e^{-\mathbf{i}\text{ad}_Y} X = \sum_{k=0}^{\infty} \frac{(-\mathbf{i})^k}{k!} \text{ad}_Y^k(X).$$

Notice that by the definition, $\text{ad}_Y^k(X) \in \mathfrak{h}$ for all k.

0 and $\mathfrak{g}$ are trivial invariant subalgebras. If $\mathfrak{g}$ has no nontrivial invariant subalgebras it's called *simple*. This is a good definition because then $e^{\mathfrak{g}}$ is a simple group. The adjoint rep of a simple algebra is irreducible. If it weren't, an invariant subspace is spanned by a subset of the generators $V = \text{span}\{T_r\}$, $V^\perp = \text{span}\{T_x\}$. The definition of 'invariant subspace' would then say $(T_A)_{xr} = 0 = -\mathbf{i} f_{Axr}$ for all $A = r, x$. Therefore there would be two invariant subalgebras with structure constants $f_{xx'x''}$ and $f_{rr'r''}$.

A one-dimensional invariant subalgebra, generated by a single element X_A, means a $\mathsf{U}(1)$ factor in the group (if the group is compact). This means $f_{ABC} = 0$ for all B, C and (therefore) also that $\text{tr} T_A^\dagger T_A = 0$ (evaluate it in the adjoint rep). So the inner product degenerates and the theory breaks down. Such $\mathsf{U}(1)$ factors are invisible to the structure constants. A Lie algebra with no abelian invariant subalgebras is called *semi-simple*. A semi-simple Lie algebra is a direct sum of simple algebras (direct sum as a vector space, and also with the implication that the commutator between different summands vanishes). This means that every X_A has some other generator with which it fails to commute, and (therefore by the total antisymmetry of f_{ABC}) every X_A appears on the right hand side of some linear combination of commutators. Since abelian factors are well-understood, from now on we will restrict our attention to such a situation without comment.

Big picture of representation theory of Lie groups. There are two approaches to representation theory of Lie groups, which can be called *tensor methods* and the *Cartan-Weyl method*. In the former, we make reps by tensoring together the fundamental rep and symmetrizing in various ways. In the latter, we choose a complete set of commuting operators in the Lie algebra (called the Cartan subalgebra) and diagonalize them; then their eigenvalues label the basis states of any rep, and we use the rest of the generators as raising and lowering operators, just like in the familiar theory of $\mathsf{SU}(2)$ representations. They both have their virtues.

[End of Lecture 14]

3.2 Irreps of $\mathsf{SO}(n)$ by tensor methods

So far we've encountered two representations of $\mathsf{SO}(n)$, the fundamental, vector representation $\mathbf{n}$ and the adjoint, of dimension $n(n-1)/2$. (For $n = 3$ they are the same rep.) Recall that a vector is something that transforms like $v^i \mapsto R^{ij} v^j$.

What happens if we take $\mathbf{n} \otimes \mathbf{n}$? An object T in the $\mathbf{n} \otimes \mathbf{n}$ is, by definition, something that transforms like

$$T^{ij} \mapsto D(R)_{ij,kl} T^{kl} = R^{ik} R^{jl} T^{kl}.$$

Such a thing is a called a 2-index (or rank-two) *tensor*. where the representation matrix $D(R)_{ij,kl}$ is $n^2 \times n^2$. Is it reducible?

Yes: if we consider just the antisymmetric bit $A^{ij} \equiv T^{ij} - T^{ji}$, the linear action of R on it preserves the antisymmetry in the indices. So this is an $n(n-1)/2$-dimensional invariant subspace.

Similarly, the symmetric bit $S^{ij} \equiv T^{ij} + T^{ji}$ maps to itself. So, so far we have $\mathbf{n} \otimes \mathbf{n} = \Lambda^2 \mathbf{n} \otimes \mathrm{Sym}^2 \mathbf{n}$. This $n(n+1)/2$-dimensional subspace is, however, itself reducible. This is because of the existence of the *invariant symbol* δ_{ij}. It is an invariant symbol in the sense that $R_{ik} R_{jl} \delta_{kl} = \delta_{ij}$ (this is the definition of $R \in \mathsf{SO}(n)$). So

$$S \equiv S^{ii} = S^{ij} \delta_{ij} \mapsto R^{ik} R^{jl} \delta_{ij} S^{kl} = S$$

– this is a 1d invariant subspace. So we've just decomposed

$$\mathbf{n} \otimes \mathbf{n} = \mathbf{1} \oplus \frac{\mathbf{n(n-1)}}{\mathbf{2}} \oplus \frac{\mathbf{n(n+1)}}{\mathbf{2}} - \mathbf{1}.$$

For $\mathsf{SO}(3)$, this is $\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$.

There is a second invariant symbol for $\mathsf{SO}(n)$, which comes from the following formula for the determinant of an $n \times n$ matrix:

$$\epsilon^{i_1 \cdots i_n} M^{i_1 j_1} \cdots M^{i_n j_n} = \epsilon^{j_1 \cdots j_n} \det M,$$

where ϵ is the completely antisymmetric Levi-Civita symbol, with $\epsilon^{12\cdots n} = 1$. Since for a rotation matrix $R^T R = \mathbb{1} \implies \det R = 1$, we have

$$\epsilon^{i_1 \cdots i_n} R^{i_1 j_1} \cdots R^{i_n j_n} = \epsilon^{j_1 \cdots j_n}. \tag{3.17}$$

Using the ϵ tensor, we can trade a p-index AS tensor $A^{i_1\cdots i_p}$ for a $(n-p)$-index tensor:

$$B^{i_{p+1}\cdots i_n} = \epsilon_{i_1\cdots i_n} A^{i_1\cdots i_p}.$$

If A is a tensor in the sense of transforming as $A^{i_1\cdots i_p} \mapsto R^{i_1 j_1}\cdots R^{i_p j_p} A^{j_1\cdots j_p}$, then using (3.17) and $R^T R = 1\!\!1$, so is B. This means that for $n=3$, the antisymmetric 2-index object $\mathbf{3}$ in the decomposition of $\mathbf{3}\otimes\mathbf{3}$ is not actually a new irrep – it's equivalent to the vector by the transformation: $A^{ij} \equiv \epsilon^{ijk} A^k$. (Which is also the adjoint rep of $\mathsf{SO}(3)$! It's as if three space dimensions is special somehow.)

More generally, who is this $\mathbf{n(n-1)/2}$ rep? It transforms as

$$A_{ij} \mapsto R_{ik} R_{jl} A_{kl} = R_{ik} A_{kl} R^T_{lj} = (RAR^{-1})_{ij}.$$

Let's write $R = e^{\theta^A \mathcal{J}^A}$, so the RHS here is $A \mapsto A + \theta^A[\mathcal{J}^A, A] + \cdots$. But now A itself is a real antisymmetric tensor, so can be expanded in the basis of $\mathcal{J}$s that we found earlier: $A = \sum_B A^B \mathcal{J}^B$ for some numbers A^B (in fact $n(n-1)/2$ of them). Therefore

$$A \mapsto A + \theta^A[\mathcal{J}^A, \mathcal{J}^B]A^B + \cdots = \theta^A f^{ABC} J^C A^B + \cdots .$$

The change in the coefficient of $\mathcal{J}^C$ in A is

$$\delta A^C = \theta^A f^{ABC} A^B = \theta^A (f_A)_C{}^B A^B$$

Thus, the generators of this representation are the matrices $(f_A)_C{}^B = f_{ABC}$, which we called the adjoint rep earlier.

3.3 Casimirs

Suppose given matrices X_A representing $\mathfrak{g}$, a semi-simple Lie algebra $[X_A, X_B] = \mathbf{i} f_{ABC} X_C$. Recall that semisimple means that $\mathrm{tr} X_A^\dagger X_B = \kappa_{AB}$ (this is called the Killing form) is invertible: $(\kappa^{-1})_{AB}\kappa_{BC} = \delta_{AC}$. Then consider the matrix

$$C_2 \equiv \kappa^{-1}_{AB} X_A X_B \ .$$

I claim that $[C_2, X_A] = 0$ for all $A = 1\cdots \dim G$ by the total antisymmetry of f_{ABC}. (To check this, I recommend using the basis where $\kappa_{AB} = \lambda\delta_{AB}$.) Therefore, if this is an irrep R, Schur tells us that $C_2 = c_2(R)1\!\!1$. Such an operator that commutes with all the generators is called a *Casimir*, and this particular one $c_2(R)$ is called the quadratic Casimir of the rep R.

For example, for $\mathsf{SU}(2)$ normalized in the usual way, this is just

$$J^2 \equiv J_x^2 + J_y^2 + J_z^2.$$

We can figure out what is the quadratic Casimir of the spin-j rep momentarily.

Actually, a theorem of Racah says that a rank-r Lie algebra has r such Casimir operators. I'll define the rank of a Lie algebra in just a moment.

3.4 Cartan-Weyl method

[Georgi; Di Francesco, Mathieu, Seneschal, chapter 13] The following procedure works for any Lie algebra. I will illustrate with $\mathsf{SU}(2)$ throughout our first pass. Normalize the generators as $\text{tr}X^A X^B = \lambda\delta^{AB}$.

The idea will be just like in quantum mechanics: we have a bunch of linear operators (like the observables of a quantum system), and we are going to diagonalize a complete set of commuting operators, and use their eigenvalues to label a basis of states. Choose a maximal subset of commuting hermitian generators

$$[H_i, H_j] = 0, \quad H_i = H_i^\dagger, \quad i = 1..r \equiv \text{rank}(G)$$

This is called a *Cartan subalgebra*. If we exponentiate them, we get a $\mathsf{U}(1)^r$ subroup of G (if G is compact, otherwise it might have some $\mathbb{R}$ factors); this is called a Cartan subgroup, or *maximal torus*. For $\mathsf{SU}(2)$, we can only diagonalize one of the generators, which we take to be J^3; $\mathsf{SU}(2)$ has rank one.

So far, this has just been a property of the algebra. In any representation R, we can then diagonalize the Cartan generators

$$H_i \left|\mu\right\rangle = \mu_i \left|\mu\right\rangle .$$

The eigenvalues μ_i are called *weights*. The weights are characteristic of R. You should think of them as a vector of *charges* under the Cartan generators. For $\mathsf{SU}(2)$, the eigenvalues of J^3 are called $m = -j, -j+1 \cdots j-1, j$ in the $2j+1$-dimensional representation labelled j.

Now what is the rest of the algebra? I claim that we can choose a basis for the rest of the algebra that diagonalizes the action of the Cartan generators – a basis of definite charge, where

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{3.18}$$

The eigenvalues α are called *roots*, and this is called the Cartan-Weyl basis for the Lie algebra. The set of roots is a property of the algebra. Notice that the complex conjugate of (3.18) is $[H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger$, so $E_\alpha^\dagger = E_{-\alpha}$ has charge $-\alpha_i$, and $-\alpha$ is also a root. The E_α are *not* hermitian. For $\mathsf{SU}(2)$, these are $J_\pm \equiv \frac{1}{\sqrt{2}}(J_1 \pm \mathbf{i}J_2)$, and (3.18) is $[J^3, J^\pm] = \pm J^\pm$, so $\alpha = \pm 1$. This equation says $J_\pm$ are eigenvectors of the adjoint action of J^3, *i.e.* of ad_{J^3}. Notice that $J_\pm^\dagger = J_\mp$. (The remaining relation is $[J_+, J_-] = J_3$.)

In everything that follows we'll use standard vector notation for these vectors of dimension r: $\mu^2 = \mu\cdot\mu = \sum_{i=1}^r \mu_i\mu_i$.

In the adjoint rep (3.13), it is perhaps more obvious that the α_i should be regarded as eigenvalues of the Cartan generators:

$$H_i \left|H_j\right\rangle = \left|[H_i, H_j]\right\rangle = 0$$

There are r states of weight zero. A rewriting of (3.18) in terms of states of the adjoint rep is:

$$H_i \left|E_\alpha\right\rangle = \left|[H_i, E_\alpha]\right\rangle = \alpha \left|E_\alpha\right\rangle$$

and the nonzero weights of the adjoint rep are the roots.

I claim that (3.18) is a good labelling – there is exactly *one* generator E_α for each root α. Demonstration later, around (3.22).

Raising and lowering for SU(2). Bear with me here as I remind us of some familiar things from SU(2); it will be crucially useful for general Lie groups momentarily. For SU(2), any finite-dimensional representation has a state of largest J^3, call it $|j, j\rangle$. This is called the *highest-weight state*. States with other J^3 eigenvalues can be obtained by acting with the lowering operator J^-:

$$J^- |j, j\rangle = N_j |j, j-1\rangle \quad \text{since} \quad J^3(J^- |j,j\rangle) = ([J^3, J^-] + J^- J^3) |j,j\rangle = (j-1) J^- |j,j\rangle .$$

In a finite-dimensional representation of SU(2), there is only so much lowering we can do until we must have $(J^-)^x |j, j\rangle = 0$ for some integer x, which is the dimension of the representation.

Notice that if we had other labels on our highest weight state $|j, j; \alpha\rangle$, we could orthogonalize them $\langle j, j; \beta | j, j; \alpha\rangle = \delta_{\alpha\beta}$, and the lowering preserves this orthogonality: $J^- |j, j; \alpha\rangle = N_j(\alpha) |j, j-1, \alpha\rangle$ and

$$\begin{aligned} \langle j, j-1; \beta | j, j-1; \alpha\rangle \, N_j^\star(\beta) N_j(\alpha) &= \langle j, j; \beta| \, J^+ J^- \, |j, j; \alpha\rangle \\ &= \langle j, j; \beta| \, [J^+, J^-] \, |j, j; \alpha\rangle = \langle j, j; \beta| \, J_3 \, |j, j; \alpha\rangle = j \delta_{\alpha\beta}. \end{aligned} \tag{3.19}$$

Since lowering by J^- reaches all the states in an irrep, there is no such extra label in an irrep.

By employing the following identities

$$J_+ J_- = J^2 - (J_3^2 - J_3), \quad J_- J_+ = J^2 - (J_3^2 + J_3) \text{ (recall that } J^2 \equiv \sum_i J_i^2 = c_2 \text{ is the Casimir)},$$

we can simplify the construction of all finite-dimensional irreps of SU(2). Since the rep is finite-dimensional, there must be a state with largest J_3 eigenvalue, call it $|j, j\rangle$. Then $0 = J^+ |j, j\rangle$, so

$$0 = \| \, J_+ |j, j\rangle \, \|^2 = \langle j, j| \, J_- J_+ \, |j, j\rangle = \langle j, j| \left(J^2 - J_3(J_3+1)\right) |j, j\rangle = \left(c_2(j) - j(j+1)\right) \langle j, j | j, j\rangle .$$

Assuming $|j,j\rangle$ is normalized (and not zero), we learn that $c_2(j) = j(j+1)$.

Now what happens if we act with the lowering operator on one of the states in this rep ($|j,m\rangle \sim J_-^{j-m}|j,j\rangle$, with $J_3|j,m\rangle = m|j,m\rangle$)? We know $J_-|j,m\rangle \propto |j,m-1\rangle$, and to figure out the norm, look at:

$$\| J_- |j,m\rangle \|^2 = \langle j,m| J_+ J_- |j,m\rangle = \langle j,m| \left(J^2 - J_3(J_3-1)\right) |j,m\rangle = (j(j+1) - m(m-1)) \langle j,m|j,m\rangle . \tag{3.20}$$

Therefore, if the states of this irrep are normalized, the three generators act on them as

$$J_3|j,m\rangle = m|j,m\rangle , \quad J_\pm |j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\, |j,m\pm 1\rangle .$$

If $j \in \mathbb{Z}/2$, then $J_-|j,-j\rangle = 0$ and the rep ends, with the $2j+1$ states with $m = \{-j, -j+1, \cdots j-1, j\}$. Otherwise, we can continue lowering past $m = -j$ and these states with $m < -j$ have negative norm (by (3.20)), so such a rep is neither finite-dimensional nor unitary.

Alternatively, we can do it (more directly but less efficiently) without using our knowledge of the quadratic Casimir. Let's normalize the state $|j-1\rangle = J^-|j\rangle / N_j$ (in the spin j rep):

$$1 = |\,|j-1\rangle\,|^2 = \langle j-1|j-1\rangle = \langle j| J^+ J^- |j\rangle / N_j^2 = \langle j| [J^+, J^-] |j\rangle / N_j^2 = j/N_j^2$$

so $N_j = \sqrt{j}$. We can go back up: $J^+|j-1\rangle = \frac{1}{N_j} J^+ J^- |j\rangle = \frac{1}{N_j}[J^+, J^-]|j\rangle = \frac{j}{N_j}|j\rangle = N_j|j\rangle$. Similarly,

$$J^-|j-k\rangle = N_{j-k}|j-k-1\rangle , \quad J^+|j-k-1\rangle = N_{j-k}|j-k\rangle .$$

The overlap factors N_{j-k}, satisfy a recursion relation:

$$N_{j-k}^2 = \langle j-k| J^+ J^- |j-k\rangle = \langle j-k| \underbrace{[J^+, J^-]}_{=J^3} |j-k\rangle + \langle j-k| J^- \underbrace{J^+|j-k\rangle}_{=|j-k+1\rangle N_{j-k+1}} = j-k+N_{j-k+1}^2 .$$

A nice solution of this recursion (from Georgi) is

$$\begin{array}{llcl} N_j^2 & & & = j \\ N_{j-1}^2 - & N_j^2 & & = j-1 \\ \vdots & \vdots & & \vdots \\ N_{j-k}^2 - & N_{j-k+1}^2 & & = j-k \\ \hline N_{j-k}^2 & & = (k+1)j - \frac{k(k+1)}{2} & = \frac{1}{2}(k+1)(2j-k) \end{array}$$

Therefore, writing $k = j-m$ in terms of the J^3 eigenvalue m, $N_m = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}$.

Lowering ends when $J^- |j-\ell\rangle = 0$ for some ℓ (where $|j-\ell+1\rangle \neq 0$), which happens when

$$0 = N_{j-\ell} = \frac{1}{\sqrt{2}}\sqrt{(2j-\ell)(\ell=1)} \quad \implies \quad \ell = 2j, \quad i.e. \ \boxed{j = \frac{\ell}{2} \in \mathbb{Z}/2}.$$

Thus, this is all the finite-dimensional irreps of $\mathsf{SU}(2)$:

$$\{|j,m\rangle\}, \quad J^3 |j,m\rangle = m\, |j,m\rangle .$$

where $j \in \mathbb{Z}/2$ labels the irrep, and $m \in \{-j, -j+1, \cdots j-1, j\}$ labels the state within the irrep. Thus the irreps of $\mathsf{SU}(2)$ have dimensions $1,2,3,4,5,\cdots$. These are: $(j=0)$ the singlet; $(j=\frac{1}{2})$ the fundamental/spinor/defining rep with generators $\vec{J} = \vec{\sigma}/2$; $(j=1)$ the adjoint/vector rep with generators $(J_i)_{jk} = \mathbf{i}\epsilon_{ijk}$; $(j=\frac{3}{2})$ spin 3/2 I don't have anything to say about; $(j=2)$ is the symmetric traceless tensor; $\cdots$ The integer-spin reps are also representations of $\mathsf{SO}(3)$; the half-integer spin reps are projective reps of $\mathsf{SO}(3)$.

Raising and lowering more generally. Now we are ready to do raising and lowering for the general Lie group. The generators $E_{\pm\alpha}$ raise and lower the weights, since

$$H_i\,(E_{\pm\alpha}|\mu\rangle) = ([H_i, E_{\pm\alpha}] + E_{\pm\alpha}\mu_i)\,|\mu\rangle = (\mu \pm \alpha)_i E_{\pm\alpha}\,|\mu\rangle .$$

So, in any rep, the roots are *differences* of the weights. Note that we don't have a notion yet of which is raising and which is lowering. Also, the state μ could in principle have other labels which I do not write.

The adjoint representation provides some useful notation to study the Lie algebra in general. $[E_\alpha, E_{-\alpha}] = \beta_i H_i$, and it remains to determine β_i. Consider

$$E_\alpha\,|E_{-\alpha}\rangle = |[E_\alpha, E_{-\alpha}]\rangle = |\beta_i H_i\rangle = \beta_i\,|H_i\rangle .$$

The second equality follows since the state has weight zero, so must be a linear combination of the Cartan generators. Using the inner product, we can determine

$$\beta_i = \frac{1}{\lambda}\langle H_i|\,E_\alpha\,|E_{-\alpha}\rangle = \frac{1}{\lambda}\mathrm{tr} H_i [E_\alpha, E_{-\alpha}] \overset{\text{IBP}}{=} \frac{1}{\lambda}\mathrm{tr} E_{-\alpha}\underbrace{[H_i, E_\alpha]}_{=\alpha_i E_\alpha} = \alpha_i \underbrace{\mathrm{tr} E_{-\alpha}E_\alpha/\lambda}_{=1} = \alpha_i.$$

Therefore $[E_\alpha, E_{-\alpha}] = \alpha \cdot H$. This is the analog of $[J^+, J^-] = J^3$ in $\mathsf{SU}(2)$.

So for any Lie algebra, $\mathfrak{g}$, each pair of roots $\pm\alpha$ produces an $\mathsf{SU}(2)$ subalgebra, which I'll call $\mathsf{su}(2)_\alpha$:

$$E^\pm \equiv |\alpha|^{-1} E_{\pm\alpha}, \qquad E_3 \equiv \frac{\alpha \cdot H}{\alpha^2}. \tag{3.21}$$

Reps of $\mathfrak{g}$ restrict to representations of $\mathsf{su}(2)_\alpha$. This simple fact is enormously constraining and will lead to a complete solution not only to the problem of the representation theory of semisimple Lie algebras, but to their classification. [End of Lecture 15]

Now we can prove that there is only one E_α for each root α: Suppose there were two, WLOG assume they are orthonormal $\langle E_\alpha | E'_\alpha \rangle = 0$. Acting with $E^- \propto E_{-\alpha}$ on $|E'_\alpha\rangle$ produces a state of weight zero:

$$E^- |E'_\alpha\rangle = \beta_i |H_i\rangle \,, \tag{3.22}$$

but by the same calculation as above, $\beta_i = \alpha_i \langle E'_\alpha | E_\alpha \rangle = 0$. This means that $|E'_\alpha\rangle$ is a lowest-weight state for the $\mathsf{SU}(2)$ subalgebra (3.21). On the other hand, $E_3 |E'_\alpha\rangle = \frac{\alpha \cdot H}{\alpha^2} |E'_\alpha\rangle = |E'_\alpha\rangle$, so this says that there is a lowest-weight state with J^3 eigenvalue $+1$. As we saw above, that doesn't happen – the lowest eigenvalue of J^3 in a rep of $\mathsf{SU}(2)$ is always ≤ 0. ■

Another general fact about the structure of the general Lie algebra is: if α is a root, then $k\alpha$ is a root only if $k = \pm 1$. Consider the action of the $\mathsf{SU}(2)$ subalgebra (3.21) on the state $|E_{k\alpha}\rangle$:

$$E_3 |E_{k\alpha}\rangle = k |E_{k\alpha}\rangle \quad \overset{\mathsf{SU}(2)}{\Longrightarrow} \quad k \in \mathbb{Z}/2$$

by our analysis of $\mathsf{SU}(2)$ reps above. If $k \in \mathbb{Z}$, then $E_{-\alpha}^{k-1} |E_{k\alpha}\rangle \propto |E'_\alpha\rangle$, another generator with the same root, which we just showed doesn't happen. If $k \in \mathbb{Z} + 1/2$, the state can be lowered to $\alpha/2$ and we can use the same argument starting with the $\mathsf{SU}(2)$ subalgebra built on $E_{\alpha/2}$ ($i.e.\alpha/2$ and α can't both be roots. ■

In general, for any rep of $\mathfrak{g}$, for any root α, the eigenvalues of the diagonal generator of the $\mathsf{SU}(2)$ subalgebra are

$$E_3 |\mu\rangle = \frac{\alpha \cdot \mu}{\alpha^2} |\mu\rangle \quad \overset{\mathsf{SU}(2)}{\Longrightarrow} \quad \frac{2\alpha \cdot \mu}{\alpha^2} \in \mathbb{Z}.$$

(Note that the state could have other labels which I don't write.) Now let's apply the representation theory of $\mathsf{su}(2)$ starting from this state. There must be some $r \in \mathbb{Z}_+$ (r is for 'raising') such that

$$\left(E^+\right)^r |\mu\rangle \neq 0, \quad \text{but} \quad \left(E^+\right)^{r+1} |\mu\rangle = 0.$$

The weight of this highest-weight state for the $\mathsf{su}(2)$ subalgebra is

$$\frac{\alpha \cdot (\mu + r\alpha)}{\alpha^2} = \frac{\alpha \cdot \mu}{\alpha^2} + r \equiv j, \tag{3.23}$$

where I've named the rep of $\mathsf{SU}(2)$ j. Similarly, there must be some $\ell \in \mathbb{Z}_+$ (ℓ is for 'lowering') such that

$$\left(E^-\right)^\ell |\mu\rangle \neq 0,, \quad \text{but} \quad \left(E^-\right)^{\ell+1} |\mu\rangle = 0.$$

The weight of this lowest-weight state is

$$\frac{\alpha\cdot(\mu-\ell\alpha)}{\alpha^2}=\frac{\alpha\cdot\mu}{\alpha^2}-\ell=-j. \tag{3.24}$$

Here we used the relationship $m_{\text{lowest}}=-m_{\text{highest}}=-j$ which holds in every $\mathsf{SU}(2)$ representation. Adding together (3.23) and (3.24) gives

$$\frac{2\alpha\cdot\mu}{\alpha^2}+r-\ell=0 \quad\implies \boxed{\frac{\alpha\cdot\mu}{\alpha^2}=-\frac{1}{2}(r-\ell)\in\mathbb{Z}/2,\quad\forall\alpha}. \tag{3.25}$$

Notice that subtracting (3.23) and (3.24) gives $r+\ell=2j$, so identifies the spin of the highest-spin representation that overlaps with the state $|\mu\rangle$. (There can be smaller-spin reps in there, too. For example, for the state $a\,|j_1,m_1\rangle+b\,|j_2,m_2\rangle$, from the definition above we can see that $j=\max(j_1,j_2)$.)[33]

(3.25) is a very powerful relation and we are going to use it *all the time* below. This notation with rs and ℓs isn't great, because we have to remember that these are the distances from $|\mu\rangle$ to the top and bottom of its $\mathsf{su}(2)_\alpha$ rep – they depend on a choice of weight μ and root α.

We are going to use these rs and ℓs a lot. Just so there is no confusion, here are their values for the states of the spin $j=3/2$ representation of $\mathsf{SU}(2)$:

m	r	ℓ	$r+\ell=2j$	$(\ell-r)/2=m$
$+\frac{3}{2}$	0	3	3	$+\frac{3}{2}$
$+\frac{1}{2}$	1	2	3	$+\frac{1}{2}$
$-\frac{1}{2}$	2	1	3	$-\frac{1}{2}$
$-\frac{3}{2}$	3	0	3	$-\frac{3}{2}$

To see the power of (3.25), let's apply it to the case of the weights of the adjoint representation, namely the roots, $\mu=\beta$:

$$\frac{\alpha\cdot\beta}{\alpha^2}=-\frac{1}{2}(r-\ell). \tag{3.26}$$

Since (3.25) applies for every root, we can also write it for the $\mathsf{su}(2)$ algebra built on $E_{\pm\beta}$, to the weight $\mu=\alpha$:

$$\frac{\beta\cdot\alpha}{\beta^2}=-\frac{1}{2}(r'-\ell'). \tag{3.27}$$

Multiplying (3.26) and (3.27) gives

$$\frac{\mathbb{Z}}{4}\ni\frac{(r-\ell)(r'-\ell')}{4}=\frac{(\alpha\cdot\beta)^2}{\alpha^2\beta^2}=\cos^2\theta_{\alpha\beta}\le 1$$

[33]What I am calling r and ℓ Georgi calls p and q, but with that notation I can never remember which one is up and which is down.

where $\theta_{\alpha\beta}$ is the angle between the two root vectors. This is an extremely strong constraint: the angle between any two root vectors falls into one of five classes:

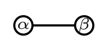
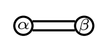
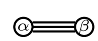
$(r-\ell)(r'-\ell')$	$\lvert\cos\theta_{\alpha\beta}\rvert$	$\theta_{\alpha\beta}$
0	0	$\pi/2$
1	$\frac{1}{2}$	$\frac{2\pi}{3}$ or $\frac{\pi}{3}$
2	$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$ or $\frac{3\pi}{4}$
3	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$ or $\frac{5\pi}{6}$
4	1	0 or π

The last case where $\cos^2\theta = 1$ says $\theta = 0, \pi$: $\theta = 0$ is ruled out by uniqueness, and we already know that both α and $-\alpha$ are roots. So it says that the only angles that can appear are the ones whose trig functions are easy to remember! (Maybe this is the reason that middle school students are forced to memorize just this particular set of values of trig functions in precalculus class.) This goes a long way toward classifying Lie algebras.

Finally, taking the ratio of (3.26) and (3.27) gives

$$\frac{\alpha^2}{\beta^2} = \frac{r'-\ell'}{r-\ell}$$

– the ratios of the lengths of roots (something independent of the annoying normalization choices) is correlated with the angles between them. So we can amplify our table as follows (I assume WLOG that α is longer than β, $\alpha^2 \geq \beta^2$):

$(r-\ell)$	$(r'-\ell')$	$\lvert\cos\theta_{\alpha\beta}\rvert$	$\theta_{\alpha\beta}$	$\frac{\alpha^2}{\beta^2}$	Dynkin notation
0	0	0	$\pi/2$	indeterminate	
1	1	$\frac{1}{2}$	$\frac{\pi}{3}$ or $\frac{2\pi}{3}$	1	
1	2	$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$ or $\frac{3\pi}{4}$	2	
1	3	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$ or $\frac{5\pi}{6}$	3	

Notice that if either $r-\ell$ or $r'-\ell'$ vanish, they must both vanish, in which case the roots are just orthogonal, and their relative lengths are not fixed. This is good because we are allowed to take tensor products, such as $\mathsf{SU}(2) \times \mathsf{SU}(2) = \exp(\mathsf{su}(2) \oplus \mathsf{su}(2))$ where the generators don't care about each other[34].

An example: $\mathsf{su}(3)$. A basis of generators of the Lie algebra $\mathsf{su}(3)$ (in the 3-dimensional fundamental representation) is made up of eight traceless hermitian matrices. (This is because $U = e^{\mathbf{i}H}$ is unitary if H is unitary and has $\det U = 1$ if H is

[34] Since $G = e^{\mathfrak{g}}$ the tensor product of Lie groups actually means direct *sum* of their Lie algebras, that is, the set of generators of the product group is the union of the generators of the factors.

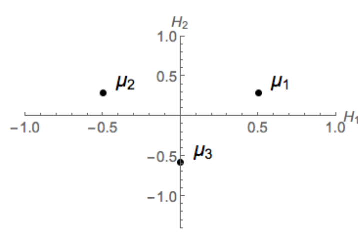
traceless.) Here is a set of them:

$$\lambda_1 = X_{12} = \begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}, \quad \lambda_2 = Y_{12} = \begin{pmatrix} & -\mathbf{i} & \\ \mathbf{i} & & \\ & & \end{pmatrix}, \quad \lambda_3 = Z_{12} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix},$$

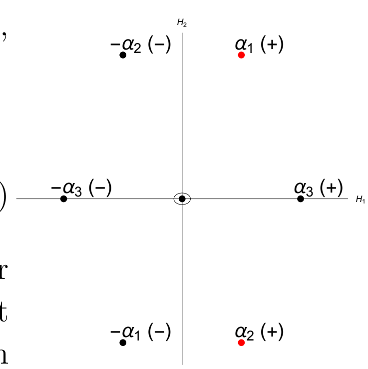
$$\lambda_4 = X_{13} = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}, \quad \lambda_5 = Y_{13} = \begin{pmatrix} & & -\mathbf{i} \\ & & \\ \mathbf{i} & & \end{pmatrix},$$

$$\lambda_6 = X_{23} = \begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix}, \quad \lambda_7 = Y_{23} = \begin{pmatrix} & & \\ & & -\mathbf{i} \\ & \mathbf{i} & \end{pmatrix}, \quad \lambda_8 = (Z_{13} + Z_{23})/\sqrt{3} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} /\sqrt{3} \; .$$

These are called the Gell-Mann matrices; you can see that they generalize the Pauli matrices, since the first three are just the Pauli matrices padded with zeros. The names X_{12}, Y_{12} etc are meant to emphasize this point. They are hermitian. You can see that λ_4 and λ_5 will also generate an $\mathsf{su}(2)$ algebra, where the third generator is $[\lambda_4, \lambda_5] = 2\mathbf{i}\lambda_{45} = \mathbf{i}(\lambda_3 + \sqrt{3}\lambda_8)$, a linear combination of our two diagonal guys. Similarly for λ_6 and λ_7. From this information we can determine the structure constants f_{ABC}. Note that although we are computing them using the matrices in a particular representation, the structure constants of $\mathsf{su}(3)$ are forever (modulo all the annoying normalization issues).



You can see that two of these guys are diagonal, and no more can be diagonalized. So we take $H_1 = \lambda_3/2, H_2 = \lambda_8/2$. (The choice of normalization is an annoying thing. With this normalization, $\mathrm{tr} T^A T^B = \frac{1}{2}\delta^{AB}$. Since they are already diagonal, we can read off the weights. The eigenvector $(1,0,0)^T$ has eigenvalues $\mu^1 = (1/2, 1/\sqrt{12})$ under (H_1, H_2). The other two are $\mu^2 = (-1/2, 1/\sqrt{12})$ and $\mu^3 = (0, -1/\sqrt{3})$.



The weights above are a property of the **3** representation. The roots, which are a more universal property of the algebra, are the differences of the weights:

$$\alpha^3 = \mu^1 - \mu^2 = (1,0), \quad \alpha^1 = \mu^3 - \mu^2 = (1, -\sqrt{3})/2,$$
$$\alpha^2 = \mu^1 - \mu^3 = (1, \sqrt{3})/2 \tag{3.28}$$

and their negatives. Notice that the roots form a regular hexagon, so the angles between them indeed fit into the list of possibilities we found. The dot and the circle at the origin represent the weights of the two Cartan generators.

Now we can construct the rest of the Cartan-Weyl basis for $\mathsf{su}(3)$:

$$E_{\pm\alpha^3} = (\lambda_1 \pm \mathbf{i}\lambda_2)/\sqrt{8}, \quad E_{\pm\alpha^2} = (\lambda_4 \pm \mathbf{i}\lambda_5)/\sqrt{8}, \quad E_{\pm\alpha^1} = (\lambda_6 \mp \mathbf{i}\lambda_7)/\sqrt{8}.$$

I chose these combinations to satisfy $[H_i, E_{\pm\alpha^a}] = \pm\alpha_i^a E_{\pm\alpha^a}$, with α^a as in (3.28). These matrices have only one nonzero (off-diagonal) entry, so they only do raising or lowering. You can check that $[E_{\alpha^1}, E_{\alpha^2} = E_{\alpha^3}$, but $E_{\alpha^2}, E_{\alpha^3}] = 0 = [E_{\alpha^3}, E_{\alpha^1}]$. Notice that the first equation is consistent with charge conservation: $\alpha^3 = \alpha_1 + \alpha_2$.

Positive and simple roots. Now we will decide who raises and who lowers (this is not a big deal, but we have to choose a convention). Choose an order for the Cartan generators (we've already done so by calling them H_1 and H_2). We declare the following very arbitrary convention: a root $\alpha > 0$ if its first nonzero entry is positive. (Note that Zee chooses a different convention in section VI.2.) So with this choice $\alpha_{a=1,2,3} > 0$ and $-\alpha_{a=1,2,3} < 0$, so E_{α^a} are raising operators and $E_{-\alpha^a}$ are lowering operators. The signs of the roots are indicated in the figure above. Now we can also say $\alpha > \beta$ if $\alpha - \beta > 0$. We can use the same convention for weights in general, and therefore we can say which is the "highest weight" – the one that is bigger than all the others (with our choice of convention). (The choice of convention will not matter in the end, I promise.)

A root is *simple* if it can't be written as a sum of two positive roots with positive coefficients. So here $\alpha^1 = (1, \sqrt{3})/2$ and $\alpha^2 = (1, -\sqrt{3})/2$ are simple, while $\alpha^3 = (1, 0) = \alpha^1 + \alpha^2$ is not. The simple roots (with our convention) are in red in the root diagram for $\mathsf{su}(3)$ above. Other conventions will lead to different simple roots (but always the same number of them for a given Lie algebra).

Building an irrep. Now here's one reward for all these definitions (the other will be an understanding of all semisimple Lie algebras from just their simple roots). Consider a highest weight state $|\phi\rangle$ in some representation, with weight μ_ϕ (that is, $H_i |\phi\rangle = (\mu_\phi)_i |\phi\rangle$). Then for any set of simple (hence positive) roots $\{\alpha_k\}$,

$$|\{\alpha_k\}\rangle_\phi \equiv E_{-\alpha_1} E_{-\alpha_2} \cdots |\phi\rangle$$

is another state in the representation – unless it's zero because we lowered too far. So I want you to think of $|\phi\rangle$ as like the Fock vacuum $|0\rangle$. Just like $|0\rangle$ is annihilated by all the annihilation operators $\mathbf{a}_\alpha |0\rangle = 0$, the statement that $|\phi\rangle$ is a highest-weight state means it is annihilated by all the raising operators $E_{+\alpha} |\phi\rangle = 0$ for any positive root α. And the lowering operators $E_{-\alpha}$ are just like the creation operators $\mathbf{a}_\alpha^\dagger |0\rangle = |\alpha\rangle$. Using the Lie algebra, the state $|\{\alpha_k\}\rangle_\phi$ is an eigenstate of the Cartan generators and has weight $\mu_\phi - \sum_k \alpha_k$.

To build a whole irrep, then, we just act with *all* sets of simple roots: $R_\phi = \text{span}\{|\{\alpha\}\rangle_\phi\}$. The inner product between two such states is

$$\langle\{\beta\}|\{\alpha\}\rangle = \langle\phi| E^\dagger_{-\beta_n} \cdots E^\dagger_{-\beta_1} E_{-\alpha_1} E_{-\alpha_2} \cdots |\phi\rangle = \langle\phi| E_{\beta_n} \cdots E_{\beta_1} E_{-\alpha_1} E_{-\alpha_2} \cdots |\phi\rangle \ .$$

In this expression the E_βs are trying to get at the vacuum on the right to annihilate it, and the only thing in their way are the $E_{-\alpha}$s. So just we have to use the commutators $[E_\beta, E_{-\alpha}] = N_{\beta,-\alpha} E_{\beta-\alpha}$ to move the annihilation operators to the right. It is tedious but systematic and finite.

Note that there can be more than one linearly-independent state with the same weight, in which case one must do Gram-Schmidt to obtain an orthonormal basis for the irrep.

Notice that this construction of the states in an irrep gives, if we want it, an explicit construction of the representation matrices for the Lie algebra (and hence the group).

[End of Lecture 16]

Now, which ϕ are allowed for finite-dimensional representations? Just like for $\mathsf{su}(2)$, if we start in a random place we'll just keep lowering into negative-norm states. We have $E_\alpha |\phi\rangle = 0$ for all positive roots α, which means $\phi + \alpha$ is not a weight for any positive root α (it is enough to check the simple roots). Applying the master equation (3.25), we therefore have $r^a = 0$ for each positive root α^a, *i.e.*:

$$\frac{2\alpha^a \cdot \phi}{(\alpha^a)^2} = \ell^a - r^a \stackrel{!}{=} \ell^a \geq 0. \tag{3.29}$$

This is the condition for ϕ to be a highest-weight state: $\frac{2\alpha^a \cdot \phi}{(\alpha^a)^2}$ must be a non-negative integer for each a. As long as ℓ^a are integers, we'll eventually reach zero by lowering.

Now notice that (3.29) is a linear condition on ϕ – the sum of two solutions is also a solution. Since the α^a for the simple roots are linearly independent, possible highest-weight states are in one-to-one correspondence with choices of ℓ^a. A basis for such highest weight vectors is $\vec{\mu}^b$ satisfying

$$\frac{2\vec{\alpha}^a \cdot \vec{\mu}^b}{(\alpha^a)^2} \stackrel{!}{=} \delta^{ab}. \tag{3.30}$$

The $\vec{\mu}^b$ are called *fundamental weights*, and any highest weight is

$$\vec{\phi} = \sum_{b=1}^{r} \ell(\phi)^b \vec{\mu}^b$$

where $\ell(\phi)^b$ are called the *Dynkin indices* of the representation. (These r integers are equivalent to the r Casimirs I mentioned in §3.3.)

If we define $\Lambda_R \equiv \{n_a \alpha^a, n_a \in \mathbb{Z}\}$ (where α^a are the simple roots) to be the *root lattice* (where all the roots live), then the definition (3.30) says that the fundamental weights form a basis for the dual lattice:

$$\Lambda_R^\star = \{m_b \mu^b, m_b \in \mathbb{Z}\} \equiv \Lambda_W$$

and this is the *weight lattice*, the space in which all weight vectors of finite-dimensional reps live.

Building irreps of $\mathsf{su}(3)$. For example, for $\mathsf{su}(3)$, our simple roots are $\alpha^1 = (1, \sqrt{3})/2, \alpha^2 = (1, -\sqrt{3})/2$ (which we've chosen to be unit vectors). The fundamental weights μ^a satisfy $\mu^a \cdot \alpha^b = \delta^{ab}$. A little algebra shows that $\mu^{1/2} = \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{6}\right)$.

Let's build some irreps of $\mathsf{su}(3)$, starting with $R_{\phi=\mu^1} \equiv R_{(1,0)}$. Recall that this means $\frac{2\alpha^a\cdot\phi}{(\alpha^a)^2} = \ell^a - 0$, where ℓ^a is the number of times we can lower with $E_{-\alpha^a}$ before reaching the lowest-weight state. The fact that $\ell^2 = 0$ means $E_{-\alpha^2}\left|\mu^1\right\rangle = 0$ – the highest weight state is already a lowest-weight state for $\mathsf{su}(2)_{\alpha^2}$. So the only thing we can do is lower with α^1, producing a state with weight $\mu^1 - \alpha^1$. What is its $\ell^a - r^a$?

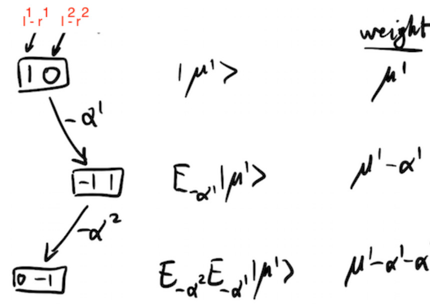
$$\frac{2\alpha^a \cdot (\mu^1 - \alpha^1)}{(\alpha^a)^2} = (1, 0) - (2, -1) = (-1, 1).$$

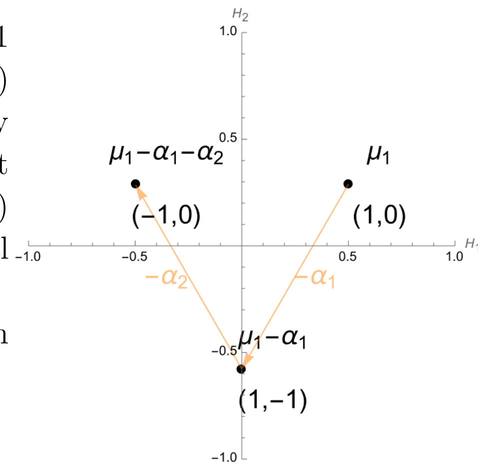
Before going on, let's record what happens to $\ell^a - r^a$ when we lower with α^b; it means we subtract

$$A_{ab} \equiv \frac{2\alpha^a \cdot \alpha^b}{(\alpha^a)^2} \overset{\mathsf{su}(3)}{=} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}_{ba}.$$

This object on the LHS, in general, is called the *Cartan matrix* of the Lie group, and encodes the inner products of the simple roots.

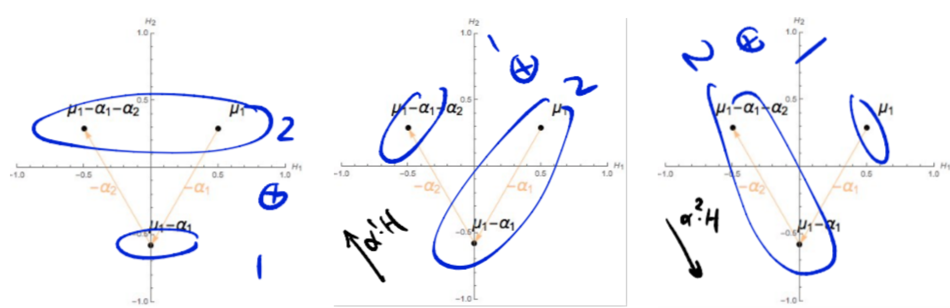
So we see that the new state $E_{-\alpha^1}|\mu^1\rangle$ is the highest weight of a doublet of $\mathsf{su}(2)_{\alpha^2}$. To be more explicit: we know that it has $r^2 = 0$, since there is no state of weight $\mu^1 - \alpha^1 + \alpha^2$ ($\alpha^2 - \alpha^1$ is not a root); therefore it has $\ell^2 - r^2 = 1 \implies \ell^2 = 1$, and it can be lowered with $E_{-\alpha^2}$ exactly once. (Notice that we also know it has $r^1 = 1$ (since we got it by lowering the highest-weight state once) and hence it has $\ell^1 - r^1 = -1$ and hence $\ell^1 = 0$ – it can't be lowered with $E_{-\alpha^1}$.) So altogether we get three states. We know they are linearly independent since they have different weights (which are the eigenvalues of the (hermitian) Cartan generators). This is the $\mathbf{3}$, the fundamental representation of $\mathsf{SU}(3)$.
In the diagram at right, I've labelled the states in the weight diagram by their values of $\ell^a - r^a$.





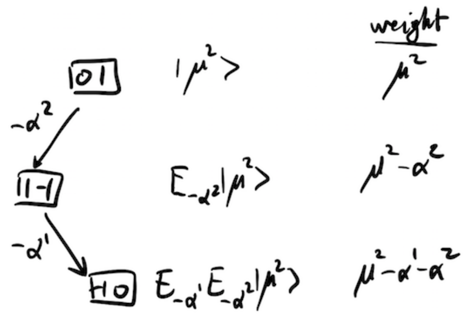
Some properties of the Cartan matrix: It is $r \times r$. A_{ab} is not symmetric in general because of the normalization by $(\alpha^a)^2$ – if there are roots of different lengths, it is not symmetric. $A_{ab} = l^a - r^a$ is twice the eigenvalue of $J^3_{\alpha^a} \equiv \frac{\alpha^a \cdot H}{(\alpha^a)^2}$ for the $\mathsf{SU}(2)_{\alpha^a}$ acting on $|\alpha^b\rangle$. Hence the entries are all integers. Its diagonal entries are all 2 (simple roots have $J_3 = 1$ because they transform in the adjoint of their own $\mathsf{SU}(2)$). The off-diagonal entries are $0, -1, -2$ or -3, and determine the angles between and relative lengths of the simple roots. The point is that A_{ba} encodes how does $|\alpha^b\rangle$ fit into reps of $\mathsf{SU}(2)_{\alpha^a}$. It is an invertible matrix since the simple roots are linearly independent and there are r of them.

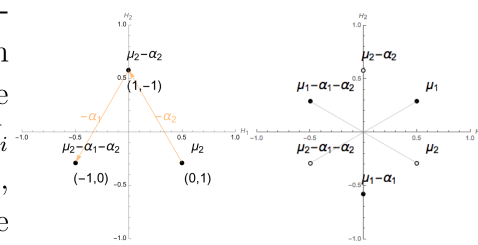
To see how the $\mathbf{3}$ of $\mathsf{SU}(3)$ decomposes under various $\mathsf{SU}(2)_\alpha$ subgroups that share a Cartan generator, we just slice the weight diagram along the corresponding axis $\alpha \cdot H$:



The three slicings correspond respectively to $\mathsf{SU}(2)_{12}$ (with generators $\lambda_{1,2,3}/2$, including the Cartan generator $H_1 = \lambda_3/2$), $\mathsf{SU}(2)_{\alpha^1}$, $\mathsf{SU}(2)_{\alpha^2}$. You can see that they all give the same decomposition $\mathbf{3} = \mathbf{2} \oplus \mathbf{1}$. As you will see on the homework, not every $\mathsf{SU}(2)$ subgroup of $\mathsf{SU}(3)$ gives the same decomposition.

What about $R_{\mu^2} \equiv R_{(0,1)}$? The same logic again produces three states, with weights which are minus the weights of the rep $R_{(1,0)}$. I claim that the representation whose weights are minus those of R is $\bar{R}$, the complex conjugate rep (as you can see in the bottom figure at right which shows them both together). $\bar{R}$ is defined in the same way as for finite groups, as the rep whose operators are the complex conjugates. In terms of the Lie algebra, this means if T_A generate R then $-T_A^\star$ generate $\bar{R}$. (The extra minus sign comes from the factor of $\mathbf{i}$ we put in to make the Lie algebra generators hermitian.) The Cartan generators in the rep $\bar{R}$ are $-H_i^\star$. But H_i are hermitian, so $H^\star$ has the same eigenvalues as H, and those of $-H_i^\star$ are indeed $-\mu_i$. In particular, the highest weight of $\bar{R}$ is minus the lowest weight of R.

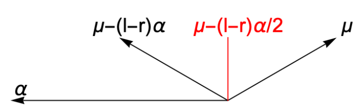




In general for $\mathsf{SU}(3)$, then, $R_{n\mu^1+m\mu^2} \equiv R_{(n,m)} = \bar{R}_{(m,n)}$, and the reps with $n = m$ are real.

Weyl group. Recall that $\mathsf{SU}(2)$ reps are symmetrical under $m \to -m$. This is true for each $\mathsf{SU}(2)_\alpha$ associated to each root, with $J_z^{(\alpha)} = \frac{\alpha \cdot H}{\alpha^2}$, so that $J_z^{(\alpha)} |\mu\rangle = \frac{\alpha\cdot\mu}{\alpha^2} |\mu\rangle$. The state with eigenvalue $-m$ in the same $\mathsf{SU}(2)_\alpha$ irrep, then, has weight $\mu - (\ell - r)\alpha$, where $\ell - r \equiv \frac{2\alpha\cdot\mu}{\alpha^2}$ as usual.

A diagram helps. This is a reflection in the plane perpendicular to α. So for any pair μ, α, this *Weyl reflection* maps the set of weights to themselves, and preserves the roots. A composition of reflections in non-parallel planes is a rotation.



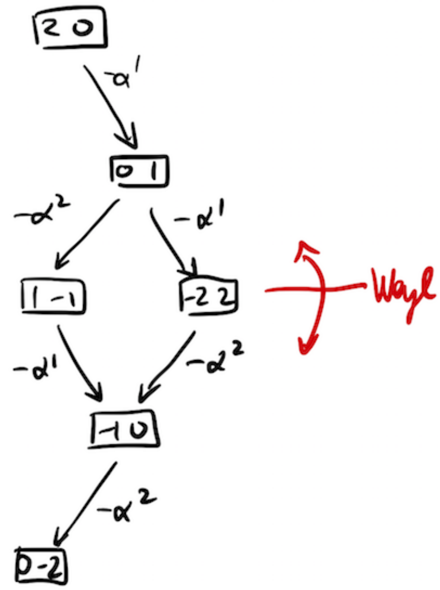
This is worth writing bigger: if μ is a weight and α is a root, then

$$\mathcal{W}_\alpha : \mu \mapsto \mu - \frac{2\alpha \cdot \mu}{\alpha^2}\alpha$$

is also a weight in the same irrep. In this way, just knowing a single weight and the simple roots we can generate a whole bunch of weights. Notice that $\mathcal{W}_\alpha(\mu)$ is not changed by rescaling $\alpha \to \lambda\alpha$ for $\lambda \in \mathbb{R}^\star$ (positive or negative, but not zero). This symmetry of the weight space explains the hexagons and triangles that appear for

SU(3). In general, when constructing a weight diagram it is a good idea to apply all Weyl reflections first – it's much easier than the process where we keep track of the $\mathsf{SU}(2)_\alpha$ irreps at each step.

Now let's be ambitious and try $R_{2\mu^1} = R_{(2,0)}$. $2\mu^1 = (1, 1/\sqrt{3})$. How did I know that the same state was obtained by $E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_1}\left|2\mu^1\right\rangle \propto E_{-\alpha_2}E_{-\alpha_1}E_{-\alpha_1}\left|2\mu^1\right\rangle$? One way to know is by Weyl reflection with α^3 (which acts by reflection in the y axis of the weight diagram), since we know that there is only one state with weight $2\mu^1 - \alpha^1$. All the other states have different weights and so are clearly orthogonal. This gives the $\mathbf{6} = \mathrm{Sym}^2\mathbf{3}$.

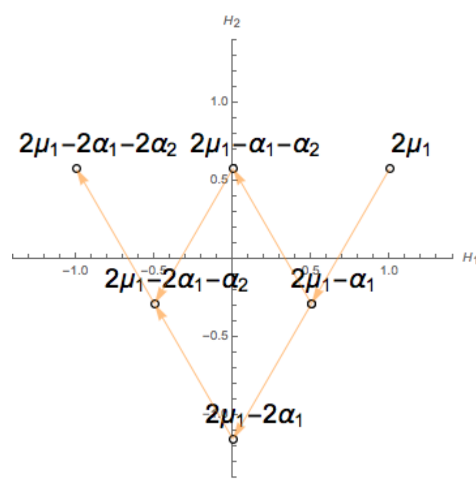


We can also see directly that the states with weight $2\mu_1 - 2\alpha_1 - \alpha_2$ reached by the two routes are linearly dependent. Start from $|A\rangle \equiv E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_1}\left|2\mu^1\right\rangle$. $E_{-\alpha_1}E_{-\alpha_2} = E_{-\alpha_2}E_{-\alpha_1} + [E_{-\alpha_1}, E_{-\alpha_2}]$. The commutator is not zero because $\alpha_1 + \alpha_2$ is a root. But it commutes with $E_{-\alpha_1}$ because $2\alpha_1 + \alpha_2$ is not a root. Therefore

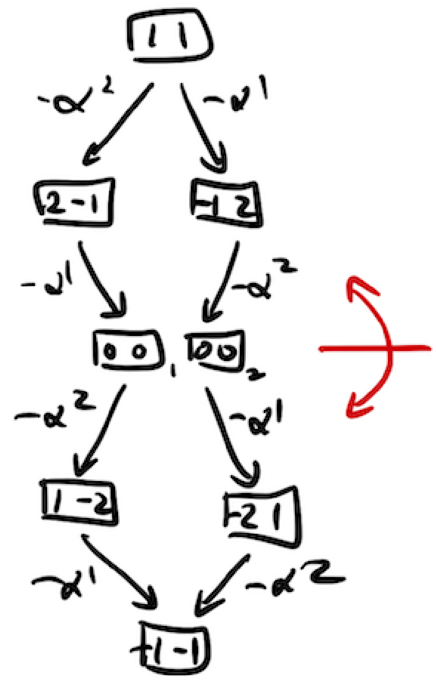
$$|A\rangle = E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_1}\left|2\mu^1\right\rangle = |B\rangle + [E_{-\alpha_1}, E_{-\alpha_2}]E_{-\alpha_1}\left|2\mu^1\right\rangle = |B\rangle + E_{-\alpha_1}[E_{-\alpha_1}, E_{-\alpha_2}]\left|2\mu^1\right\rangle$$

where $|B\rangle \equiv E_{-\alpha_2}E_{-\alpha_1}E_{-\alpha_1}\left|2\mu^1\right\rangle$ is the other state in question. But we know that $E_{-\alpha_2}\left|2\mu^1\right\rangle = 0$ because $\left|2\mu^1\right\rangle$ is a highest weight state for $\mathsf{SU}(2)_{\alpha_2}$ with weight zero. Therefore

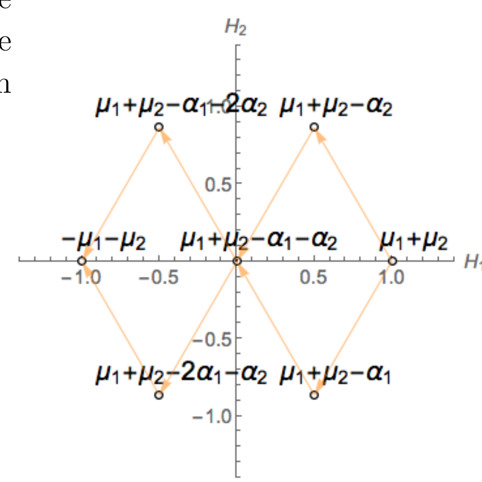
$$|A\rangle = |B\rangle + E_{-\alpha_1}E_{-\alpha_1}E_{-\alpha_2}\left|2\mu^1\right\rangle = 2\,|B\rangle\,.$$



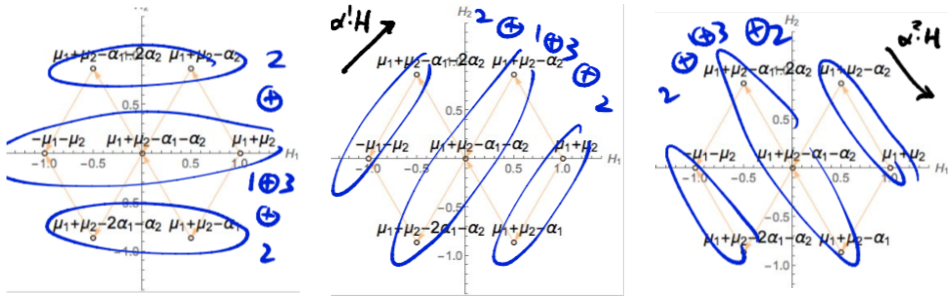
[End of Lecture 17]



What about $R_{\mu^1+\mu^2} = R_{(1,1)}$? Here something funny happens. Observe that $\mu^1 + \mu^2 = \alpha^1 + \alpha^2$ is a root vector. This is the adjoint representation, whose weights are the roots. In this case we know that the states with zero weight must be two-fold degenerate, because there are two Cartan generators, $|H_1\rangle, |H_2\rangle$. You can also check explicitly that $E_{-\alpha^1}E_{-\alpha^2}|\mu^1+\mu^2\rangle$ and $E_{-\alpha^2}E_{-\alpha^1}|\mu^1+\mu^2\rangle$ are linearly independent.
For the adjoint rep, we know a priori that there can be no degeneracy at any other weight, because we proved that there is a unique generator for each nonzero root.



To see how the adjoint of $\mathsf{SU}(3)$ decomposes under various $\mathsf{SU}(2)_\alpha$ subgroups that share a Cartan generator, we just slice the weight diagram along the corresponding axis $\alpha \cdot H$:



$$(3.31)$$

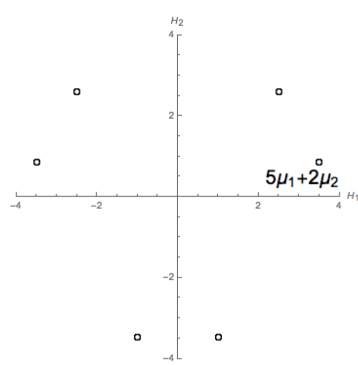
So the irreps of $\mathsf{SU}(3)$ are labelled by a pair of non-negative integers. A way to arrange this information is as a Young diagram with at most two rows: for each $(1,0)$ we put a column with one box and for each $(0,1)$ we put a column with two boxes (and put all the 2-box columns on the left). So for example,

$$\mathbf{3} = (1,0) = \square,\ \ \bar{\mathbf{3}} = (0,1) = \begin{smallmatrix}\square\\\square\end{smallmatrix},\ \ \mathbf{6} = (2,0) = \square\square,\ \ \bar{\mathbf{6}} = (0,2) = \begin{smallmatrix}\square\square\\\square\square\end{smallmatrix},\ \ \mathbf{8} = (1,1) = \begin{smallmatrix}\square\square\\\square\ \end{smallmatrix},\ \ \mathbf{10} = (3,0) = \square\square\square,\ \ \cdots$$

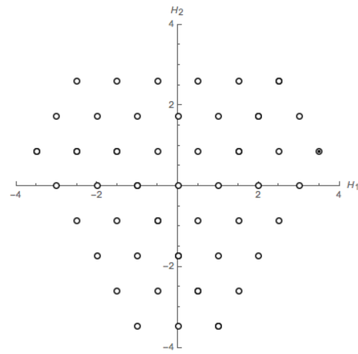
I claim that this notation is consistent with the symmetrization properties we ascribed to Young diagrams earlier (rows are symmetrized and columns are antisymmetrized). In particular I claim that $R_{2\mu^1} = \mathbf{6} = \text{Sym}^2\mathbf{3}$. What is the highest weight of $\text{Sym}^2\mathbf{3}$? The Cartan generators act on the tensor product as $H_i \otimes \mathbb{1} + \mathbb{1} \otimes H_i$, so their eigenvalues add. The state $|\mu^1\rangle \otimes |\mu^1\rangle$ is symmetric (so it appears in $\text{Sym}^2\mathbf{3}$) and is the highest weight, with weight $2\mu^1$. This argument works to show that $\text{Sym}^n\mathbf{3} = \underbrace{\square\square\square\square\square}_{n}$ for any n.

Similarly, I claim that $\Lambda^2\mathbf{3} = \bar{\mathbf{3}}$. (Notice that the dimensions work out.) The state $|\mu^1\rangle \otimes |\mu^1\rangle \in \mathbf{3} \otimes \mathbf{3}$ is projected out by the antisymmetrization, so the highest weight is $\mu^1 + (\mu^1 - \alpha^1)$ where $\mu^1 - \alpha^1$ is the next-highest-weight. You can check that this is $\mu^1 - \alpha^1 = \mu^2$, so $\Lambda^2\mathbf{3} = R_{\mu^2} = \bar{\mathbf{3}}$.

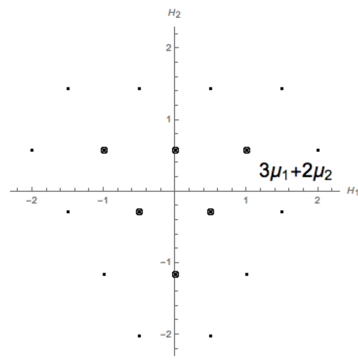
One piece of advice about the larger irreps of $\mathsf{SU}(3)$: if you apply the Weyl group to the highest weight, you get the boundary of the weight diagram – all possible highest weights that would arise from different conventions, and hence also the lowest weights for each convention. For example, for $R_{(5,2)}$, this gives the figure at right.



The full weight diagram can then be filled in by acting with the lowering operators and stopping when you get to the boundary determined by the Weyl orbit above. The only catch is that some of the roots have a multiplicity larger than one. As you move in from the outer layer, the multiplicity increases by one with each step; the triangular regions in the middle all have the same multiplicity.



Here is a weight diagram for $R_{(3,1)}$ which indicates the multiplicity. I made it just by plotting the Weyl orbits of the highest weight μ, and of $\mu-\alpha_1, \mu-(\alpha_1+\alpha_2), \mu-2(\alpha_1+\alpha_2)$.



3.5 Cartan-Weyl method, continued: everything from the simple roots

[Georgi, §8] Suppose someone hands us the simple roots $\{\alpha\}$ of some Lie algebra. (That someone will be Dynkin.) There are r of them, and they are linearly independent.

All roots from the simple roots. Any positive root is of the form $\phi_k = \sum_\alpha k_\alpha \alpha$, $k_\alpha \geq 0$, and where $k \equiv \sum_\alpha k_\alpha$ is a measure of how composite the root is. But which objects of this form are actually roots?

We can answer this by induction on k. For $k = 1$, any ϕ_1 is just a simple root. Suppose we know the roots ϕ_k for $k \leq \kappa$. Then consider $E_\alpha |\phi_\kappa\rangle$ (or equivalently $[E_\alpha, E_{\phi_\kappa}]$); this would give a root $\phi_{\kappa+1} = \phi_\kappa + \alpha$. As an element of a rep of $\mathsf{SU}(2)_\alpha$, the state $|\phi_\kappa\rangle$ has $\frac{2\alpha\cdot\phi_\kappa}{\alpha^2} = \ell - r$ (recall that this is twice the $J^3_\alpha \equiv \frac{\alpha\cdot H}{\alpha^2}$ eigenvalue of the state). This would be useless without other information, but we independently know ℓ, because we know how to make ϕ_κ from $\phi_{k<\kappa}$. Therefore we know r. If $r > 0$, then acting on it with E_α gives another state in the $\mathsf{SU}(2)_\alpha$ irrep, and hence $\phi_\kappa + \alpha$ is also a root.

For example, consider $\kappa = 1$. $\phi_1 = \beta$ is a simple root. We know that the state $|\beta\rangle$ has $\ell = 0$ with respect to any $\mathsf{SU}(2)$ by definition of simple root: β is not the sum of any other positive roots. So the state $|\beta\rangle$ has $\frac{2\alpha\cdot\phi_1}{\alpha^2} = \frac{2\alpha\cdot\beta}{\alpha^2} = \ell - r = -r$. So $r = 0$ if

$\alpha \cdot \beta = 0$, which means that in this case $\alpha + \beta$ is not a root. Otherwise $\alpha + \beta$ is a root.

This procedure gets all the roots because any $\phi_{\kappa+1} = \phi_\kappa + \alpha$ for some simple root α. If it were not so, $E_{-\alpha} |\phi_{\kappa+1}\rangle = 0$ for all α (since otherwise this would produce a state we could lower by α to get $|\phi_\kappa\rangle$), which means $|\phi_{\kappa+1}\rangle$ would be a highest-weight state for every $\mathsf{SU}(2)_\alpha$. But that means $\frac{\alpha \cdot \phi_{\kappa+1}}{\alpha^2} \leq 0, \forall \alpha$ which means $\phi_{\kappa+1}^2 \leq 0$. ■

For $\mathsf{SU}(3)$, $\alpha^{1,2} = (1, \pm\sqrt{3})/2$, and the Cartan matrix is $A_{ba} = \frac{2\alpha^a \cdot \alpha^b}{(\alpha^a)^2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}_{ba}$. The -1s on the off-diagonal mean that $r = 1$ for $|\alpha^2\rangle$ with respect to $\mathsf{SU}(2)_{\alpha^1}$ (or $|\alpha^1\rangle$ with respect to $\mathsf{SU}(2)_{\alpha^2}$), so we conclude that $\alpha^1 + \alpha^2$ is a root. (And we proved that there is one generator for each nonzero root, there is no question of multiplicity for the adjoint.) So $|\alpha^1 + \alpha^2\rangle$ has $\ell = 1$ for *e.g.* $\mathsf{SU}(2)_{\alpha^1}$, and $\ell - r = A_{11} + A_{12} = 2 - 1 = 1$. We conclude it has $r = 0$, and therefore $2\alpha^1 + \alpha^2$ is *not* a root. The same happens for $\alpha^1 + 2\alpha^2$. So that's all the positive roots, as we found earlier starting from the Gell-Mann matrices.

The whole algebra from the simple roots. Recall that $\frac{\alpha \cdot \mu}{\alpha^2} + r = j$, $\frac{\alpha \cdot \mu}{\alpha^2} - \ell = -j$, where j is the spin of the largest rep of $\mathsf{SU}(2)_\alpha$ that overlaps with $|\mu\rangle$, and that the generators of $\mathsf{SU}(2)_\alpha$ are $J^3_\alpha = \frac{\alpha \cdot H}{\alpha^2}$, $J^\pm_\alpha = E_{\pm\alpha}/|\alpha|$. And in the adjoint rep, each nonzero weight (*i.e.* root) labels a unique state $|\beta\rangle = |E_\beta\rangle$. So if we know r and ℓ (and hence $r + \ell = 2j$) for some such state with respect to $\mathsf{SU}(2)_\alpha$, then we know $J^3_\alpha |\beta\rangle = \frac{\alpha \cdot \beta}{\alpha^2} |\beta\rangle$, which means $|\beta\rangle = \eta \left| j, \frac{\alpha \cdot \beta}{\alpha^2} \right\rangle$, where only the relative phase η is not fixed. This means moreover that we know how $E_{\pm\alpha}$ act on this state.

This information is enough to construct the whole Lie algebra from just the simple roots. Once we know all the roots, we know how the Cartan generators act: $[H_i, E_\alpha] = \alpha_i E_\alpha$. The only thing we really don't know is the normalization in $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$. But this we can fix as follows. Consider $\mathsf{SU}(3)$ for example, and consider the state E_{α^2}. On the one hand, we have

$$J^+_{\alpha^1} |E_{\alpha^2}\rangle = \frac{E_{\alpha^1}}{|\alpha^1|} |E_{\alpha^2}\rangle = E_{\alpha^1} \left|\alpha^2\right\rangle = |[E_{\alpha^1}, E_{\alpha^2}]\rangle .$$

On the other hand, with respect to $\mathsf{SU}(2)_{\alpha^1}$, this state $|\alpha^2\rangle$ has $\ell = 0, r = 1$, and hence $j = (\ell + r)/2 = 1/2, m = (\ell - r)/2 = -1/2$. Therefore

$$J^+_{\alpha^1} |E_{\alpha^2}\rangle = J^+_{\alpha^1} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \frac{1}{\sqrt{2}} \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \frac{\eta}{\sqrt{2}} |E_{\alpha^1+\alpha^2}\rangle .$$

Here η is a phase which is actually arbitrary and can be set to 1. I will keep it around to show that it doesn't matter. Therefore

$$|E_{\alpha^1+\alpha^2}\rangle = \sqrt{2}\eta \, |[E_{\alpha^1}, E_{\alpha^2}]\rangle \qquad \text{or} \qquad E_{\alpha^1+\alpha^2} = \sqrt{2}\eta [E_{\alpha^1}, E_{\alpha^2}].$$

Note that $\alpha_3 = \alpha^1 + \alpha^2$ is the other positive root of $\mathsf{SU}(3)$.

Slicking up this procedure a bit, you can see that it is the same as what we did to construct the adjoint rep of $\mathsf{SU}(3)$ earlier. Starting from a positive root $|\phi_k\rangle = \left|\sum_b k_b \alpha^b\right\rangle$ of compositeness $k = \sum_b k_b$, raising with E_{α^b} takes

$$k_b \to k_b + 1, k \to k + 1, \ell^a - r^a \to \ell^a - r^a + A_{ab} \ .$$

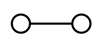
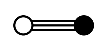
The vertical axis of that figure is the compositeness, starting from $k = 0$ for the Cartan generators (and calling $k < 0$ for the negative roots).

Once we know how to get all the positive roots as commutators of simple roots, we can get all of the commutators by the Jacobi identity. For example,

$$[E_{-\alpha^1}, E_{\alpha^1+\alpha^2}] = \sqrt{2}\eta[E_{-\alpha^1}, [E_{\alpha^1}, E_{\alpha^2}]] \overset{\text{Jacobi}}{=} \frac{\eta}{\sqrt{2}} E_{\alpha^2}.$$

(In doing so, don't forget that $\alpha^1 - \alpha^2$ is not a root, and so $[E_{\alpha^1}, E_{-\alpha^2}] = 0$.) Or

$$[E_{-\alpha^2}, E_{\alpha^1+\alpha^2}] = \sqrt{2}\eta[E_{-\alpha^2}, [E_{\alpha^1}, E_{\alpha^2}]] \overset{\text{Jacobi}}{=} -\frac{\eta}{\sqrt{2}} E_{\alpha^1}.$$

Dynkin diagrams. Since the simple roots and their geometry determine the whole algebra, it's useful to have a notation from which it is easy to read off their properties. A Dynkin diagram associates to each simple root a circle. Then for each pair of simple roots, we connect them in a way that encodes the angle between them. This information is also correlated with their relative lengths. Shorter roots are indicated with darker circles. (It will turn out that only two lengths are possible in a simple algebra, so there are only filled or empty circles.) So the number of circles is the rank. The Dynkin diagram encodes all the information about the algebra, in particular the Cartan matrix. Some examples: $\mathsf{SU}(2)$:  $\mathsf{SU}(3)$:  G_2: 

Example: G_2. From the Dynkin diagram, we can read off that the Cartan matrix for G_2 is $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$. Explicit simple roots are

$$\alpha^1 = (0, 1), \quad \alpha^2 = (\sqrt{3}, -3)/2, \tag{3.32}$$

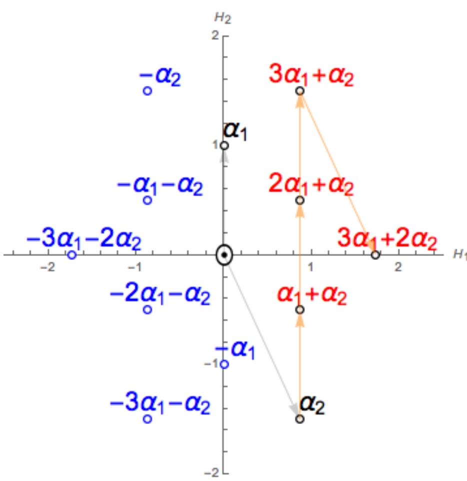
which you can check indeed have

$$\frac{2\alpha^1 \cdot \alpha^2}{(\alpha^1)^2} = -3, \quad \frac{2\alpha^1 \cdot \alpha^2}{(\alpha^2)^2} = -1, \tag{3.33}$$

and hence $\cos\theta_{\alpha^1\alpha^2} = -\frac{\sqrt{3}}{2}$ and $\theta_{\alpha^1\alpha^2} = 150°$.

Let's build all the roots. (3.33) says that $|\alpha^2\rangle$ has $\ell = 0, r = 3$ with respect to α^1, (so we can raise $|\alpha^2\rangle$ at most 3 times with E_{α^1}) while $|\alpha^1\rangle$ has $\ell = 0, r = 1$ with respect to α^2 (so we can raise $|\alpha^1\rangle$ only once with E_{α^2}). So far $\phi_2 = \alpha^1 + \alpha^2, \phi_3 = 2\alpha^1 + \alpha^2, \phi_4 = 3\alpha^1 + \alpha^2$ are roots, but $\alpha^1 + 2\alpha^2$ and $4\alpha^1 + \alpha^2$ are not. I've labelled the roots by their 'compositeness' $k = \sum_\alpha k_\alpha$. These are the only roots at $k = 2, 3, 4$ since we've ruled out $\alpha^1 + 2\alpha^2$, and $2\alpha^1 + 2\alpha^2 = 2(\alpha^1 + \alpha^2)$ is twice a root and hence not a root[35].

Since $4\alpha^1+\alpha^2$ is ruled out, the only possibility for $\phi_5 = 3\alpha^1 + 2\alpha^2$. We'd get this from $\phi_4 = 3\alpha^1 + \alpha^2$ by acting with E_{α^2}. With respect to $\mathsf{SU}(2)_{\alpha^2}$, $|\phi_4\rangle$ has $\ell - r = 3A_{12} + A_{22} = -1$. It has $\ell = 0$ since $3\alpha^2$ is not a root. Therefore it has $r = 1 > 0$ and we conclude that it can be raised by E_{α^2} to get a new root $\phi_5 = 3\alpha^1 + 2\alpha^2$. But because $r = 1$ (and not larger) this is the end of the line. More directly we can see that $3\alpha^1 + 3\alpha^2$ or $4\alpha^1 + 2\alpha^2$ would be integer multiples of roots and therefore not roots. In the diagram at right, I indicate the simple roots in black, the rest of the positive roots in red, the negative roots in blue, and the two weights associated with the two Cartan matrices $|H_{1,2}\rangle$ as a dot and a circle at the origin. The orange arrows indicate the trajectory by which we constructed all the positive roots. So we see that G_2 is $6 + 6 + 2 = 14$ dimensional.



3.6 Classification of simple Lie algebras

[Georgi §20.1, Zee §VI.5] The simple roots of a simple Lie algebra $\mathfrak{g}$ have the following (necessary and sufficient) properties:

(A) They are $r = \text{rank}(\mathfrak{g})$ linearly independent vectors $\{\alpha_i\}$.

(B) The off-diagonal entries of the Cartan matrix (their matrix of inner products, up to normalization), can only be certain non-positive integers: $\frac{2\alpha_i \cdot \alpha_j}{\alpha_i^2} \in \{0, -1, -2, -3\}$.

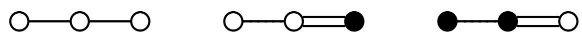
(C) They are *indecomposable*. This is a requirement of simplicity and just means that the Dynkin diagram is connected.

From here the classification of simple Lie algebras[36] is just geometry.

[35] Alternatively, we can see more directly that $2\alpha^1 + 2\alpha^2$ can't be a root. We would have to get it by acting with E_{α^2} on $|\phi_3 = 2\alpha^1 + \alpha^2\rangle$. With respect to $\mathsf{SU}(2)_{\alpha^2}$ this state has $\ell - r = 2A_{12} + A_{22} = 0$. But it has $\ell = 0$ since $2\alpha^1$ is not a root. So it has $r = 0$, and can't be raised by E_{α^2}.

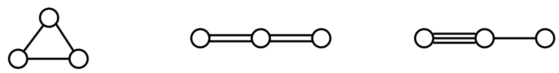
[36] Note that we don't have to specify 'compact' in this classification. Each of these algebras can be associated with various groups by allowing the parameters s in $e^{\mathbf{i}sX}$ to be variously real or imaginary or complex. When s are real and $X = X^\dagger$, we get compact simple groups.

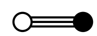
Lemma 1: The only possibilities at rank 3 are



Proof: For three linearly independent vectors, $\alpha_{i=1..3}$, the sum of the angles between them is less than 2π: $\sum_{i<j} \theta_{\alpha^i \alpha^j} < 2\pi$. But here the angles are chosen from $\theta_{\alpha^i\alpha^j} \in \{90°, 120°, 135°, 150°\}$, and at most one can be 90° lest the algebra be decomposable.

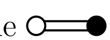
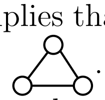
The following just barely fail this condition, since the angles add up to exactly 2π, and hence the simple roots would be coplanar and not linearly independent (they satisfy conditions B and C but not A):

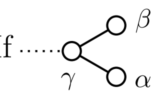


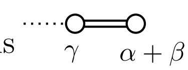
The powerful bit is that we can apply lemma 1 to any subdiagram of 3 connected nodes – it has to be one of these. We immediately conclude that a diagram with $r \geq 3$ cannot have triple lines, and therefore the only diagram with a triple line is G_2 

Note, by the way, that a subdiagram of the Dynkin diagram determines a subalgebra of $\mathfrak{g}$.

Lemma 2: If the Dynkin diagram contains two nodes connected by a single line, then the diagram obtained by smooshing together those two nodes into one node is also allowed.
Proof: Call the roots associated with the two nodes α and β. First of all, $\alpha \cdot \beta / \alpha^2 = -\frac{1}{2}$, and $\alpha^2 = \beta^2$, so $\alpha + \beta$ has the same length as α and β, $(\alpha+\beta)^2 = \alpha^2$. Lemma 1 implies that no node connects to both α and β: a node γ connected to α has $\gamma \cdot \beta = 0$, and a node δ connected to β has $\delta \cdot \alpha = 0$. This means $\gamma \cdot (\alpha + \beta) = \gamma \cdot \alpha, \delta \cdot (\alpha + \beta) = \delta \cdot \beta$, so replacing the two nodes with the node $\alpha + \beta$ produces a diagram that still satisfies A,B,C. (In fact, it's the Dynkin diagram for a subalgebra of the original one: recall that if the angle between α and β is 120° then $\alpha + \beta$ is also a root.)

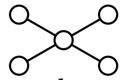
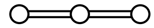
This implies that no diagram has more than one double-line  and no diagram has a loop . This is because they could be smooshed using Lemma 2 into a configuration that would contradict Lemma 1.

Lemma 3. If 

is allowed then so is 

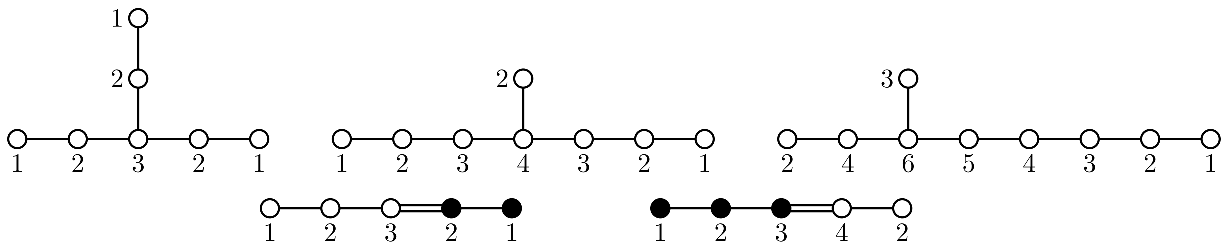
(I am not distinguishing between black and white nodes here, and the stuff to the left of γ is arbitrary.)

Here $\alpha \cdot \beta = 0$, $(\alpha + \beta)^2 = \alpha^2 + \beta^2 = 2\alpha^2$ and $\frac{2\alpha \cdot \gamma}{\alpha^2} = \frac{2\alpha\cdot\gamma}{\gamma^2} = \frac{2\beta\cdot\gamma}{\beta^2} = \frac{2\beta\cdot\gamma}{\gamma^2} = -1$. So $\frac{2(\alpha+\beta)\cdot\gamma}{\gamma^2} = -2$, $\frac{2(\alpha+\beta)\cdot\gamma}{(\alpha+\beta)^2} = -1$ which says were $\alpha + \beta$ a simple root, there would be a double line between $\alpha + \beta$ and γ.

From this we conclude that if  were OK, then  would be. Combining with Lemma 2, we see that only one junction is allowed.

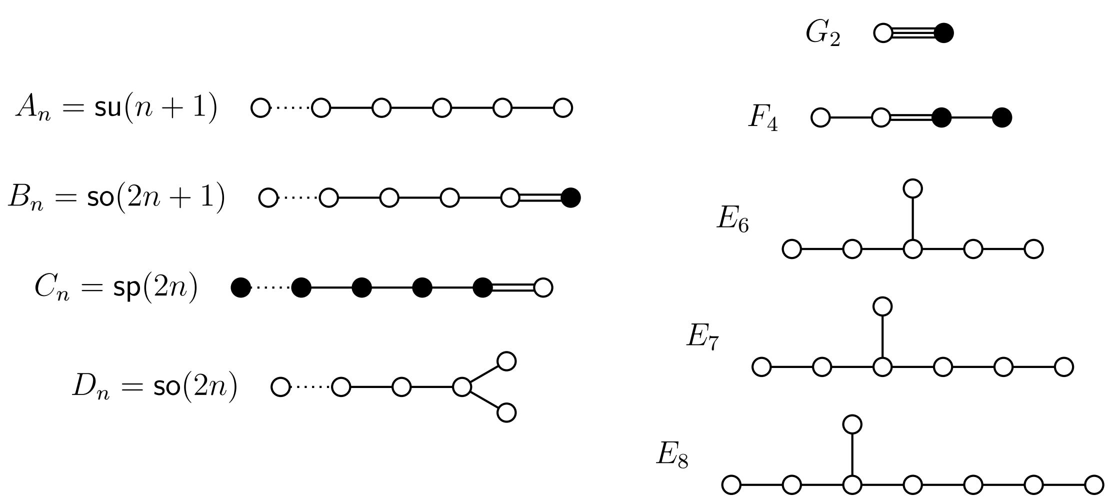
A nice way to restate Lemmas 3 and 1 together (see Zee p. 391 for a different proof) is that no more than three lines can come out of any node of a Dynkin diagram.

Lemma 4 is where it gets a little ugly. Recall that the Cartan matrix for a semisimple Lie algebra is invertible, meaning that it has no kernel. A vector annihilated by K would imply a linear relation between the simple roots. The following monsters correspond to K matrices with a kernel:



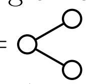
The proof is: define the number next to α_a in the diagram to be ξ_a. I claim that $\sum_a \xi_a \alpha_a = 0$, so that the simple roots would not be linearly independent. In each case, we could see this by showing that $0 = \left(\sum_a \xi_a \alpha_a\right)^2$. Actually these sets of numbers are not so mysterious: as you can check, they are solutions to $K_{ab}\xi_b = 0$ where K_{ab} is the Cartan matrix associated with the would-be Dynkin diagram. These vectors ξ_b are like harmonic functions on the would-be Dynkin diagram. Their existence implies a linear relation among the roots.

This leaves the following classification of four infinite families (the classical Lie algebras) and five *exceptional* Lie algebras[37]:

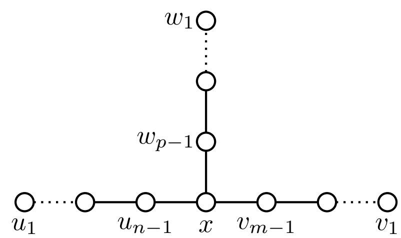


[37] I got the Tikz for the Dynkin diagrams from here.

It remains to show that the labelling of the infinite families is consistent with our previous definitions of these groups. We'll do that in §3.7.

At low rank there are a few collisions: $A_1 = B_1 = C_1$ are all just a single node. $B_2 = C_2$ are the same diagram; this is the statement that $\mathsf{sp}(4) = \mathsf{so}(5)$. $D_3 =$  $= A_3$, that is $\mathsf{so}(6) = \mathsf{su}(4)$. If we keep removing the middle node we get $D_2 = A_1 \times A_1$, not a simple algebra. And there is no D_1.

A nice way to discover $E_{6,7,8}$ is described in Zee's book (§VI.5). Consider any Dynkin diagram with a single junction. Call the roots on each branch $u_1 \cdots u_{n-1}, u_n = x, v_1 \cdots v_{m-1}, v_m = x, w_1 \cdots w_{p-1}, w_p = x$, where x is the root associated to the junction. For convenience, normalize all the roots to unit length. The vectors



$$u = \sum_{k=1}^{n-1} k u_k, \quad v = \sum_{k=1}^{m-1} k v_k, \quad w = \sum_{k=1}^{p-1} k w_k$$

(just like in the forbidden diagrams above) satisfy some simple properties that you can read off the Dynkin diagram:

$$u \cdot v = 0, v \cdot w = 0, w \cdot u = 0, u^2 = \frac{1}{2}n(n-1), v^2 = \frac{1}{2}m(m-1), w^2 = \frac{1}{2}p(p-1),$$

$$x \cdot u = -\frac{1}{2}(n-1), x \cdot v = -\frac{1}{2}(m-1), x \cdot w = -\frac{1}{2}(p-1).$$

Now demanding that $s^2 > 0$, where s is the projection of x onto the orthogonal complement of the space spanned by u, v, w gives

$$1 < \frac{1}{n} + \frac{1}{m} + \frac{1}{p}.$$

At least one of the three integers must be < 3. Besides $(n, m, p) = (n, 2, 2)$ which is $\mathsf{so}(2n+4)$ there are only three solutions $(3, 3, 2), (3, 4, 2), (3, 5, 2)$ which are $E_{6,7,8}$.

Some cultural remarks. Many other mathematical objects are classified by Dynkin diagrams. A Dynkin diagram (and its corresponding algebra) is called *simply laced* if there are no double or triple lines – these are the ADE cases. Discrete subgroups of $\mathsf{SU}(2)$ (which includes the symmetries of the platonic solids) have an ADE classification by simply-laced Dynkin diagrams. (A are the cyclic groups, D are the dihedral groups, and E are the TOI groups – tetrahedral, octahedral and icosahedral. In fact these are all also subgroups of $\mathsf{SO}(3)$, and hence rotational symmetries of three-dimensional objects. The TOI groups are the symmetries of the platonic solids.)

There is also an ADE classification of singularities of complex 2-manifolds. The last two classifications are related by the fact that the singularity associated with a given Dynkin diagram X can be realized as a quotient $\mathbb{C}^2/\Gamma_X$, where Γ_X is the associated discrete subgroup of $\mathsf{SU}(2)$. (This is a space with holonomy $\Gamma_X \subset \mathsf{SU}(2)$ and so preserves some supersymmetry in string compactification.) String theory unifies all three of the classifications I've discussed by the fact that compactification of (type IIA) string theory on a space with a singularity of type X produces a gauge theory with gauge group X. (A gauge theory has a vector field for each generator of the Lie algebra of X.)

The way this gauge theory arises is a beautiful thing: the gauge bosons associated with the Cartan subalgebra H_i are visible in supergravity, as (zero-energy, Ramond-Ramond[38]) states of the superstring. But the gauge bosons associated with the raising and lowering operators E_α come from D2-branes wrapping shrinking 2-cycles inside the 4-manifold. The fact that these gauge bosons are charged under the Cartan subgroup comes from the fact that D-branes carry Ramond-Ramond charge. Since the Cartan-Weyl labelling of the generators of the algebra is arbitrary (we could always pick a different set of r generators to be the Cartan), this means that strings and D-branes are somehow fungible. For more on this, take a look at this review.

Another thing with an ADE classification is modular invariant partition functions of unitary minimal models of 2d conformal field theory. See, *e.g.* Di Francesco et al, *Conformal Field Theory*, aka the big yellow book or this free, briefer, and more accessible book.

[38] I mention this name since it will come up again when we talk about spinor representations.

3.7 The classical groups

3.7.1 $A_{n-1} = \mathsf{su}(n)$

$U = e^{\mathbf{i}H}$ is unitary if $H = H^\dagger$ and special unitary ($\det U = 1$) if $\mathrm{tr} H = 0$, so $\mathsf{su}(n) = \{$hermitian, traceless $n \times n$ matrices $\}$. There are $n^2 - 1$ of these. You might think we need to find some analog of the Gell-Mann matrices, but it is actually much better to directly construct the Cartan-Weyl basis.

For the Cartan subalgebra $\mathfrak{h}$, we can just use the $n-1$ diagonal ones, in some basis:

$$\mathfrak{h} = \{a_i h_i | \sum_i a_i = 0\}.$$

Here h_i is the diagonal matrix with zeros everywhere but a 1 in the ii entry, $(h_i)_{jk} = \delta_{ij}\delta_{ik}$. Resist the temptation to choose an explicit representation of the $n-1$ independent h_a – this is where all the ugly complications in the discussion in Georgi and Zee come from.

The raising and lowering operators are also simple: consider the matrix $E_{(ij)}$ which has a 1 in the ij entry and zeros everywhere else (if you insist: $(E_{(ij)})_{kl} = \delta_{ik}\delta_{jl}$). These are eigenvectors of the Cartan generators:

$$[a \cdot h, E_{(ij)}] = (a_i - a_j) E_{(ij)}.$$

$a \cdot h$ is the diagonal matrix with a_i in the ii entry. The coefficient of a_ℓ in that last equation is

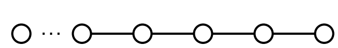
$$[h_\ell, E_{(ij)}] = (\delta_{\ell i} - \delta_{\ell j})\, E_{(ij)} \equiv (e_i - e_j)_\ell E_{(ij)}$$

where e_i is the vector with a 1 in the ith entry and zeros everywhere else (if you insist: $(e_i)_\ell = \delta_{i\ell}$). This says that $e_i - e_j \equiv \alpha_{ij}$ is a root for each $i \neq j$. Notice that these are n-dimensional vectors, but they all lie in the hyperplane perpendicular to $\sum_i e_i$: $\sum_i e_i \cdot \alpha_a = 0$. All the complications come from solving this equation to make explicit $(n-1)$-vectors. The weights of the fundamental representation are just $e_a, a = 1..n$.

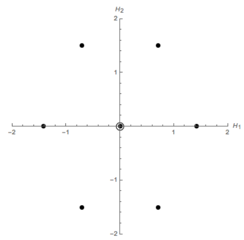
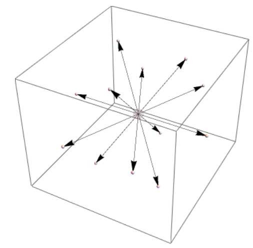
Choose a convention for positivity where the positive roots are $e_i - e_j, i < j$. Notice that there are $\frac{n(n-1)}{2}$ positive roots, $\frac{n(n-1)}{2}$ negative roots, and $n-1$ Cartan generators, giving $n^2 - 1$ generators altogether – everybody is accounted for. The simple roots are $\alpha_a \equiv e_a - e_{a+1}, a = 1..n-1$.

The nonzero off-diagonal entries in the Cartan matrix are

$$\frac{2\alpha_k \cdot \alpha_{k+1}}{\alpha_k^2} = \frac{2}{2}(e_k - e_{k+1}) \cdot (e_{k+1} - e_{k+2}) = -1.$$

So we confirm that the Dynkin diagram is 

The drawable examples are[39]

su(3) :  su(4) : 

What are the fundamental weights, μ^b? They satisfy $\frac{2\alpha^a\cdot\mu^b}{(\alpha^a)^2} = \delta^{ab}$. The solution is strikingly simple: $\mu^b = \sum_{a=1}^b e_a$.

Where does this come from? First consider $\mu^1 = e_1$. This is the highest weight of the fundamental representation. How did I know this? Well, the eigenvectors of the $n \times n$ matrices H_i are just $|j\rangle$, with eigenvalues $\delta_{ji} = (e_j)_i$. With our convention, $|1\rangle$ is the highest weight state, so the highest weight is e_1.

Now consider the $\Lambda^m \mathbf{n}$ rep. The highest weight state is

$$(|1\rangle \otimes |2\rangle \otimes |3\rangle \cdots) - (|2\rangle \otimes |1\rangle \otimes |3\rangle \cdots) \pm \cdots \equiv |123\cdots\rangle$$

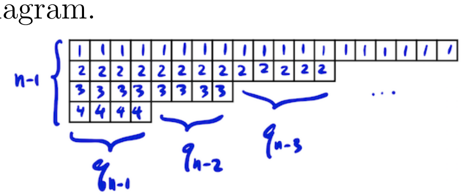
where we choose the m highest-weight vectors and antisymmetrize them. This state has weight $\sum_{a=1}^m e^a = \mu^m$, exactly the mth fundamental weight.

Let's think about the states of this irrep a bit more. Letting

$$|i_1 \cdots i_m\rangle = (|i_1\rangle \otimes |i_2\rangle \otimes \cdots) - (|i_2\rangle \otimes |i_1\rangle \otimes \cdots) \pm \cdots ,$$

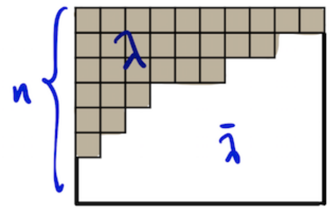
a general state is $|A\rangle = A^{i_1\cdots i_m}|i_1 \cdots i_m\rangle$. The components of the wavefunction are an antisymmetric m-index tensor. This rep has dimension $\begin{pmatrix} n \\ m \end{pmatrix}$.

As always, the general irrep then has highest weight $\sum_k q_k \mu^k, q_k \in \mathbb{Z}_{\geq 0}$. Now the wavefunction has q_k sets of k indices, antisymmetric within each set, and you can probably guess that we can represent it with a Young diagram.
The highest weight state occurs when all the indices associated with the first row are equal to 1 (hence, symmetric), those in the second row are equal to 2, and so on. This reproduces the weight $\sum_k q_k \mu^k$.



[39]Note that to draw the root diagram for su(4) I did have to succumb to choosing a basis for the hyperplane $\sum_i e^i = 0$ in order to project down the 4-component vectors. A good basis is $(\frac{e^1-e^2}{\sqrt{2}}, \frac{e^1+e^2-2e^3}{\sqrt{6}}, \frac{e^1+e^2+e^2-3e^4}{\sqrt{12}})$. Notice that these are the entries of the Cartan generators in Georgi and Zee.

The diagram associated with the conjugate representation of $\mathsf{SU}(n)$, $\bar{R}_\lambda = R_{\bar{\lambda}}$, where $\bar{\lambda}$ is the Young diagram obtained as follows. Draw a box of height n and width $\sum_k q_k$ and put the diagram in the upper left corner of the box. Whatever's left is the dual diagram (rotated by π). Note that this agrees with what we found for $\mathsf{SU}(3)$. More on Young diagrams and reps of $\mathsf{SU}(n)$ later.



As a simple consequence of the notion of conjugate tableau, we can see that the highest weight of the $\bar{n}$ representation is $\mu^{n-1} = \sum_{a=1}^{n-1} e_a$. This vector has the same projection onto the plane perpendicular to $\sum_a e_a$ as $-e_1$. So the weights of the $\mathbf{n}$ are $-e_a$.

[End of Lecture 18]

3.7.2 $\mathsf{so}(N)$ (B_n, D_n)

Recall that the generators of $\mathsf{so}(N)$ are the antisymmetric imaginary matrices, of which $N(N-1)/2$ are linearly independent. Unfortunately none of these are diagonal, but we can choose as Cartan generators the generators of $\mathsf{SO}(2)$ rotations in 2d subspaces. You see that odd and even will be different, since for odd N there will be a 1d subspace left over.

$N = 2n$ **even:** So the Cartan generators are

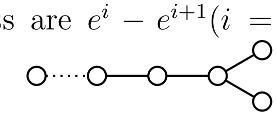
$$(H_m)_{jk} = -\mathbf{i}\left(\delta_{j,2m-1}\delta_{k,2m} - \delta_{j,2m}\delta_{k,2m-1}\right) = \sigma^2 \otimes \text{projector onto } m\text{th 2d subspace} .$$

The eigenvalues of H_m are ± 1 and 0, so the weights of the $\mathbf{2n}$ rep are $\pm e^i, i = 1..n$. One way to get the roots, then, is just to take differences of these weights. This gives

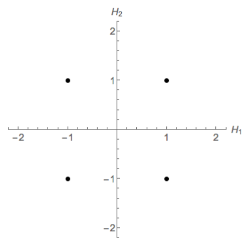
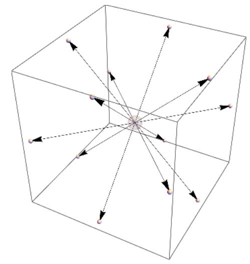
$$\{\pm e^i \pm e^j, i \neq j\}$$

where the $\pm$ are uncorrelated. Notice that $\pm 2e^i$, which would take the weight e^i to the weight $-e^i$, is not a root. This is because the eigenvectors of, say H_1, with evals ± 1, are $(1, \pm\mathbf{i}, 0, 0\cdots)^T$. But no rotation can take $(1, +\mathbf{i}, \cdots)$ to $(1, -\mathbf{i}, \cdots)$ (and preserve everyone else) – this would require the operation $\text{diag}(1, -1, 1, 1, \cdots)$ which has determinant -1.

Notice that, including the n Cartan generators, there are $n + 4\frac{n(n-1)}{2} = n(2n-1)$ states in the adjoint rep, which agrees with $\dim \mathsf{so}(2n)$.

The positive roots are $\{e^i \pm e^j, i < j\}$ and the simple roots are $e^i - e^{i+1} (i = 1..n-1), e^{n-1} + e^n$. This gives the Dynkin diagram $D_n = \mathsf{so}(2n)$ 

The drawable examples are[40]:

$\mathsf{so}(4) = \mathsf{su}(2) \times \mathsf{su}(2)$:  $\mathsf{so}(6)$: 

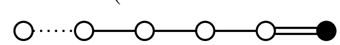
You can see that $\mathsf{so}(4)$ just falls apart into two copies of $\mathsf{su}(2)$.

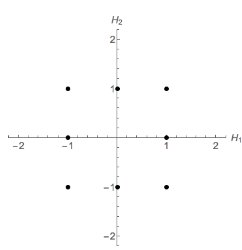
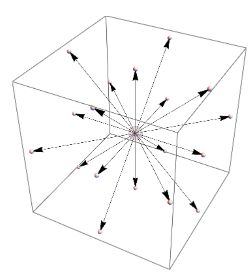
The fundamental weights are

$$\mu^b = \sum_{a=1}^{b} e_a,\ b < n-1, \mu^{n-1} = \frac{1}{2}\left(e^1 + e^2 + \cdots + e^{n-1} - e^n\right), \mu^n = \frac{1}{2}\left(e^1 + e^2 + \cdots + e^{n-1} + e^n\right). \tag{3.34}$$

Again the first $n-2$ come from $\Lambda^b \mathbf{2n}$. The last two are special and produce spinor representations. The factor of $\frac{1}{2}$ means that these reps do not appear in tensor products of the fundamental.

$N = 2n+1$ **odd:** For $\mathsf{so}(2n+1)$, we add an extra 1d block, but we don't add a new Cartan generator – the Cartan subalgebra is exactly the same as for $\mathsf{so}(2n)$. But there are new roots, which connect the 2d subspaces to the 1d subspace (which has weight zero for all n of the Cartan generators). Thus the extra roots are $\pm e^i$. Altogether, the nonzero roots are $\{\pm e^i \pm e^j\ (i \neq j), \pm e^i\}$. Including the n Cartan generators, there are $n + 4\frac{n(n-1)}{2} + 2n = n(2n+1)$ states in the adjoint rep, which agrees with $\dim \mathsf{so}(2n+1)$.

The positive roots are $\{e^i \pm e^j\ (i < j), e^i\}$ and the simple roots are $e^i - e^{i+1}$ ($i = 1..n-1$) and e^n. This reproduces the Dynkin diagram $B_n = \mathsf{so}(2n+1)$ 

$\mathsf{so}(5)$:  $\mathsf{so}(7)$: 

The fundamental weights are

$$\mu^b = \sum_{a=1}^{b} e_a,\ b < n, \quad \text{and } \mu^n = \frac{1}{2}\left(e^1 + e^2 + \cdots + e^{n-1} + e^n\right).$$

The last one is the highest weight for the spinor representation.

[40] A mathematica notebook for looking at the 3d root diagrams from different angles is here

3.7.3 $C_n = \mathsf{sp}(2n)$

The group $\mathsf{Sp}(2n)$ is made of $2n \times 2n$ matrices preserving an antisymmetric form $\omega : V \otimes V \to \mathbb{C}$ (where V is a $2n$-dimensional vector space, the carrier space for the fundamental rep): $M \in \mathsf{Sp}(2n)$ if $\omega(Mv, Mw) = \omega(v, w)$ for all $v, w \in V$. (In components, $\omega(v, w) \equiv \omega_{ij} v^i w^j$.) What does this mean for the Lie algebra? If $M = e^{\mathbf{i}X}$, with X small, the condition is $\omega(Xv, w) + \omega(v, Xw) = 0$. Choosing $\omega = Y \otimes 1\!\!1 = \begin{pmatrix} 0 & -\mathbf{i}1\!\!1_n \\ \mathbf{i}1\!\!1_n & 0 \end{pmatrix}$ (where $Y = \sigma^y$; note that a different choice of AS matrix here would not change the group theory) the condition on X is

$$Y X^T Y + X = 0. \tag{3.35}$$

Demanding a unitary representation, we can take X to be hermitian, and expand it in Pauli matrices. Noting that $Y\left(\sigma^i\right)^T Y = -\sigma^i$, the solution to (3.35) is

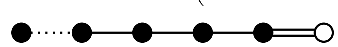
$$X = \sigma^i \otimes S^i + 1\!\!1_2 \otimes A \tag{3.36}$$

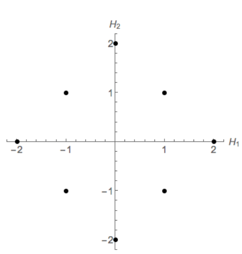
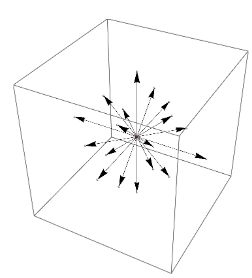
where S^i are real symmetric and A is imaginary antisymmetric. The dimension of $\mathsf{Sp}(n)$ is therefore $3\frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n(2n+1)$.

A good choice of Cartan subalgebra is just to pick the diagonal ones of (3.36), namely

$$H_m = \sigma^3 \otimes h_m = \begin{pmatrix} h_m & 0 \\ 0 & -h_m \end{pmatrix}$$

where h_m is defined as above to be the matrix with a 1 in the mm entry and zeros everywhere else. The state $|i\rangle$ is an eigenstate of H_m with eigenvalues $+\delta_{im}, i \leq n, -\delta_{im}, i > n$. The nonzero weights of the **2n** are then $\pm e^i$, just like for $\mathsf{SO}(2n)$. Does this mean the roots are the same as those for $\mathsf{so}(2n)$? No: the roots are $\pm e^i \pm e^j, i \neq j$ as before, but *also* $\pm 2e^i$. The latter appears because now the eigenvectors with eigenvalues ± 1 under (for example) H_1 are just $(100000)^T$ and $(000100)^T$ (for $n = 3$) and they *are* related by a $\mathsf{Sp}(2n)$ transformation.

So the positive roots are $\{e^i \pm e^j \ (i < j), 2e^i\}$. Check that we reproduce $\dim \mathsf{Sp}(2n) = n(2n+1)$ from the n Cartan generators plus these $2(n(n-1)+n)$ where the 2 counts the negative roots. The simple roots are $e^i - e^{i+1}$ $(i = 1..n-1)$ and $2e^n$. This matches the Dynkin diagram $C_n = \mathsf{sp}(2n)$ 

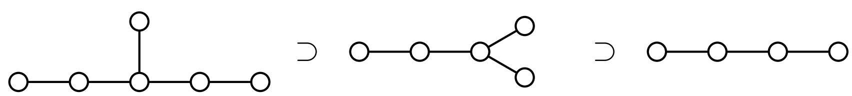
$\mathsf{sp}(4) = \mathsf{so}(5)$:  $\mathsf{sp}(6)$: 

You can see that the root diagram for $\mathsf{sp}(4)$ is just that of $\mathsf{so}(5)$ rotated and rescaled. The normalization of the roots depends on our convention for normalizing the Cartan generators, and thus is not a priori meaningful. You can check that all of their representations are the same.

3.8 Regular subalgebras

While we're on the subject of Dynkin diagrams: some subalgebras are easy to read off of the Dynkin diagram – just take any subdiagram. This will produce a subalgebra which shares Cartan generators and roots. Such a subalgebra is called *regular*.

An important example in high energy theory is the sequence of regular subalgebras:



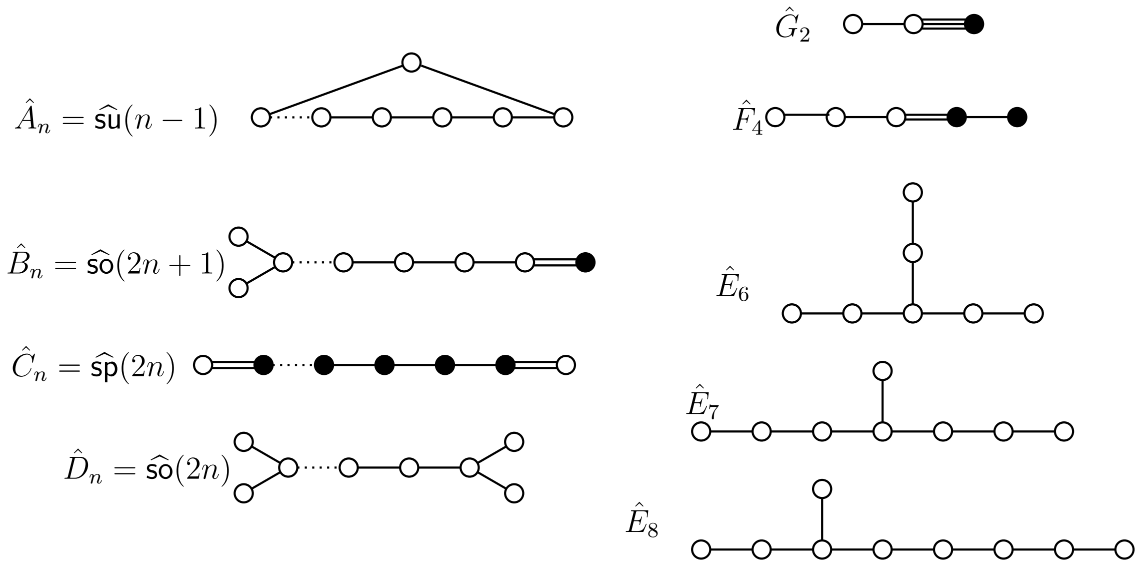
$E_6 \supset \mathsf{SO}(10) \supset \mathsf{SU}(5)$, the last step of which we'll understand in detail in the next section.

Not even every regular subalgebra arises this way: for example, G_2 has an $\mathsf{SU}(3)$ subalgebra with simple roots $\alpha_2, 3\alpha_1 + \alpha_2$ (where $\alpha_{1,2}$ are the simple roots of G_2 in (3.32)), but this is not a associated with a sub-diagram. Another example which does not arise just by leaving out nodes is $G_2 \subset \mathsf{SU}(4)$. If you just forget about one of the Cartan generators of $\mathsf{SU}(4)$, *i.e.* project its root diagram onto the subspace spanned by $H_{1,2}$ you find exactly the root diagram of G_2 (notice $14 = 15 - 1$). (See this mathematica notebook where I discovered this by accident.)

A *maximal* subalgebra is one with the same rank. Here is a nice trick for reading off maximal regular subalgebras. Add to the Dynkin diagram an extra node for $\alpha_0 \equiv$ the lowest root. Since α_0 is not linearly independent of the others, this diagram will violate condition A. But if we remove any one of the nodes from this *extended Dynkin diagram*, we'll get an allowed Dynkin diagram, for a regular maximal subalgebra of the original algebra. For example, the lowest root of $\mathsf{su}(n)$ (in the notation and convention for positivity of subsection §3.7) is $e^n - e^1$. The lowest root of B_n and D_n is $-e^1 - e^2$. The lowest root of C_n is $-2e^1$.

(Interpreted differently, the extended Dynkin diagram is related to an (infinite dimensional) affine Lie algebra; I learned about this from here but probably there is a more elementary reference.)

The extended Dynkin diagrams are:



From the extended diagram for $\mathsf{so}(2n+1)$ you can see the obvious fact $\mathsf{SO}(2n) \subset \mathsf{SO}(2n+1)$. From the extended diagram for G_2 you can see the $\mathsf{SU}(3)$ subgroup. And you can see, for example, that F_4 has a B_4 subgroup and a $A_1 \times C_3$ subgroup.

3.9 Spinor representations

Spinor representations of the algebra $\mathsf{so}(N)$ are projective representations of the group $\mathsf{SO}(N)$. They exist because $\mathsf{SO}(N)$ is not simply connected; its covering space is a group with the same algebra called $\mathsf{Spin}(N)$. They fit in the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathsf{Spin}(N) \to \mathsf{SO}(N) \to 1,$$

as in our discussions of projective reps in (2.7). But no one will be confused about what you mean if you say 'spinor reps of $\mathsf{SO}(N)$'.

We could directly construct the spinor representations of $\mathsf{so}(N)$ starting with the highest weights we found in §3.7, but we will do it in another way where we get to use our knowledge of fermions.

Imagine we have $2n$ *majorana zeromodes*[41]:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad \gamma_i^\dagger = \gamma_i, \quad i = 1 \cdots 2n \ . \tag{3.37}$$

These are operators acting on some Hilbert space. Our next job is to identify the structure of this Hilbert space. The fact that δ_{ij} appears suggests some connection with $\mathsf{SO}(2n)$.

To make this look more familiar, and define

$$c_a \equiv \frac{1}{2}\left(\gamma_{2a-1} + \mathbf{i}\gamma_{2a}\right), \text{(and therefore)} \ \ c_a^\dagger \equiv \frac{1}{2}\left(\gamma_{2a-1} - \mathbf{i}\gamma_{2a}\right) \quad a = 1..n.$$

Then (3.37) implies that these operators satisfy

$$\{c_a, c_b^\dagger\} = \delta_{ab}, \quad \{c_a, c_b\} = 0,$$

the canonical fermion creation-annihilation algebra, for n fermion modes. In particular $(c_a^\dagger)^2 = 0$ is an implementation of Pauli exclusion.

Now we know what the structure of the Hilbert space is: there is a vacuum, $|0\rangle$, with no fermions: $c_a |0\rangle = 0, \forall a$. Then we can create a single fermion in mode a by $c_a^\dagger |0\rangle$. To specify the general state, we must say whether each mode is occupied or unoccupied.

$$\mathcal{H} = \text{span}\{|s_1 \cdots s_n\rangle\} \tag{3.38}$$

where for reasons that will become clear momentarily, I've labelled the states by $s_a = \pm\frac{1}{2}$ ($+\frac{1}{2}$ for unoccupied states, $-\frac{1}{2}$ for occupied states). $c_a^\dagger c_a |s_1 \cdots s_n\rangle = (\frac{1}{2} - s_a) |s_1 \cdots s_n\rangle$. So the dimension of $\mathcal{H}$ is 2^n.

[41] Incidentally, realizations of this situation (with the label i associated with controllable particles) is being vigorously sought by many people in connection with quantum computing. Also incidentally, this is called a *Clifford algebra*.

Actually a connection between the Clifford algebra and $\mathsf{SO}(2n)$ can be made very directly as follows. Let

$$T^{ij} = \frac{1}{2}\mathbf{i}\gamma_i\gamma_j. \tag{3.39}$$

Note that $\mathbf{i}\gamma_i\gamma_j = (\mathbf{i}\gamma_i\gamma_j)^\dagger$ is hermitean. I claim that (just using the Clifford algebra (3.37)), these operators satisfy the $\mathsf{so}(2n)$ Lie algebra, (3.15). You can show this directly for the T^{ij}.[42]

But then we see that the Hilbert space (3.38) provides a representation of the Lie algebra $\mathsf{so}(2n)$. (Is it then a representation of $\mathsf{SO}(2n)$? Almost: it's a spinor (projective) representation.) Its dimension is 2^n.

Actually this rep of $\mathsf{so}(2n)$ is reducible. This is because of the operator $\gamma_{2n+1} \equiv C\gamma_1 \cdots \gamma_{2n} \equiv \gamma_F$. γ_F anticommutes with all of the majoranas: $\{\gamma_F, \gamma_i\} = 0, \forall i$. Therefore it commutes with all the generators $\frac{\mathbf{i}}{2}\gamma^i\gamma^j$ (it is an intertwiner). Notice that C can be chosen so that $\gamma_{2n+1} = \gamma_{2n+1}^\dagger$ and $\gamma_{2n+1}^2 = 1$ so that (3.37) can be satisfied with $i, j = 1 \cdots 2n+1$. Since $\gamma_F^2 = \mathbb{1}$, its two eigenspaces have $\gamma_F = \pm 1$ – these are are 2^{n-1}-dimensional invariant subspaces which are irreps of $\mathsf{so}(2n)$.

We can make contact with the Cartan-Weyl method. A cartan subalgebra is generated by $\{H_a \equiv \frac{1}{2}\mathbf{i}\gamma^{2a-1}\gamma^{2a}, a = 1..n\}$. Note that $c_a^\dagger c_a = \frac{1}{2}\left(1 + \mathbf{i}\gamma^{2a-1}\gamma^{2a}\right)$ are the fermion occupation number operators. The eigenvalues of the H_a on the spinor states are $s_a = \pm\frac{1}{2}$. That is, the weight vectors for the spinor rep are $\frac{1}{2}\left(\pm e^1 \pm e^2 \pm \cdots \pm e^n\right)$.

The parity operator is $\gamma_F = \text{sign}\left(H_1 H_2 \cdots H_n\right)$. Thus, the number of (-1)s in the weight vector is equal to γ_F. The highest weight of the irrep with an even number of minus signs is $\frac{1}{2}\sum_{a=1}^n e^a$, and the highest weight of the irrep with an odd number of minus signs is $\frac{1}{2}\sum_{a=1}^{n-1} e^a - \frac{1}{2}e^n$. These are exactly the last two fundamental weights of $\mathsf{SO}(2n)$ that we found in (3.34).

For the raising and lower operators, consider the following operators acting on the fermion Hilbert space:

$$H_a = \frac{1}{2}\mathbf{i}\gamma^{2a-1}\gamma^{2a} = c_a^\dagger c_a - \frac{1}{2}, \quad E_{ab} \equiv c_a^\dagger c_b, \quad E'_{ab} \equiv c_a^\dagger c_b^\dagger \;\; (a \neq b). \tag{3.40}$$

E_{ab} removes a fermion in mode b and creates one in mode a. For example, E_{12} takes $\left|-\frac{1}{2}, +\frac{1}{2}, \cdots\right\rangle$ to $\left|\frac{1}{2}, -\frac{1}{2}, \cdots\right\rangle$, so these two states differ by the root vector $e_1 - e_2$. You

[42]The best way to do this is to consider the object $\Gamma_A \equiv \frac{1}{2}A_{ij}T^{ij}$, where $A_{ij} = -A_{ji}$ is an antisymmetric matrix (hence parametrizes an element of $\mathsf{SO}(2n)$). Then show that

$$[\Gamma_A, \Gamma_B] = \Gamma_{[A,B]}.$$

This shows that the map $A \to \Gamma_A$ is a representation of $\mathsf{SO}(2n)$. (It also works if there is an odd number of γs.)

can check that

$$[H_a, E_{bc}] = (\delta_{ab} - \delta_{ac})\, E_{bc} = (e_b - e_c)_a E_{bc}.$$

This is the Cartan-Weyl form of $\mathsf{SU}(n)$ (it says the roots are $e_a - e_b$). Note that $(E_{bc})^\dagger = E_{cb}$. So this is an $\mathsf{SU}(n)$ algebra (if we throw away $\sum_a H_a$ – the total particle number – which commutes with everyone and generates a $\mathsf{U}(1)$ subgroup). This is an $\mathsf{su}(n) \subset \mathsf{so}(2n)$ subalgebra, which will be useful later. It generates the subgroup which preserves the pairings between the majoranas that we've chosen (the "complex structure"). That is, it preserves the particle number $\sum_a c_a^\dagger c_a$.

How do the spinor reps decompose under $\mathsf{SO}(2n) \supset \mathsf{SU}(n)$? The state $|0\rangle$ with no particles has $\tilde{H}_a \equiv H_a - \frac{1}{2}$ eigenvalue 0 and is annihilated by the E_{ab} – it is a singlet under this $\mathsf{SU}(n)$. The states with one particle form an n dimensional irrep, $A_a c_a^\dagger |0\rangle$; The states with two particles form an n dimensional irrep, $A_{ab} c_a^\dagger c_b^\dagger |0\rangle$, $A_{ab} = -A_{ba}$; the states with k particles form the rep $\Lambda^k \mathbf{n}$, $A_{a_1 a_2 \cdots a_k} c_{a_1}^\dagger c_{a_2}^\dagger \cdots c_{a_k}^\dagger |0\rangle$. The states with an even (odd) number of particles make up the positive (negative) chirality spinor rep of $\mathsf{so}(2n)$. For example, for $n = 5$, we've just shown that

$$\mathbf{16}^+ = \mathbf{1} \oplus \mathbf{10} \oplus \bar{\mathbf{5}}, \quad \mathbf{16}^- = \mathbf{5} \oplus \bar{\mathbf{10}} \oplus \mathbf{1}.$$

This is worth a table:

# of particles	state	irrep of SU(n=5)
0	$\|0\rangle$	$\mathbf{1}$
1	$A_a c_a^\dagger \|0\rangle$	$\mathbf{n} = \mathbf{5}$
2	$A_{ab} c_a^\dagger c_b^\dagger \|0\rangle$	$\mathbf{n(n-1)/2 = 10}$
3	$A_{abc} c_a^\dagger c_b^\dagger c_c^\dagger \|0\rangle = \tilde{A}_{de} c_d c_e \|1\rangle$	$\bar{\mathbf{n}}(\bar{\mathbf{n}} - \mathbf{1})/\mathbf{2} = \bar{\mathbf{10}}$
4	$A_e c_e \|1\rangle$	$\bar{\mathbf{n}} = \bar{\mathbf{5}}$
5	$\|1\rangle$	$\mathbf{1}$

It's worth saying a little more about what happens as we add more particles, for example, in the case of $n = 5$. Let's introduce the 'plenum' (opposite of vacuum) state which is totally full of particles: $|1\rangle \equiv c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger c_5^\dagger |0\rangle = \frac{1}{n!} \epsilon_{a_1 \cdots a_n} c_{a_1}^\dagger \cdots c_{a_n}^\dagger |0\rangle$. Then we can write the $\bar{1}0$ states as

$$A_{abc} c_a^\dagger c_b^\dagger c_c^\dagger |0\rangle = \epsilon_{abcde} \tilde{A}_{de} c_a^\dagger c_b^\dagger c_c^\dagger |0\rangle = \tilde{A}_{de} c_d c_e |1\rangle \,.$$

(Maybe there should be some multiplicative factors.)

Now back to the rest of the raising and lowering operators in (3.40). E'_{ab} creates fermions in modes a and b (if none are there already). This is like a Cooper-pair operator in a superconductor, which breaks particle number symmetry. For example,

E'_{12} takes $\left|-\frac{1}{2},-\frac{1}{2},\cdots\right\rangle$ to $\left|\frac{1}{2},\frac{1}{2},\cdots\right\rangle$, so these two states differ by the root vector e_1+e_2. You can check that

$$[H_a, E'_{bc}] = (\delta_{ab} + \delta_{ac})\, E_{bc} = (e_b + e_c)_a E_{bc}.$$

That is, these operators are associated with the rest of the $\mathsf{so}(2n)$ roots $e_a + e_b, a \neq b$. Note that $E_{bc} = -E_{cb}$ but $(E'_{bc})^\dagger = c_c c_b$ is associated with the root $-e_b - e_c$. So altogether we have the n cartan generators, the $n(n-1)$ E_{ab}s and the $n(n-1)$ $E'_{ab}, (E'_{ab})^\dagger$s, which gives $n(2n-1)$ generators altogether – we didn't miss anyone.

To get $\mathsf{SO}(2n+1)$, just let $\gamma_{2n+1} \equiv C\gamma_1 \cdots \gamma_{2n} = \gamma_F$, with C chosen as above so that $\gamma_{2n+1} = \gamma_{2n+1}^\dagger$ and $\gamma_{2n+1}^2 = 1$. Then (3.37) is satisfied with $i, j = 1 \cdots 2n+1$. The generators (3.39) (with i, j now running up to $2n+1$) then satisfy the $\mathsf{so}(2n+1)$ Lie algebra, and the *same* Hilbert space we've been talking about all along also gives a representation of (the double-cover of) $\mathsf{SO}(2n+1)$. This is an irrep of $\mathsf{so}(2n+1)$. This is consistent with the fact that we found a single fundamental weight of $\mathsf{so}(2n+1)$ with factors of $\frac{1}{2}$.

Matrix representation. Regard the 2^n dimensional Hilbert space

$$\mathcal{H} = \text{span}\{|s_1 \cdots s_n\rangle = |s_1\rangle \otimes |s_2\rangle \cdots \otimes |s_n\rangle\} = \otimes_{a=1}^n \mathcal{H}_a$$

where $\mathcal{H}_a$ is a single qubit. On this space, we have a set of Pauli matrices for each 'site' $a = 1..n$, $\vec{\sigma}_a |s_1 \cdots s_n\rangle = \sum_{s'_a} (\vec{\sigma})_{s_a s'_a} |s_1 \cdots s'_a \cdots s_n\rangle$. That is, $\vec{\sigma}_a = 1\!\!1 \otimes \cdots \vec{\sigma} \cdots \otimes 1\!\!1$, where the $\vec{\sigma}$ is in the ath entry. In this representation, the Cartan generators are just $H_a = \frac{1}{2} Z_a$. Note that these satisfy $H_a^2 = 1\!\!1/4$; since the Cartan generators could be anyone, this means that all the generators in this rep satisfy $(T_{ij})^2 = 1\!\!1/4$.

Our next goal is to write all of the $\mathsf{SO}(2n+1)$ generators (and hence the $\mathsf{SO}(2n)$ subgroup) as matrices on this collection of n qubits. Consider the raising operators

$$E_a \equiv T_{2a-1,2n+1} - \mathbf{i} T_{2a,2n+1} = \mathbf{i}\frac{1}{2}\left(\gamma_{2a-1} - \mathbf{i}\gamma_{2a}\right)\gamma_F = \mathbf{i} c_a^\dagger \gamma_F.$$

Essentially $E_a \sim c_a^\dagger$, up to a sign. These satisfy $\{E_a, E_b\} = 0$ (just like $\{c_a^\dagger, c_b^\dagger\} = 0$.

On the basis states $|s_1 \cdots s_n\rangle$, E_a acts like σ_a^+. BUT the modes associated with different sites do not commute, rather they anticommute $\{E_a, E_b\} = 0$. The trick to finding a matrix representation is to attach a string of Zs:

$$E_1 = \sigma_1^+ \tag{3.41}$$
$$E_2 = Z_1 \sigma_2^+ \tag{3.42}$$
$$E_3 = Z_1 Z_2 \sigma_3^+ \tag{3.43}$$
$$\vdots \qquad E_a = Z_1 Z_2 \cdots Z_{a-1} \sigma_a^+, a = 1..n \tag{3.44}$$

(In many-body physics this is called a Jordan-Wigner transformation, which in general relates a spin system to a fermionic system.) The hermitian generators are

$$T_{2a-1,2n+1} = \frac{1}{2} Z_1 \cdots Z_{a-1} X_a, \quad T_{2a,2n+1} = \frac{1}{2} Z_1 \cdots Z_{a-1} Y_a.$$

The general element of $\mathsf{SO}(2n+1)$ is then

$$T_{ij} = -\mathbf{i}[T_{i,2n+1}, T_{j,2n+1}], \quad i \neq j \neq 2n+1. \tag{3.45}$$

Here is an interesting question that we can now answer: For which groups $\mathsf{SO}(N)$ is the spinor representation real? Recall that a unitary rep R is not complex if $\exists S \in R^{\otimes 2}$ such that $T_A = -S T_A^\star S^{-1} = -S T_A^T S^{-1}$. I claim that the relevant S for the 2^n-dimensional spinor rep of $\mathsf{SO}(2n+1)$ is

$$S = S^{-1} = \prod_{a \text{ odd}}^{n} Y_a \prod_{b \text{ even}}^{n} X_b,$$

that is, this satisfies

$$T_{a,2n+1} = -S T_{a,2n+1}^\star S^{-1}, \; a = 1..n$$

and hence by (3.45) the same for the rest of the generators of $\mathsf{SO}(2n+1)$.

Recall that when such an S exists, when S is symmetric R is real (that is, there is a basis where the representation matrices are real), and when S is antisymmetric, R is pseudoreal.

n	S	symmetry of S
1	Y_1	AS
2	Y_1X_2	AS
3	$Y_1X_2Y_3$	S
4	$Y_1X_2Y_3X_4$	S

This pattern repeats mod 4. This gives the following for the spinor reps R of $\mathsf{SO}(N)$, N odd:

n mod 4	$2n+1$ mod 8	G	sym of S	R is
4	1	$\mathsf{SO}(8k+1)$	S	real
1	3	$\mathsf{SO}(8k+3)$	AS	pseudoreal
2	5	$\mathsf{SO}(8k+5)$	AS	pseudoreal
3	7	$\mathsf{SO}(8k+7)$	S	real

Now what about $\mathsf{SO}(2n)$, where the $\mathbf{2^n} = \mathbf{2^{n-1}_+} \oplus \mathbf{2^{n-1}_-}$ is reducible into eigenspaces of γ_F? A real rep of $\mathsf{SO}(2n+1)$ will be complex as a rep of $\mathsf{SO}(2n)$ if the intertwiner S fails to commute with γ_F. In terms of the matrix representation, γ_F acts as $\Gamma = \prod_a Z_a$.

$$S\Gamma S^{-1} = \underbrace{Y_1X_2Y_3X_4\cdots}_{n \text{ of these}} \underbrace{Z_1Z_2Z_3Z_4\cdots}_{n \text{ of these}} \underbrace{Y_1X_2Y_3X_4\cdots}_{n \text{ of these}} = (-1)^n \Gamma.$$

Here we used the fact that each X or Y anticommutes with Z: $XZ = -ZX, YZ = -ZY$, so we pick up n signs in moving them through Γ. Therefore, when n is odd, complex conjugation maps the odd chirality rep to the even chirality rep – each of them is complex.

G	n mod 4	$2n$ mod 8	$2n+1$ mod 8	S	R is
$\mathsf{SO}(8k+1)$	4		1	is Sym	real
$\mathsf{SO}(8k+2)$	1	2		takes R to $\bar{R}$	complex
$\mathsf{SO}(8k+3)$	1		3	is AS	pseudoreal
$\mathsf{SO}(8k+4)$	2	4		is AS	pseudoreal
$\mathsf{SO}(8k+5)$	2		5	is AS	pseudoreal
$\mathsf{SO}(8k+6)$	3	6		takes R to $\bar{R}$	complex
$\mathsf{SO}(8k+7)$	3		7	is Sym	real
$\mathsf{SO}(8k)$	4	8		is Sym	real

In the table I've set $8k+m = 2n$ or $2n+1$ (for m even and odd respectively).

The pattern repeats mod 8. This strange mod 8 behavior is called Bott periodicity.

A similar analysis applies to spinor reps of the Lorentz group in various dimensions.

[End of Lecture 19]

4 Brief encounters

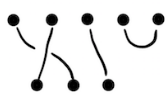
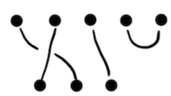
4.1 Tensor methods by diagrams

[Zee §V.2, Georgi 1st ed §X, Cvitanovic]

4.1.1 Diagrammatic methods, and SO(n) for non-integer n

[Much of the following discussion comes from this beautiful paper.] Consider, for amusement and edification, the following ... abstract situation (the actual word for it is *category*). You will see that it has strong parallels with the representation theory of a group. The *objects* (which are like the irreps) are collections of some number of points arranged in a row, so

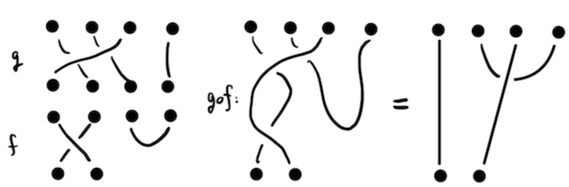
$$[0] = \emptyset, [1] = \bullet, [2] = \bullet\,\bullet, \cdots .$$

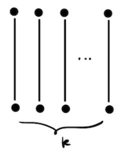
The *morphisms* – maps between the objects – (which are like invariant symbols) are string diagrams. So for example, an example of a morphism from [2] to [2] looks like:  .  is a morphism from [3] to [5]. (So we regard time as proceeding vertically.) Only the connectivity will matter:  =  . Moreover, we're not going to care about whether the strands go over or under each other (that's another kettle of worms, relevant to the study of anyons). We can make a vector space out of these little monsters, so we allow linear combinations, like

$\frac{1}{2}$  $+\frac{1}{2}$  $-$ 

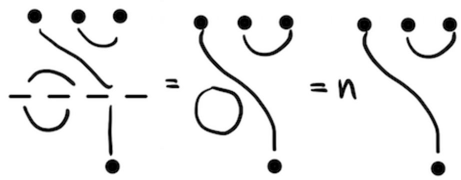
(I could put the diagrams inside kets, but I will not. Pretend I did if you prefer.) So for example, the space of morphisms from $[2] \to [2]$ is a 3d vector space.

Moreover, there is a product on this vector space (so it is an algebra, called the *Brauer algebra*) which is just composition – we stack the diagrams on top of each other and get a new one:

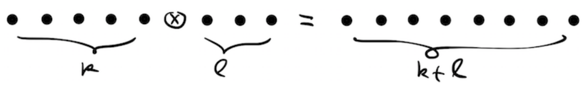


The product has an identity operation, which is the string with parallel strands,  Any morphism $[0] \to [0]$ is just a number.

One more rule: if we make a closed loop when stacking, we replace it by the number n:



Tensor products. We can take tensor products of the objects: $[k] \otimes [\ell] \equiv [k+\ell]$



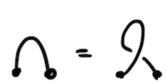
The analog of the trivial rep, which has $\mathbf{1} \otimes \mathbf{a} = \mathbf{a}$ for all $\mathbf{a}$ is $[0]$. We can also take tensor product of the morphisms – just put the diagrams next to each other.

Diagrammatic notation for $\mathsf{O}(n)$. Recall that $\mathsf{O}(n)$ is the maximal group preserving the invariant symbol (or tensor) $\delta_{IJ} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Draw this symbol as  . Here we've regarded it as an invariant symbol on $\mathbf{n} \otimes \mathbf{n}$, a singlet in two copies of the defining, vector representation. It is an invariant symbol because of this special property of $\mathsf{O}(n)$; in particular the $\mathbf{n}$ is a real representation $\mathbf{n} \simeq \bar{\mathbf{n}}$. We can use the same object as an invariant symbol in some other ways: It's also the identity operator on the the $\mathbf{n}$

$$\delta_I^J : \mathbb{R}^n \to \mathbb{R}^n$$ 

$$\delta^{IJ} : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n$$ 

It satisfies some identities:

$$\delta_{IJ} = \delta_{JI}$$ 

$$\delta_{IJ}\delta^{JI} = \delta_{II} = n$$  $= n$

So maybe you believe me that the diagrammatic notation has something to say about reps of $\mathsf{O}(n)$. There are three important questions I haven't answered yet: how do we take direct sums? What's an irrep? And what is the dimension of a representation in this language? Recall that a unitary representation, such as $\mathbf{n}^{\otimes k}$, can be completely decomposed as

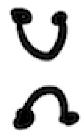
$$R = \oplus_{\text{irreps},a} (\mathbf{a} \oplus \mathbf{a} \oplus \cdots) = \oplus_{\text{irreps},a} V_a \otimes \mathbf{a}$$

some number of copies of each irrep. Moreover, this decomposition means that the identity on R has a decomposition

$$\mathbb{1}_R = \sum_{a,i_a} P_{\mathbf{a},i_a}, \qquad P_{\mathbf{a},i_a} P_{\mathbf{b},j_b} = \delta_{ab}\delta_{ij} P_{\mathbf{a},i_a}$$

where $i_a = 1 \cdots \dim V_a$. So the decomposition into irreps is accomplished by this resolution of the identity by (as many as possible) orthogonal projectors.

Let's look at the vector space of morphisms from $[2] \to [2]$. As we saw above, there are three linearly-independent diagrams:

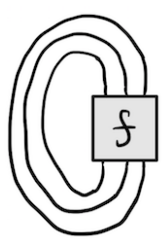
$T_1 \equiv$  , $T_2 \equiv$  , $T_3 \equiv$  .

It is a fun exercise to check that

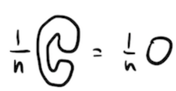
$$P_1 = T_3/n, \quad P_2 = \frac{1}{2}(T_1 + T_2) - T_3/n, \quad P_3 = \frac{1}{2}(T_1 - T_2)$$

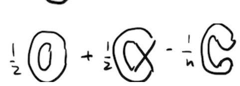
are orthogonal projectors $P_i P_j = \delta_{ij} P_j$. And since there are three of them this is a maximal set. This should look familiar: we saw above that $\mathbf{n}^{\otimes 2} = \mathbf{1} \oplus \text{Sym}^2_{\text{traceless}}\mathbf{n} \oplus \Lambda^2\mathbf{n}$.

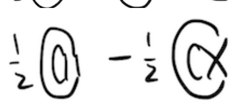
What about the dimensions of the irreps? Once we have the projectors, it's easy: $\dim R_a = \text{tr} P_a$. For a morphism $f : [k] \to [k]$, the trace is defined as

$\text{tr} f \equiv$ 

So

$\dim R_1 = \text{tr} P_1 =$  $= 1,$

$\dim R_2 = \text{tr} P_2 =$  $= \dfrac{n(n+1)}{2} - 1,$

$\dim R_3 = \text{tr} P_3 =$  $= \dfrac{1}{2} n(n-1)/2,$

just as we found before.

The identity operator $\mathrm{id}_{[k]}$ for any k has a unique such decomposition (up to re-labellings, just like irreps). (This is a theorem of our colleague Wenzl in the math department.)

Notice that these statements are true even if n is not an integer! The representation theory of $\mathsf{O}(n)$ makes perfect sense for any real number n. There are a few funny things as we vary n. One is that sometimes we'll find that the dimension is zero. You can see this in the formulae above for $\mathrm{tr} P_3 = n(n-1)/2$ (which vanishes for $n = 0, 1$) and $\mathrm{tr} P_2 = (n+2)(n-1)/2$ (which vanishes for $n = 1, -2$). This is not a problem, it just means that there is no such representation (there's no AS tensor with zero or one index, and there's no traceless symmetric tensor with one index).

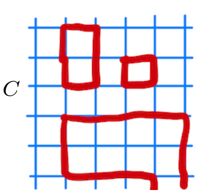
Here's a motivation for caring about $n \in \mathbb{R} \setminus \mathbb{Z}$. Consider the $\mathsf{O}(n)$ model of a magnet with n components. It has many microscopic realizations; importantly, the microscopic details don't matter for the universal physics. One way to construct its partition function is to integrate over n-component unit vectors $s^i(x)$ at each point in a spatial lattice:

$$Z = \prod_x \int ds^i_x e^{-\beta \sum_{\langle x,y\rangle} \vec{s}_x \cdot \vec{s}_y} \simeq \prod_x \int ds^i_x \prod_{\langle xy \rangle} \left(1 + \kappa s^i_x s^i_y\right). \tag{4.1}$$

Here $\langle x, y\rangle$ denotes a pair of neighboring points. It is $\mathsf{O}(n)$ invariant ($s^i_x \mapsto R^i_j s^j_x$) since the indices are contracted with δ_{ij}.

In the second step I appealed to universality to rearrange things in a convenient way, so that we can now do the integrals over the spins in (4.1)[43]:

$$Z = \sum_{\text{collections of closed loops, } C} n^{\#\text{ of components of } C} \tilde{\kappa}^{\text{total length of } C} \tag{4.2}$$

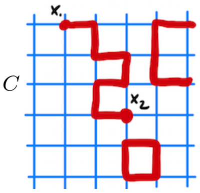


(where $\tilde{\kappa} \propto \kappa$). If we did the same manipulation for a correlation function, we would

[43] I used the following integration table for the integral over s on the unit sphere

$$\int ds 1 = 1, \quad \int ds s^i = 0, \quad \int ds (s^i)^2 = \frac{1}{4\pi} \int_0^\pi \cos^2\theta \sin\theta = \frac{1}{6\pi}.$$

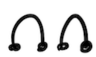
find, *e.g.*

$$\langle s_{x_1} s_{x_2} s_{x_3} s_{x_4} \rangle = \sum_{\text{collections of loops, } C_x} n^{\#\text{ of components of } C} K^{\text{total length of } C}$$ 

where now the loops $C_{x,i}$ must end at the points x_a (and the indices must match up).

Here's one reason this rewriting (which is called a loop model) is useful: in the representation (4.2), there is no need for n to be an integer. It's just a coupling constant. For example, when we set $n \to 0$, we disallow all loops – this produces a sum over self-avoiding random walks.

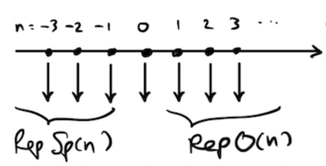
So it is interesting to ask about what is the *symmetry* of such a model with $n \notin \mathbb{Z}_+$. And actually, for most purposes of doing physics, we don't really care about the group itself, we care about its representation theory. For example, in field theory (such as the field theory description valid near the critical points of (4.1)), the fields transform in irreps (which generalize to 'objects'), and their correlation functions are linear combinations of invariant symbols of those irreps (which generalize to 'morphisms'). So for example, the four-point function above is

$$\langle s_{x_1} s_{x_2} s_{x_3} s_{x_4} \rangle = c_1(x_1 \cdots x_4)$$  $$+ c_2(x_1 \cdots x_4)$$  $$+ c_3(x_1 \cdots x_4)$$ 

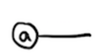
where the coefficients c_i are ordinary functions. So we can regard it as a morphism from $[4] \to [0]$. To get a number out, we can contract with a morphism from $[0] \to [4]$, so for example

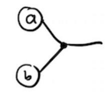
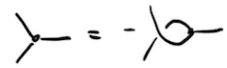
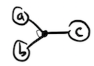
$$\langle s_{x_1} s_{x_2} s_{x_3} s_{x_4} \rangle \circ$$  $$= (n^2 c_1 + n c_2 + n c_3) \cdot (\text{empty diagram}).$$

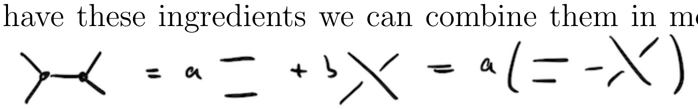
What about $n < 0$? Here comes a real shocker: the things we find in this way for representations $\mathsf{O}(n)$ with $n < 0$ reproduce exactly the structure of $\mathsf{Sp}(|2n|)$! The idea is that taking $n \leftrightarrow -n$ interchanges symmetrization and antisymmetrization (consider for example the formulae for $\dim \Lambda^2 \mathbf{n} = \frac{n(n-1)}{2}$ and $\dim \mathrm{Sym}^2 \mathbf{n} = \frac{n(n+1)}{2}$). Recall that the adjoint of $\mathsf{O}(n)$ is $\Lambda^2 \mathbf{n}$, while the adjoint of $\mathsf{Sp}(|2n|)$ is the symmetric combination.



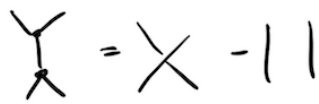
4.1.2 The epsilon tensor

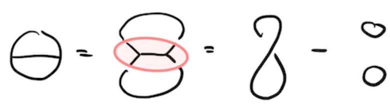
Let's think about the case of $\mathsf{SO}(3)$ for a moment – ordinary vectors in 3-space. Denote a vector $\vec{a}$, with components a_i by  . So the inner product $\vec{a} \cdot \vec{b} = a_i b_i = a_i b_j \delta^{ij}$

is  . In the special case of SO(3) (not O(3)), there is another invariant tensor, namely the Levi-Civita symbol. The cross product $(\vec{a} \times \vec{b})_i = \epsilon_{ijk} a_j b_k$ is  . That the epsilon tensor is antisymmetric can be written like  . Notice that this notation makes it manifest that $(\vec{a} \times \vec{b}) \cdot \vec{c} =$  is cyclically symmetric.

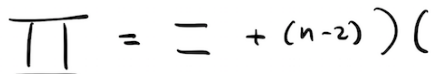
Once we have these ingredients we can combine them in more complicated ways. For example,  One way to know the first step must be true is that there are only so many indices, and the external indices on the left must be chosen from the same subset of two as those on the right. Alternatively, it is because there are no other invariant tensors. The second step follows by antisymmetry of epsilon. By computing one component we get $a = -1$. This is the identity $\epsilon_{ijm}\epsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$.

Other identities that look complicated in components follow easily:

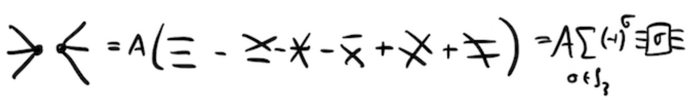


 $= 3 - 3^2 = -6.$

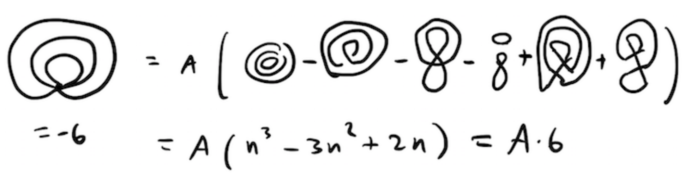
 $= (1-3)$— $= -2$—

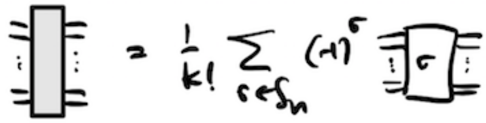


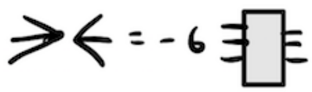
An interesting one is:



This follows since the indices on the left must be a permutation of those on the right, and the result is determined just by the sign of the permutation. To determine the coefficient, take the trace of the BHS



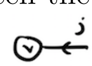
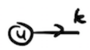
to find $A = -1$. In terms of  we've shown that

 .

Here is a useful perspective on the epsilon tensor in $\mathsf{SO}(3)$: recall that it is the matrix elements of the generators of the fundamental representation of $\mathsf{SO}(3)$ (up to a factor). So we can regard two of the lines coming out of it as indices in the fundamental, and the third line as labelling which generator it is. It is an invariant tensor connecting two copies of the fundamental with the adjoint. In this special case all three of these representations are the same, but we'll see next that the same idea applies in other cases where they are not.

4.1.3 Diagrams for $\mathsf{SU}(n)$

These methods are very flexible. In fact, the lines can represent states in any representation of any group. Tensors that connect the lines are invariant tensors.

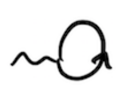
[A nice reference is this one by Stefan Keppeler] For example, the same methods work for $\mathsf{SU}(n)$ just by making the strings oriented, to distinguish between the $\mathbf{n}$ and the $\bar{\mathbf{n}}$ – that is, just draw arrows on the lines. A vector v in the $\mathbf{n}$ is $v^j =$  and a vector u in the $\bar{\mathbf{n}}$ is $u_k =$ . Here I distinguish between upper and lower indices for the first time. The arrows point away from the upper indices. The inner product (the singlet in $\mathbf{n} \otimes \bar{\mathbf{n}}$) is $u \cdot v = u_k v^k = u_k \delta^k_j v^j =$ . The invariant tensor we used here is $\delta^j_k =$ .

Another representation of the same group can be distinguished by a different kind or color of line. For example, we can denote a state in the adjoint representation with a wiggly line:  This is a real representation so there is no arrow.

Another invariant tensor that we can't escape is the generators of the representation:

$$(T^A)^j_k =$$ 

The fact that the generators of $\mathsf{SU}(n)$ are traceless is this picture:

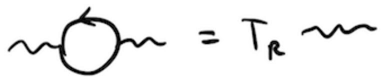
$$\mathrm{tr} T^A = (T^A)^j_j = \quad (4.3)$$ 

The commutator is

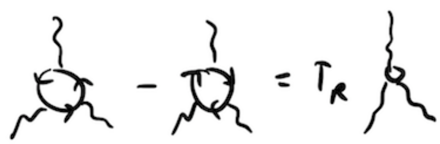
$$[T^A, T^B]^j_k = \mathbf{i} f^{ABC} \left(T^C\right)^j_k =$$  .

This produces a new invariant tensor  in the $\mathbf{adj}^{\otimes 3}$, which is completely antisymmetric, the tensor of structure constants, also known (up to a factor) as the generators of the adjoint rep.

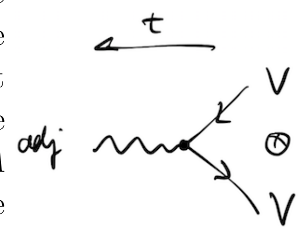
The normalization of the generators is

$$\mathrm{tr} T^A T^B = \left(T^A\right)^j_k \left(T^B\right)^k_j =$$  . (4.4)

Contracting both sides of (4.5) with T^C and taking trace, we have

 . (4.5)

How could we have discovered T^A?
It's a theorem that for every rep V, the adjoint appears in the decomposition of $V \otimes \bar{V}$. into irreps. In fact this statement can be understood diagramatically quite easily: it's just the statement that the generators of the Lie algebra $(T^A)^b_a$ in any rep V comprise an invariant tensor. Here a is a V index, b is a $\bar{V}$ index, and A labels the generator, *i.e.* A is an **adj** index. So this tensor can be read as a map from $V \otimes \bar{V} \to \mathbf{adj}$.



Decompose $\mathbf{n} \otimes \bar{\mathbf{n}}$ into irreps by decomposing $\mathbb{1}_{\mathbf{n}\otimes\bar{\mathbf{n}}} = \sum_a P_a$ into orthogonal projectors. Here's one[44]: $P_1 = c)($. It's hermitian and satisfies $P_1^2 = c^2)()(= nc^2 P_1$ from which we infer $c = 1/n$. Cvitanovic thinks of the equation $)()(= n)($ as an eigenvalue equation for the operator $)($; one of the eigenspaces of an invariant tensor such as $)($ on $V \otimes \bar{V}$ must be the adjoint rep.

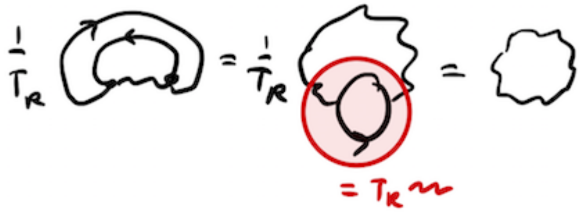
In the case of $\mathsf{SU}(n)$ in fact the rest of the representation is the image of $P_{\mathrm{adj}} =$ c  . This satisfies

$$P^2_{\mathrm{adj}} = c^2$$  $$= c T_R P_{\mathrm{adj}}$$

[44] Note that time is going to the left here.

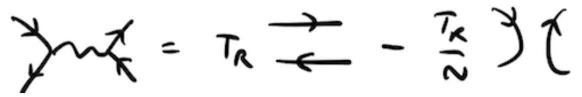
from which we infer $c = 1/T_R$. In components, this is the awful equation $(P_{\text{adj}})^{j\ell}_{km} = \left(T^A\right)^j_k \left(T^A\right)^\ell_m / T_R$. These two projectors are orthogonal $P_1 P_{\text{adj}} = 0$, as a consequence of (4.3).

What are the dimensions of the associated representations? $\dim R_1 = \text{tr} P_1 = 1$.

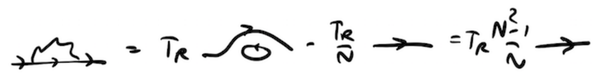
$\dim R_{\text{adj}} = \text{tr} P_{\text{adj}} =$  $= n^2 - 1.$

So we learn that  .

In practice, this identity is most useful in the form

 (4.6)

For example, suppose we want to know $c_2(\square)$, the second Casimir for the fundamental representation. This is

$$\left(T^A T^A\right)^j_\ell = \left(T^A\right)^j_k \left(T^A\right)^k_\ell \overset{\text{Schur}}{=} c_2(\square)\delta^j_\ell =$$ 

It's a little harder to compute $c_2(\text{adj})$

 $= c_2(\text{adj})$ 

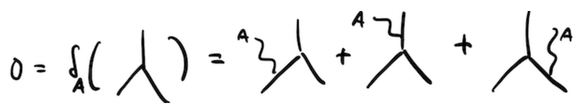
(to get rid of the internal gluon lines, first use (4.5) and then use (4.6), then use (4.4)) but the answer is $c_2(\text{adj}) = 2T_R n$.

For $\mathsf{SU}(n)$ (as opposed to $\mathsf{U}(n)$), there is another invariant tensor, namely a vertex with n lines coming out (and its conjugate which has n lines going in) associated with the $\epsilon_{i_1 \cdots i_n}$ symbol.

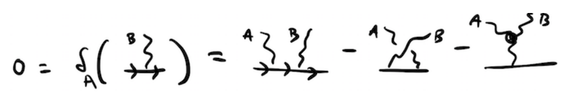
Now I have to make a confession. These diagrams we've been drawing look a lot like Feynman diagrams for a gauge theory. The straight lines for the fundamental or antifundamental look like charged fermion lines (like electrons or quarks) and the wiggly lines for the adjoint look like gauge boson lines (like photons or gluons). This not an accident. In perturbative quantum field theory (QFT), we associate an amplitude (the probability amplitude for the process of which the diagram is a cartoon) to such diagrams. The values of the diagrams we've been drawing give the *color factors* in QCD

Feynman diagrams (if you set $n = 3$). The rest of the amplitude in the actual QCD Feynman diagram comes from spacetime stuff. Actually, it is an interesting question to ask to what extent the rest of the stuff in a Feynman diagram in a relativistic QFT can also be regarded as a birdtrack diagram for the Poincaré group. (!)

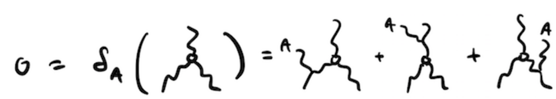
A comment about invariant tensors. An invariant tensor d (with say 3 indices for definiteness) satisfies $d^{abc}v_a w_b x_c = d^{abc}v'_a w'_b x'_c$ where $v'_a = U_a^{a'} v_{a'}$ etc is how v transforms under G. The infinitesimal version of this, with $U = e^{\mathbf{i}\theta^A T^A}$ (in whatever rep v is in, and similarly for w and x), says that



where as above the wiggly lines are insertions of the generators (again, in whatever rep the legs are in). You can check that T^A_{ij} (the generators in any rep) comprise an invariant tensor by this definition[45]:



is just the Lie algebra itself. The condition that the structure constants f_{ABC} are an invariant tensor



is the Jacobi identity. For more on this see the book by Cvitanovic. He takes this point of view quite far. For example, demanding that the only 'primitive' invariant tensors are δ_i^j and d_{ijk} a 3-index antisymmetric tensor (primitive means that making new tensors by products of these (like $d_{ijk}d^{jk\ell} = c\delta_i^\ell$) must be proportional to these again) he derives the representations of G_2. A good brief account is his strangely-titled summary Tracks, Lie's and Exceptional Magic where he derives the dimensions of reps of E_6 from the assumption of a 3-index symmetric tensor.

4.1.4 Identical particles and Young diagrams for $\mathsf{SU}(n)$

Wave functions of identical particles provide a deep connection between the irreps of the (finite) symmetric group S_n and certain Lie groups. The idea is simple. Suppose we have a particle in the fundamental representation of $\mathsf{SU}(n)$, $|\psi\rangle \in \text{span}\{|i\rangle\,,\ i = 1..n\} \equiv \square$,

$$|\psi\rangle \mapsto D(g)\,|\psi\rangle\,.$$

[45] About the signs: the minus in front of the second term is because the generator of $\bar{R}$ is $-(T_R^A)^\star$. The minus in front of he last term is because the generator of the adjoint is $(T^A_{\mathbf{adj}})_{BC} = -\mathbf{i}f_{ABC}$.

Forget about the position degree of freedom, which plays no role here. Now suppose we have instead k such particles. If the particles are distinguishable, the state of the system forms a tensor product representation of $\mathsf{SU}(n)$

$$\square^{\otimes k} \ni |\psi_1\rangle \otimes \cdots \otimes |\psi_k\rangle \mapsto D(g)\,|\psi_1\rangle \otimes \cdots \otimes D(g)\,|\psi_k\rangle \equiv D^{(k)}(g)\,|\psi_1\rangle \otimes \cdots \otimes |\psi_k\rangle\,.$$

The very existence of indistinguishable particles tells us two things: this representation is reducible, and the symmetrized or antisymmetrized states live in invariant subspaces.

Notice that this tensor product space $\square^{\otimes k}$ *also* carries a representation of S_k, by

$$R(\pi)\,|\psi_1\rangle \otimes \cdots \otimes |\psi_k\rangle = |\psi_{\pi_1}\rangle \otimes \cdots \otimes |\psi_{\pi_k}\rangle\,.$$

Furthermore, $[R(\pi), D^{(k)}(g)] = 0$ for all g, π. So Schur's lemma says $R(\pi)$ is a constant on irreps of G and $D^{(k)}(g)$ is a constant on irreps of S_k. This means that irreps of S_k in $\square^{\otimes k}$ are compatible with irreps of $\mathsf{SU}(n)$. This gives a correspondence, then, between Young diagrams and irreps of $\mathsf{SU}(N)$.

The simplest example is the product of two qubits (spin-half particles) $\mathbf{2}\otimes\mathbf{2} = \mathbf{1}\oplus\mathbf{3}$. $P_{\yng(1,1)}(V_\mathbf{2} \otimes V_\mathbf{2}) = V_1, P_{\yng(2)}(V_\mathbf{2} \otimes V_\mathbf{2}) = V_\mathbf{3}$.

Only tableaux with n or fewer vertical boxes can appear as representations of $\mathsf{SU}(n)$.

What is the dimension of the representation of $\mathsf{SU}(n)$ labelled by a given Young diagram? It is *not* the same as the dimension of the corresponding representation of S_k! Rather it is the number of ways of placing numbers from the set $\{1 \cdots n\}$ in the diagram, while preserving the order in rows, and strict order in columns. For example, for $\mathsf{SU}(3)$,

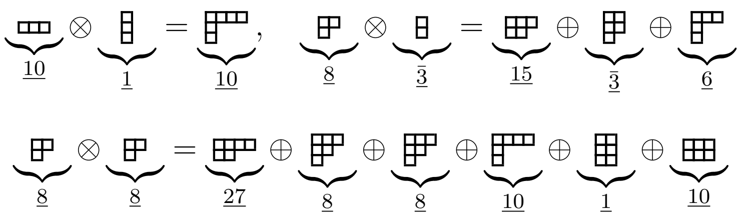
$$\young(32,1),\ \young(33,1),\ \young(32,2),\ \young(33,2),\ \young(31,1),\ \young(31,2),\ \young(22,1),\ \young(21,1)$$

shows that $\yng(2,1)$ is the $\mathbf{8}$ of $\mathsf{SU}(3)$.

More generally, the formula for the dimension of R_λ as a rep of $\mathsf{SU}(n)$ is given by the *factors over hooks rule*: $\dim(R_\lambda) = \frac{f_\lambda}{h_\lambda}$. The denominator is the product of the hooks, which determines the dimension of the corresponding rep of S_k (where k is the number of boxes) – recall that in terms of h_λ, this is $\dim R_\lambda^{S_k} = \frac{k!}{h_\lambda}$. The numerator is obtained by placing an n in the top right box, and placing a number one larger in each box moving right, and one smaller in each box moving down; For example, for $\mathsf{SU}(2)$: $\young(234,12)$; for $\mathsf{SU}(3)$: $\young(345,23)$. Then f_λ is the product of these numbers. The examples give dimensions 2 and 15, respectively, for the associated representation. Notice that you automatically get zero if you try to stack more than n boxes. (A proof of this statement can be found in §6.2 of Fulton and Harris.)

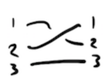
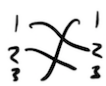
Multiplying irreps. The secret to using Young diagrams to decompose tensor product representations is the following: Put the bigger diagram on the left. Now stick the boxes of the right one onto the left one in every way that does not explicitly violate the symmetry of the original diagram[46].

If you get a stack of n boxes in a column, you can erase them (since that's a singlet). If you get a stack of more than n boxes in a column, that's nothing, it doesn't contribute. Always check that the dimensions add up. For example, in $\mathsf{SU}(3)$,

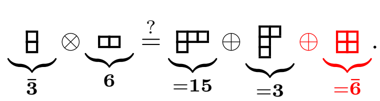


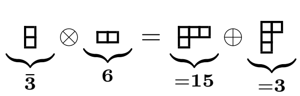
Now you can check my claim about the dimension formula by comparing $\yng(2) \otimes \yng(1) = \yng(2,1) \oplus \yng(3)$ – for general n, the LHS has dimension $\frac{n(n-1)}{2} + n$. The RHS has dimension $\dim R_{\yng(2,1)} + \frac{n(n-1)(n-2)}{6}$ which gives $\dim R_{\yng(2,1)} = \frac{n(n-1)(n+1)}{3}$. This agrees with $f_{\yng(2,1)} = n(n+1)(n-1)$ and $h_{\yng(2,1)} = 3$.

Young projectors with birdtracks and Schur-Weyl duality. Birdtracks can represent *either* representation theory of a Lie group *or* the symmetric group. Using this fact, we can make explicit representations of the *Young projectors* (which accomplish the antisymmetrization along columns and symmetrization along rows of a given Young diagram) using birdtracks.

By the above remark, note that birdtracks provide a notation for the group algebra of the permutation groups: For example (12) in S_3 is (with time going to the left)  , and (123) is  . We can compose permutations by stacking the birdtracks (now I am composing them from right to left instead of vertically to keep

[46] Georgi gives a method to implement this constraint without overcounting: when multiplying tableaux A and B, put 1s in the top row of B, 2s in the second row of B and so on. When you attach the boxes of B to A, do it row by row, top row first. Then only keep diagrams where: reading from right to left and top to bottom, the number of 1s is greater than or equal to the number of 2s, which are in turn greater than or equal to the number of 3s, and so on. But this rule actually fails to exclude some things. For example (the numbers below the diagrams are for $\mathsf{SU}(3)$, for example):

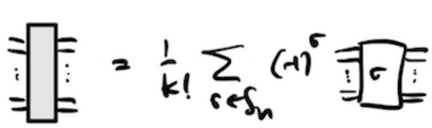


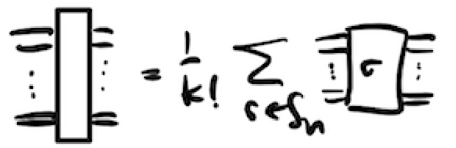
(The correct answer is  .) As you can see by adding up the dimensions, the last term should be absent; this is because the two boxes we're adding are symmetrized, and therefore cannot be stacked on top of each other. But using Georgi's rule it seems to me they would both have 1s in them and would be allowed. Perhaps I've misunderstood his rule.

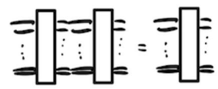
you on your toes):

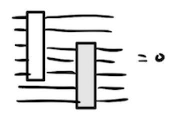
$$(12)(123) = $$  $$= (13).$$

And this is a notation for the group *algebra* because we can consider formal combinations of the diagrams with arbitrary coefficients, as we've been doing. An important set of operators in this algebra are the symmetric and antisymmetric projectors:





As you can check, they are projectors:  (and similarly for the AS one).

Concatenating two of these, we get zero if two or more legs overlap:  .

Recall from §2.6 that Young diagrams with k boxes (*i.e.* partitions of k, which I'll denote $\lambda \vdash k$) are in one-to-one correspondence with irreps of S_k. And recall that the Young tableaux (obtained by filling in the diagrams with numbers $1 \cdots k$, increasing to the right and downward) provide a basis for the associated irrep. Now we can associate to each Young tableau[47] $\hat{\lambda}$ an operator $Y_{\hat{\lambda}}$ in the group algebra of S_k where k is the number of boxes:

$$Y_{\hat{\lambda}} \equiv \frac{1}{h_\lambda} s_{\hat{\lambda}} a_{\hat{\lambda}} \tag{4.7}$$

where $s_{\hat{\lambda}}$ is the row symmetrizer, $a_{\hat{\lambda}}$ is the column antisymmetrizer, and the denominator is the product of hooks of the diagram λ. For example, $Y_{\square\square\square}$ is just the symmetric projector on 3 lines, and $Y_{\boxminus}$ is just the antisymmetric projector on 3 lines. For these I didn't have to specify the numbers in the boxes because there was only one way to do it – these are associated with the one-dimensional irreps of S_3. A more nontrivial example is the diagram ⊞ which labels the **2** of S_3, so there are two tableaux, and so two projectors:

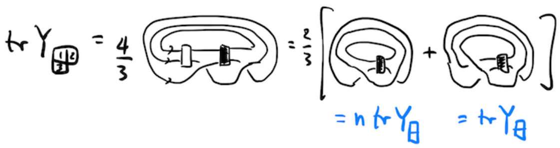
[47] I put a hat to distinguish the tableau from the diagram – removing the hat forgets the numbers in the boxes.

Either of them projects onto a copy of the $\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}$ rep of $\mathsf{SU}(n)$. Note that if we switched the order of a and s in (4.7) it would just mix up which tableau maps to which copy.

So I claim that the operator defined this way is a projector $Y_{\hat{\lambda}} : \mathbf{n}^{\otimes k} \to \mathbf{n}^{\otimes k}$ into the associated irrep of $\mathsf{SU}(n)$. For different Young diagrams, $\lambda \neq \mu$ the projectors are orthogonal $Y_{\hat{\lambda}} Y_{\hat{\mu}} = 0$ (for any choice of tableaux). (Within the subspace associated with a given Young diagram, the projectors associated with different tableaux are not necessarily orthogonal.) And I claim that if you compute the dimension of the image $\mathrm{tr}Y_{\hat{\lambda}}$, you'll get the factors-over-hooks rule. For example,

$$= \frac{1}{2}(n^2+n) = \frac{n(n+1)}{2} \quad \checkmark.$$

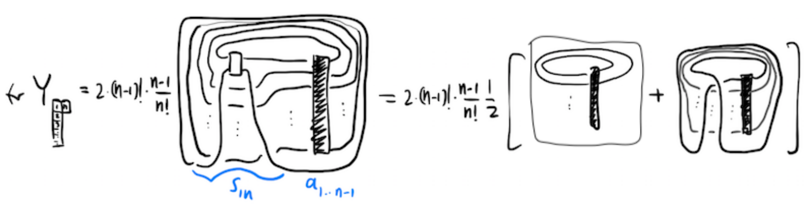
Similarly, $\mathrm{tr}Y_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{n(n-1)}{2}$.



$$= \frac{2}{3}\left(n\mathrm{tr}Y_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + 1 \cdot \mathrm{tr}Y_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right) = \frac{n(n^2-1)}{3} \quad \checkmark.$$

For the case of $n = 3$, this is the 8-dimensional adjoint rep.

For any n, we get a copy of the adjoint of $\mathsf{SU}(n)$ as the image of $Y_{\widehat{\begin{smallmatrix}\square&\square\\\square\\\square\\\square\end{smallmatrix}}}$, for any choice of tableau with this shape. You can check that the trace of this operator is indeed $n^2 - 1$:



$$= \frac{2(n-1)}{n}\frac{1}{2}[n\mathrm{tr}a_{n-1} + \mathrm{tr}a_{n-1}]$$

where $a_k \equiv a_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ (k boxes) is the totally antisymmetric projector on k lines. From here we can use $\mathrm{tr}_n a_k = \binom{n}{k}$, which is derived on the homework.

There are $\dim R_\lambda^{S_k}$ different tableaux for the given diagram λ. Specifying the tableau $\hat{\lambda}$ says *which* of the copies of the irrep of $\mathsf{SU}(n)$ associated to the diagram λ we pick out. This shows that as a representation of $\mathsf{SU}(n) \times S_k$,

$$\mathbf{n}^{\otimes k} = \bigoplus_{\lambda \vdash k} R_\lambda^{\mathsf{SU}(n)} \otimes R_\lambda^{S_k} \tag{4.8}$$

(where $\lambda \vdash k$ means λ is a tableau with k boxes (*i.e.* it specifies a partition of k) and in particular taking dimensions of both sides,

$$n^k = \sum_{\lambda \vdash k} \dim R_\lambda^{\mathsf{SU}(n)} \dim R_\lambda^{S_k}.$$

These statements are associated with the names Schur and Weyl, and the beautiful equation (4.8) is called Schur-Weyl duality. I recently encountered some of its applications in quantum information theory, here and here.

4.2 Group integration and characters

Earlier I said that all the statements we proved by averaging over a finite group would also be true for compact groups, by replacing $\frac{1}{|G|}\sum_{g\in G}...$ with $\int_G dg$ in the proofs.

Let's think about $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$ for example. The character of some element g in the spin-j representation is $\chi_j(g) = \mathrm{tr}_{V_j} D(g)$. Now recall that conjugation changes the axis but not the angle of a rotation. Since the character is a class function, we can just evaluate on the Cartan subgroup where $D(\psi) = e^{\mathbf{i}\psi J_z}$. We get

$$\chi_j(\psi) = \sum_{m=-j}^{j} \langle j,m| \, e^{\mathbf{i}\psi J_z} \, |j,m\rangle = \sum_{m=-j}^{j} e^{\mathbf{i}\psi m}.$$

We can do some manipulations to write this compactly as

$$\chi_j(\psi) = \frac{\sin\left(j+\frac{1}{2}\right)\psi}{\sin\frac{\psi}{2}}$$

but actually the first expression is more useful. The fact that the coefficient of each term is 1 expresses the fact that there is no nontrivial multiplicity. Checks: $\chi_j(\mathbb{1}) = 2j+1 = \dim V_j$, $\chi_0(\psi) = 1$. These formulae work just as well for $j \in \mathbb{Z}/2$.

The utility of characters rests on their orthogonality with respect to the inner product:

$$\langle \chi_1, \chi_2 \rangle \equiv \frac{1}{|G|}\sum_g \bar{\chi}_1(g)\chi_2(g) \rightsquigarrow \int_G d\mu(g)\bar{\chi}_1(g)\chi_2(g).$$

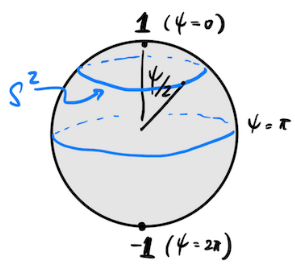
The crucial step in the proofs (that any rep was unitary and of the grand orthogonality theorem) is that $\Lambda_X \equiv \int d\mu(g) D^\dagger(g) X D(g)$ (for any matrix X of the right shape) is an intertwiner: $D^\dagger(h)\Lambda_X D(h) = \Lambda_X$:

$$D^\dagger(h)\left(\int d\mu(g) D^\dagger(g) X D(g)\right) D(h) = \int d\mu(g) D^\dagger(gh) X D(gh) = \int d\mu(kh^{-1}) D^\dagger(k) X D(k) \stackrel{!}{=} \Lambda_X.$$

This will be true if $d\mu(kh^{-1}) = d\mu(k)$ for all $h \in G$. The idea is that the measure must be independent of where we are in the group. The measure with this property $d\mu(g) = d\mu(g')$ is called the *Haar* measure, or group-invariant measure.

Some examples: for $\mathsf{SO}(2) = \mathsf{U}(1)$, the Haar measure is just $d\mu(e^{\mathbf{i}\theta}) = d\theta$. It is invariant under $\theta \to \theta + \delta\theta$, and the group is compact because $\theta \in [0, 2\pi)$. For $\mathsf{SO}(1,1)$, the Lorentz group in $1+1$ dimensions, the measure is $d\eta$, where η is the rapidity, the boost parameter, and this is noncompact since $\eta \in \mathbb{R}$.

What about $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$? Recall that the $\mathsf{SU}(2)$ group manifold is the 3-sphere: $U = w\mathbb{1} + \mathbf{i}\vec{x}\cdot\vec{\sigma} \in \mathsf{SU}(2)$ iff $1 = w^2 + \vec{x}^2$. In terms of the axis-angle parameterization, $U = e^{\mathbf{i}\vec{\psi}\cdot\frac{\vec{\sigma}}{2}} = \cos\frac{\psi}{2} + \mathbf{i}\hat{\psi}\cdot\sigma\sin\frac{\psi}{2}$ (with $\vec{\psi} \equiv \psi\hat{\psi}$), so we can identify $w = \cos\frac{\psi}{2}, \vec{x} = \sin\frac{\psi}{2}\hat{\psi}$. The group-invariant measure is just the round measure on the 3-sphere[48]. Therefore the measure is $\sin^2\frac{\psi}{2}d\psi\sin\theta d\theta d\varphi$, where θ, φ are polar coordinates for $\hat{\psi}$. A class function is independent of θ and φ, so we can forget that part.



Now about the range of ψ. Notice that as $\psi \in [0, 2\pi)$, $w = \cos\psi/2$ takes its full range from 1 to -1, and we cover the whole 3-sphere. This is the $\mathsf{SU}(2)$ manifold. The element $U(\vec{\psi})$ of $\mathsf{SU}(2)$ maps the same element of $\mathsf{SO}(3)$ as $-U(\psi, \hat{\psi}) = U(\psi + 2\pi, \hat{\psi})$. (Recall that if $X = \vec{x}\cdot\vec{\sigma}$, then $U^\dagger X U = \vec{x}'\cdot\sigma$ is a rotation: $x'_i = R_i^j x_j$ has the same length $x^2 = X^2$ – but it's a rotation by angle ψ, not $\psi/2$.) Therefore, the points $\psi = 0$ and $\psi = 2\pi$ are identified in $\mathsf{SO}(3)$, and we need only integrate $\psi \in [0, \pi)$ to cover the $\mathsf{SO}(3)$ group manifold. (Note that $\psi = \pi$ is not a boundary in $\mathsf{SO}(3)$; rather, the range $\psi \in [0, \pi)$ is a fundamental domain for the action of the $\mathbb{Z} : U \to -U$ by which we must quotient to get $\mathsf{SO}(3)$.)

You can check that in $\mathsf{SO}(3)$,

$$\langle\chi_j|\chi_{j'}\rangle = \int d\mu(g)\bar{\chi}_j(g)\chi_{j'}(g) \tag{4.9}$$

$$= c\int_0^\pi d\psi\sin^2\psi/2\frac{\sin(j+\frac{1}{2})\psi}{\sin\frac{\psi}{2}}\frac{\sin(j'+\frac{1}{2})\psi}{\sin\frac{\psi}{2}} \tag{4.10}$$

$$= c\int_0^\pi d\psi\sin(j+\frac{1}{2})\sin(j'+\frac{1}{2}) = c\frac{\pi}{2}\delta_{jj'} \tag{4.11}$$

so we should choose the constant normalizing the measure as $c = \frac{2}{\pi}$. For $\mathsf{SU}(2)$ we would integrate $\int_0^{2\pi} d\psi$ and hence the constant would differ by a factor of two.

[48] Notice that the group manifold $\mathsf{SU}(2)$ has *two* $\mathsf{SU}(2)$ symmetries, namely $h \to g_L h g_R^{-1}$, left-action and right action. These combine to form $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R = \mathsf{SO}(4)$, the symmetry of the 3-sphere, under which $(w, \vec{x})$ is in the fundamental $\mathbf{4} = (\mathbf{2}_L, \mathbf{2}_R)$ representation.

As usual, characters represent the representation ring. For example, the fact that $R_j \otimes \mathbf{2} = R_{j-\frac{1}{2}} \oplus R_{j+\frac{1}{2}}$ (for $j > 0$) can be seen from the fact that

$$\chi_j(\psi)\chi_{\frac{1}{2}}(\psi) = \sum_{m=-j}^{j} e^{\mathbf{i}\psi m}\left(e^{-\mathbf{i}\psi/2} + e^{+\mathbf{i}\psi/2}\right) = \sum_{m=-j-\frac{1}{2}}^{j+\frac{1}{2}} e^{\mathbf{i}\psi m} + \sum_{m=-j+\frac{1}{2}}^{j-\frac{1}{2}} e^{\mathbf{i}\psi m} = \chi_{j-\frac{1}{2}}(\psi) + \chi_{j+\frac{1}{2}}(\psi).$$

More generally, the $\mathsf{SU}(2)$ fusion rules are

$$\chi_{j_1}\chi_{j_2} = \sum_{j_3=-|j_1-j_2|}^{j_1+j_2} \chi_{j_3} \tag{4.12}$$

with every multiplicity equal to one.

The $\mathsf{SU}(2)$ characters are sometimes called Chebyshev functions (polynomials in $z \equiv \cos\theta \equiv \cos\frac{\psi}{2}$):

$$U_{s-1}(z = \cos\theta) = \frac{\sin s\theta}{\sin\theta},$$

with $s = 2j+1$. This means that the Chebshev functions satisfy the $\mathsf{SU}(2)$ fusion rules (4.12). They have lots of nice properties and are very useful for numerical interpolation and spectral methods of solving differential equations, see *e.g.* Boyd, *Chebyshev and Fourier Spectral Methods*, or Trefethen, *Spectral Methods in Matlab*.

One of the results that follows from compactness is the Great Orthogonality Theorem. In the context of Lie groups, this is called the Peter-Weyl theorem:

$$\int d\mu(g)\left(D^a_{ij}(g)\right)^\star D^b_{lm}(g) = \frac{1}{d_a}\delta^{ab}\delta_{il}\delta_{jm}.$$

The proof is the same as in the finite case, with the replacement of sums by integrals. For example, it implies that the Chebyshev polynomials are orthogonal with respect to the integration measure given above for $\mathsf{SU}(2)$. More generally it implies that the characters of irreps form an orthonormal basis for class functions on the group manifold.

Given an explicit representation of a Lie group, it is quite simple to compute the character. The crucial fact[49] is that any element of G can be conjugated to an element $h = e^{\mathbf{i}\theta_a H_a}$ of the Cartan subgroup $T = \mathsf{U}(1)^r$. So the characters are really just functions of r angles.

One reason to care about the characters of irreps of Lie groups is that they answer (or at least record the answer to) the annoying question about the multiplicities of

[49] This is Theorem 26.16 of Fulton-Harris. Unfortunately their proof sketch is rather geometric and appeals to a fixed-point theorem. Try to come up with a more elementary argument.

various weights. Let $z_a = e^{\mathbf{i}\theta_a}, a = 1..\text{rank}(G)$ parametrize the Cartan torus. Then since the weights are eigenvectors of the Cartan generators:

$$\chi_R(z) = \text{tr}_R e^{\mathbf{i}\theta\cdot H} = \sum_\mu n_R(\mu) e^{\mathbf{i}\theta\cdot\mu}.$$

So we can extract the multiplicity $n_R(\mu)$ *e.g.* by an integral over T.

For the case of $\mathsf{SU}(n)$ it is convenient to parametrize the Cartan torus in terms of n z_a satisfying $\prod_{a=1}^n z_a = 1$. Just looking at the weights e_a of the fundamental of $\mathsf{SU}(n)$, its character is

$$\chi_{\square}(z) = \sum_a z_a. \tag{4.13}$$

Similarly, since the weights of the completely antisymmetric k-index rep are $e^{a_1} + e^{a_2} + \cdots + e^{a_k}, a_1 < a_2 < \cdots < a_k$, its character is

$$\chi_{\text{(column of } k \text{ boxes)}}(z) = \sum_{a_1 < a_2 < \cdots < a_k} z_{a_1} z_{a_2} \cdots z_{a_k}. \tag{4.14}$$

For the special case of $k = n-1$ (the $\bar{n}$ rep), we can use $\prod_a z_a = 1$ to rewrite this as $\chi_{\bar{n}}(z) = \sum_a z_a^{-1}$ in agreement with the fact that its weights are $-e_a$. More generally $\chi_\lambda(z) = S_\lambda(z)$ is a *Schur polynomial.* These are symmetric polynomials, symmetrized or antisymmetrized according to the Young diagram, of which (4.13) and (4.14) are special cases[50]. Those with at most n boxes provide a basis for the space of symmetric functions on n variables. (See Fulton and Harris, §6.1 and §A.1.)

There are many beautiful formulae for the characters of irreps of compact Lie groups. For example, for any Lie group, the *Weyl character formula* says that character for the representation R_λ with the highest weight μ is

$$\chi_{R_\mu}(z) = \frac{A_{\mu+\rho}(z)}{\Delta(z)} \quad \text{with}$$

$$A_\mu(z) \equiv \sum_{W\in\mathcal{W}} (-1)^W e^{\mathbf{i}\theta\cdot W(\mu)},$$

50

$$\chi_\lambda(z) = S_\lambda(z_1 \cdots z_n) \equiv \frac{\det_{ij} z_j^{\lambda_i+n-i}}{\det_{ij} z_j^{n-i}}$$

where $\lambda = \lambda_i e_i$ is the highest weight of the representation. Recall from §3.7.1 that the shape of the Young diagram is related to the highest-weight vector by

$$\lambda = e_1(q_1 + q_2 \cdots q_{n-1}) + e_2(q_2 + \cdots q_{n-1}) + \cdots + q_{n-1}e_{n-1} + 0e_n,$$

where q_k is defined in the figure at the end of that section. Taking $\lim_{z_a\to 1} \chi_\lambda(z) = \dim R_\lambda^{\mathsf{SU}(n)}$, this formula reproduces the factors-over-hooks rule.

$$\Delta(z) \equiv A_\rho(z) = \prod_{\alpha \in R_+} \left(e^{\mathbf{i}\alpha\cdot\theta/2} - e^{-\mathbf{i}\alpha\cdot\theta/2} \right) \tag{4.15}$$

with R_+ the set of positive roots, and $\rho \equiv \frac{1}{2}\sum_{\alpha\in R_+}\alpha$. Here $\mathcal{W}$ is the Weyl group, $(-1)^W$ means the sign of the determinant of W. For example for $\mathsf{SU}(n)$, $\Delta(z) = \prod_{i<j}(z_i - z_j)$ is the vandermonde determinant or descriminant (use the relation $\prod_a z_a = 1$ repeatedly). For $\mathsf{SU}(n)$, the Weyl group is just S_n. See Fulton and Harris for much more.

The *Weyl integration formula* relates group integrals of class functions to integrals over the Cartan torus T:

$$\int_G d\mu(g) F(g) = \frac{1}{|\mathcal{W}|}\int_T d\mu(z)\Delta(z)\overline{\Delta(z)}F(z) \tag{4.16}$$

with Δ as in (4.15). For the case of $\mathsf{SU}(2)$, $|\Delta|^2 = \sin^2\frac{\psi}{2}$, and this is our previous formula. Combined with the Weyl character formula (4.16) says that the inner product between characters of $\mathsf{SU}(n)$ is

$$\langle \chi_{R_1}, \chi_{R_2}\rangle = \int_G d\mu(g)\bar{\chi}_{R_1}(g)\chi_{R_2}(g) = \int_T d\mu(z)\overline{A_{\lambda_1+\rho}(z)}A_{\lambda_2+\rho}(z)s.$$

4.3 Unification

4.3.1 Isospin and flavor SU(3)

[Georgi §5, 11, Zee §V.1, V.3]

The purpose of this brief subsection is twofold. One goal is to explain an important use of Lie groups in particle physics, where $\mathsf{SU}(2)$ and also $\mathsf{SU}(3)$ are realized as *approximate* symmetries of the strong interactions. The second goal is to gently introduce the some ideas about symmetries in quantum field theory. I emphasize that this is an almost-criminally brief account of a huge subject.

Isospin. The fractional mass difference between the proton and the neutron is tiny:

$$\frac{M_n - M_p}{M_p} = \frac{939.6 - 938.6}{938.6}\frac{\text{GeV}}{\text{GeV}} = 0.0014.$$

If we ignore the fact that one is electrically charged and the other is not, which the strong interactions do ignore, there is an approximate symmetry that rotates them into each other. The nucleon $N = \begin{pmatrix} p \\ n \end{pmatrix}$ is a doublet, **2** of some global $\mathsf{SU}(2)$ symmetry called isospin.

A bit of technology which is useful much more generally: Let's write neutron and proton states as

$$|p, \alpha\rangle = a^\dagger_{\frac{1}{2},\alpha} |0\rangle\,, \quad |n, \alpha\rangle = a^\dagger_{-\frac{1}{2},\alpha} |0\rangle$$

where α is any other quantum numbers of the particle, such as its momentum. $|0\rangle$ is a vacuum state, annihilated by all the annihilation ops $a|0\rangle = 0$. $m_I = \pm\frac{1}{2}$ are eigenvalues of the isospin generator J^3. Nucleons are fermions, so the creation operators satisfy $\{a_{m\alpha}, a^\dagger_{m'\alpha'}\} = \delta_{mm'}\delta_{\alpha\beta}, \{a_{m\alpha}, a_{m'\alpha'}\} = 0$. The point of the creation and annihilation operators as usual is that now we can think about states with many nucleons:

$$|n \text{ nucleons}, m_1\alpha_1, m_2\alpha_2 \cdots m_n\alpha_2\rangle = a^\dagger_{m_1\alpha_1} a^\dagger_{m_2\alpha_2} \cdots a^\dagger_{m_n\alpha_n} |0\rangle\,.$$

On one-particle states, the isospin generators act as $\hat{T}^a |m\alpha\rangle = |m'\alpha\rangle \left(\frac{1}{2}\sigma^a\right)_{m'm}$ and on the vacuum they act as $\hat{T}^a |0\rangle = 0$. To match these, they must act on the creation operators as

$$[\hat{T}^a, a^\dagger_{m\alpha}] = a^\dagger_{m'\alpha} \left(\frac{1}{2}\sigma^a\right)_{m'm}$$

which is solved by

$$\hat{T}^a = a^\dagger_{m'\alpha} \left(\frac{\sigma^a}{2}\right)_{m'm} a_{m\alpha} + \cdots$$

where the $\cdots$ is terms that commute with $a, a^\dagger$. So now we know how multiparticle states transform. You can check that these operators $\hat{T}^a$ satisfy the $\mathsf{SU}(2)$ Lie algebra, $[\hat{T}^a, \hat{T}^b] = \mathbf{i}\epsilon^{abc}\hat{T}^c$. (And if we replaced $\frac{\sigma^a}{2}$ by matrices comprising a unitary representation of some other Lie algebra, you can guess what would happen.)

There are other particles other carry isospin. In some description, pions (pseudoscalar spin-0 bosons) mediate the force between nucleons, a short-ranged strong attraction which holds together nuclei. Allowed (isospin-symmetric) processes include $p \to n+\pi^+, n \to p+\pi^-$. The pions must therefore transform under isospin. Comparing the isospin of the BHS of these reactions we have

$$I = \frac{1}{2} \ni \frac{1}{2} \otimes I_\pi = (\frac{1}{2} + I_\pi) \oplus |\frac{1}{2} - I_\pi|$$

This essentially requires that $\pi^\pm$ have $I_\pi = 1$ and therefore also have a neutral partner π^0. The pions $\pi_\pm, \pi_0$ transform as a triplet $\mathbf{3}$ of $\mathsf{SU}(2)_{\text{isospin}}$. (The electric charge is $Q = I_3 + \frac{1}{2}Y$ where $Y = 1$ for nucleons and $Y = 0$ for pions; Y is a new quantum number called hypercharge.)

We can follow the same logic as above and write

$$\big|1\ \pi \text{ of isospin } J^3 = m \in \{-1, 0, 1\}\ \big\rangle = b^\dagger_{m\alpha} |0\rangle\,.$$

Pions are bosons, so $[b_{m\alpha}, b^\dagger_{m'\alpha'}] = \delta_{\alpha\alpha'}\delta_{mm'}, [b, b] = 0$, and the vacuum is the same vacuum as above, also annihilated by $b\,|0\rangle = 0$. The isospin generators are then

$$\hat{T}^a = \hat{T}^a_{\text{nucleons}} + b^\dagger_{m'\alpha}\,(J^a_1)_{m'm}\,b_{m\alpha} + \cdots = \sum_{\text{particle types, } x} a^\dagger_{xm\alpha}\left(J^a_{j_x}\right)a_{xm'\alpha}$$

where j_x is the spin of particle type x and the statistics of a_x are chosen appropriately. These guys still satisfy the Lie algebra, $[\hat{T}^a, \hat{T}^b] = \mathbf{i}\epsilon^{abc}\hat{T}^c$.

The fact that isospin is an approximate symmetry means that the hamiltonian is

$$H = H_0 + \Delta H, \qquad [H_0, T^a] = 0.$$

The terms in H_0 include QCD interactions, and kinetic terms with equal quark masses. ΔH, which does not commute with T^a, contains quark mass differences and electromagnetic interactions.

Isospin can be used to make strong predictions for ratios of observables without any knowledge of the complicated underlying strong dynamics. As one example of very many, it predicts that (the deuteron d is a boundstate of p and n with isospin zero (and spin 1))

$$\frac{\sigma(p + p \to d + \pi^+)}{\sigma(p + n \to d + \pi^0)} \simeq \frac{|\langle 1,1|1,1\rangle|^2}{|\langle 1,0|\left(|1,0\rangle - |0,0\rangle\right)/\sqrt{2}|^2} = 2.$$

This follows from the fact that the isospin state of two protons is $|p,p\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle = |1,1\rangle$, while that of proton plus neutron is $|p,n\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{|1,0\rangle - |0,0\rangle}{\sqrt{2}}$.

Another example is

$$\frac{\sigma(\pi^+ + p \to \Delta^{++} \to \pi^+ + p)}{\sigma(\pi^- + p \to \Delta^0 \to \pi^- + p)} \simeq 3. \tag{4.17}$$

In the numerator is the cross-section to create a particular resonance at $E \sim 1.232$ GeV, which (since $\pi^+ + p$ is the isospin state $|1,1\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle$) must have $I_3 = 3/2$ and therefore isospin 3/2 and therefore 3 other partners (of which Δ^0 is one). You can figure out using $\mathsf{SU}(2)$ Clebsch-Gordon coefficients what is the isospin state of $\pi^- + p$, and thereby find (4.17).

Isospin constrains the couplings between pions and nucleons. Writing the pion fields as a matrix

$$\pi \equiv \vec{\pi}\cdot\vec{\sigma} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$$

they transform under isospin as $\pi \to U\pi U^\dagger$. This means that the combination

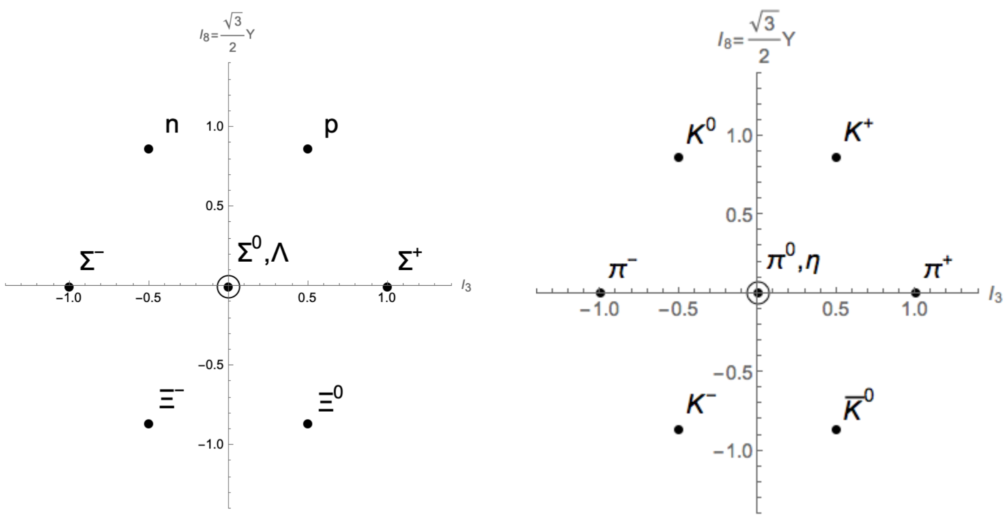
$$g\bar{N}\pi N = g\bar{N}_i\pi^i_jN^j = g\left(\bar{p}\pi^0p - \bar{n}\pi^0n + \sqrt{2}\left(\bar{p}\pi^+n + \bar{n}\pi^-p\right)\right)$$

is an isospin-invariant allowed interaction – several couplings are determined in terms of the coupling g.

Eight-fold way. In 1950 another nucleon-like-particle ($\equiv$ baryon) was discovered, with $M_\Lambda \sim 1.115$ GeV. You might have thought that we should try to put this in an $\mathsf{SU}(3)$ triplet with (n, p, Λ), but no. Instead, a whole bunch of other baryons was found with similar (a bit larger) masses, all spin-half fermions. The new ones fit into an isospin triplet ($\Sigma^{\pm,0}$) and an isospin doublet ($\Xi^\pm$), while the Λ is an isospin singlet.

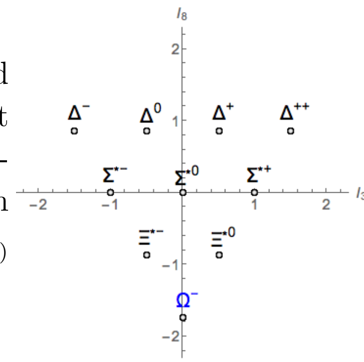
Moreover, a bunch new pion-like particles ($\equiv$ mesons) were found, with a bit larger masses. These were two isospin doublets K^+, K^0 and $K^-, \bar{K}^0$ and an isospin singlet η.

Now, from having done homework 9, you know that each of these things fit into a copy of the **8** of $\mathsf{SU}(3)$ (recall the figures (3.31)), one for baryons and one for mesons:

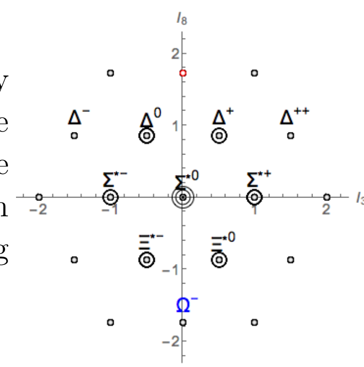


The new quantum number in these weight diagrams is proportional to the hypercharge $I_8 = \frac{\sqrt{3}}{2} Y$. (Sometimes people also speak of *strangeness*, which is hypercharge plus baryon number (baryons carry baryon number 1).) It is also conserved by the strong interactions. The particles that carry nonzero values of the new quantum number are heavier because the whole $\mathsf{SU}(3)$ is not as good an approximation as $\mathsf{SU}(2)$. (We'll see that this is because the strange quark is quite a bit heaver than the up and down quarks, $m_s - m_d \gg m_u - m_d$.)

A bunch of other baryons (with spin 3/2) had also been found as short-lived resonances (such as the Δs I mentioned at (4.17)). At one point there were nine of them, and Gell-Mann and Neeman predicted that there would be a tenth (the Ω^-) to fill out this weight diagram for the $\mathbf{10} = R_{(3,0)}$ of $\mathsf{SU}(3)$, where it was duly found.



There was a time when other $\mathsf{SU}(3)_f$ multiplets could equally well have accommodated the existing baryons. At right is the $R_{(2,2)} = \mathbf{27} = \boxplus\!\square$ representation, for example. But people (Gerson Goldhaber) had looked for the state associated with the red circle, which would have been produced in scattering of K^+ off a neutron, and it was absent.



Gell-Mann Okubo mass formulae. [Zee §V.4, Georgi §11] An important aspect of their prediction is that they had a good estimate for what the mass should be, as well as all the quantum numbers. This arises by considering the possible mass terms that we can write for the baryon fields

$$B \equiv \lambda^A B^A/\sqrt{2} = \begin{pmatrix} \Sigma^0/\sqrt{2} + \Lambda^0/\sqrt{6} & \Sigma^+ & p \\ \Sigma^- & -\Sigma^0/\sqrt{2} + \Lambda^0/\sqrt{6} & n \\ \Xi^- & \Xi^0 & -2\Lambda^0/\sqrt{6} \end{pmatrix}.$$

First we ask which terms $\mathrm{tr} B^\dagger B$ are consistent with $\mathsf{SU}(3)_f$, and then we ask about the leading terms that break $\mathsf{SU}(3)_f \to \mathsf{SU}(2)_{\text{isospin}}$, and do perturbation theory in these terms. The idea is there are two symmetry-breaking terms ($\mathrm{tr} B^\dagger I_8 B$ and $\mathrm{tr} B^\dagger B I_8$)[51] with unknown coefficients, but there are four unknown masses of isospin multiplets. A similar analysis can be made of the meson masses and of the couplings between mesons and nucleons.

To estimate the splittings in the $\mathbf{10}$, note that the matrix element of the perturbing hamiltonian is of the form $\langle \mathbf{10}| I_8 |\mathbf{10}\rangle$. But in $\mathbf{8} \otimes \mathbf{10} = \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{27} \oplus \mathbf{35}$ there is a unique $\mathbf{10}$, and therefore there's only one mass term with these quantum numbers,

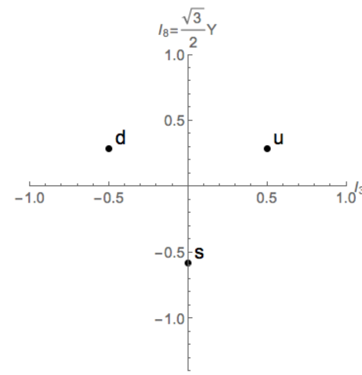
[51] Here I am cheating a little bit, and using some non-group-theory information to say that the $\mathsf{SU}(3)_f$ breaking interaction transforms like the $\mathbf{8}$. In principle there could also have been an interaction transforming like the $\mathbf{27}$. But with hindsight, the breaking is caused by the mass term of the strange quark, which transforms as $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$.

and we expect that mass splitting in the decuplet is proportional to the hypercharge: $M_\Sigma - M_\Delta = M_\Xi - M_\Sigma = M_\Omega - M_\Xi$. The known masses were: $M_\Delta \simeq 1230, M_\Sigma \simeq 1385, M_\Xi \simeq 1530$ in MeV, so the prediction was $M_\Omega \simeq 1680$ MeV. The answer is 1672!

4.3.2 Quarks and color

[Georgi, §11.4, 16]

Quarks. The lovely weight diagrams are an improvement but it's still a bit of a mess. Actually the weight diagram at right explains the whole thing:



The idea is to introduce new particles – quarks – in the **3** of Gell-Mann's $\mathsf{SU}(3)_f$, which is now called *flavor* $\mathsf{SU}(3)$. The baryons are made out of three quarks, $B_{abc} = q_a q_b q_c$ (with $a, b, c \in u, d, s$ labelling the flavor). Since

$$\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \left(\ydiagram{1,1} \oplus \ydiagram{2}\right) \otimes \ydiagram{1} = \ydiagram{1,1,1} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \ydiagram{3}, \tag{4.18}$$

that is, $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$, this produces both baryon multiplets we showed. And the mesons are made from a quark and an antiquark $m_a^b = q_a \bar{q}^b$. Since $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$, this contains the octet of the pions and kaons (and also multiplets of spin-1 mesons).

There are two problems, which solve each other: in order for the baryons to be fermions, the quarks must be fermions. But then only the antisymmetric combinations would survive, unless they had some other quantum numbers in which to antisymmetrize. The second problem is: why don't we see quarks, which (as you can see from $Q = I_3 + Y/2$) would have to carry fractional electric charge.

Color. The solution is that there are a lot more quarks: in addition to being a triplet of Gell-Mann's $\mathsf{SU}(3)_f \supset \mathsf{SU}(2)_{\text{isospin}}$, the quarks also comprise a triplet of another, totally distinct $\mathsf{SU}(3)_c$, called *color* $\mathsf{SU}(3)$, so the quarks carry two labels $q_{ai}, a = 1..3 = u, d, s$ labels the flavor and $i = 1..3 = r, g, b$ labels the color (actually, they have spin-half so there is a third spin label). This quantum number is called color and it behaves in a strange way: finite-energy excitations are colorless. This demand that finite-energy excitations are color-neutral explains why the particles we see are made of three quarks or quark-antiquark. Regarding the decomposition (4.18) as multiplication in $\mathsf{SU}(3)_c$ now, we see that the product of three fundamentals contains a color singlet – this is the baryon $\epsilon_{ijk} q^i q^j q^k$. And of course there is a color singlet in $\mathbf{3} \otimes \bar{\mathbf{3}}$ – this is the meson $q^i \bar{q}_j \delta_i^j$.

In the previous equations I've suppressed the other quantum numbers of the quark fields, and only shown their color indices. If you put them back you can see that this also explains why we don't see the other parts of (4.18) which have the wrong properties under exchange of quarks.

Why should physical states be singlets of color? Here is an extremely heuristic explanation. Recall that in E&M, the force between two charges is proportional to $q_1 q_2$. E&M is a gauge theory with gauge group $\mathsf{U}(1)$, which means that charge is a number. In more general gauge theories with gauge group G, the analog of 'charge' is a representation label. For two particles, each in a rep $R_{a=1,2}$ of $\mathsf{SU}(3)_c$

$$\hat{T}_A \left|i\right\rangle_1 = \left|j\right\rangle_1 \left(T_A^{R_1}\right)^i_j, \quad \hat{T}_A \left|x\right\rangle_2 = \left|y\right\rangle_2 \left(T_A^{R_2}\right)^x_y .$$

On a two-particle state $v_{ix} \left|i\right\rangle_1 \otimes \left|x\right\rangle_2$, the form of the interactions is

$$\propto \sum_A \hat{T}_A^{R_1} \otimes \hat{T}_A^{R_2} \equiv H.$$

(You can regard the diagram as a birdtrack diagram if you wish. Or, for the field theorists: you can regard the diagram as a Feynman diagram, and we are not thinking about the spatial dependence of the gluon propagator.) This hamiltonian has a symmetry: $\hat{T}_A \equiv \hat{T}_A^{R_1} + \hat{T}_A^{R_2}$ has $[\hat{T}_A, H] = 0$. In fact, we can write H completely in terms of the quadratic Casimirs, $C_2 = \hat{T}_A \hat{T}_A \equiv T^2$:

$$H = \frac{1}{2}\left(T^2 - T_1^2 - T_2^2\right).$$

The second and third terms are fixed by the properties of the constituent particles, but the T^2 depends on how they combine. It is minimized by small representations, and in particular the singlet. This is a generalization (and an extreme version of) the statement that opposite charges attract. So for example, $q\bar{q}$ can form a singlet, and this minimizes their interaction energy. And in a set of three quarks, the pairwise interactions are minimized when the three form the totally antisymmetric state (in which each pair forms a $\bar{\mathbf{3}} \in \mathbf{3} \otimes \mathbf{3}$ which has $T^2 = 4/3$, while the $\mathbf{6}$ has $T^2 = 10/3$).

This crude explanation is a sort of cartoon of the phenomenon of color confinement in quantum chromodynamics (QCD), the gauge theory of $G = \mathsf{SU}(3)_{\text{color}}$. A gauge theory with gauge group G is a quantum field theory with a vector field A_μ^A for each generator T^A of the lie algebra of G, and various matter fields which transform in representations of G. The Lie algebra data completely determines the most important interactions. Different gauge theories have different vacuum behavior – electromagnetism has almost-free charges interacting with a propagating massless photon, but QCD confines its charges into color singlets.

4.3.3 Grand Unification

[Zee §IX.2, IX.3, Georgi §18]

The Standard Model. The Standard Model of particle physics is a gauge theory with gauge group[52] $G_{\text{SM}} = \mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)$. We can label the representations of the fermion fields by $(D, d)_{R_1}$ where D is a rep of $\mathsf{SU}(3)$, d is a rep of $\mathsf{SU}(2)$ and R_1 is the charge under the $\mathsf{U}(1)$. This means

$$[\hat{T}_A, a^\dagger_{x,i}] = a^\dagger_{y,i} \left(T^D_A\right)_{yx}, \quad [\hat{R}_A, a^\dagger_{x,i}] = a^\dagger_{x,j} \left(R^d_A\right)_{yx}, \quad [S, a^\dagger_{xi}] = s a^\dagger_{xi}.$$

So T_A, R_A, S are the generators of $\mathsf{SU}(3)$, $\mathsf{SU}(2)$ and $\mathsf{U}(1)$ respectively. The $\mathsf{SU}(3)$ is color as above. The $\mathsf{SU}(2) \times \mathsf{U}(1)$ is the electroweak theory, which is a unification of electromagnetism and the weak interactions which I don't have time to explain. $S = Y/2$ is proportional to the hypercharge, and the electric charge is $Q = R_3 + S$.

In this notation, a generation of the standard model is

$$R_{\text{gen}} = (\mathbf{3}, \mathbf{1})_{2/3} \oplus (\mathbf{3}, \mathbf{1})_{-1/3} \oplus (\mathbf{1}, \mathbf{1})_{-1} \oplus (\bar{\mathbf{3}}, \mathbf{2})_{-1/6} \oplus (\mathbf{1}, \mathbf{2})_{1/2} \tag{4.19}$$

corresponding, respectively, to $(u_R, d_R, e_R, q^c_L, (e^c_L, \nu^c_L) \equiv e^c_L)$. These are all right-handed Weyl fermions (the c turns a left-handed field into a right-handed one). Notice that this is a complex representation – if you computed its Frobenius-Schur indicator you would get zero. This one line is a summary of 30 years worth of work. (For slightly more information see Zee §IX.1.)

There is also a scalar field transforming as a doublet of $\mathsf{SU}(2)$, the Higgs field.

There could be a (heavy) right-handed neutrino, which would be in the rep $(\mathbf{1}, \mathbf{1})_0$, a singlet under everything.

Grand unification. Question: does there exist a compact Lie group $G \supset G_{\text{SM}}$ such that the representations (4.19) unify, perhaps into a single irrep?

The group would have to have rank ≥ 4, since we already have $\text{rank}\,(G_{\text{SM}}) = 4$, with Cartan generators $\{T_3, T_8, R_3, S\}$.

A hint that this might be the case is that $\text{tr}_{R_{\text{gen}}} S = \frac{2}{3} \cdot 3 - \frac{1}{3} \cdot 3 - 1 - \frac{1}{6} \cdot 6 + \frac{1}{2} \cdot 2 = 0$, the $\mathsf{U}(1)$ generator is traceless, as it must be if it is a subgroup of a compact group.

[52] or perhaps a quotient of this by $\mathbb{Z}_6$

Here's the simplest possible guess with rank 4: $\mathsf{SU}(5) \supset \mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)$. The idea is that the generators would fit together as

$$\begin{pmatrix} & & & & \\ T_A & & & & \\ & & & & \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & R_A & \\ & & & & \end{pmatrix}, \quad \begin{pmatrix} -1/3 & & & & \\ & -1/3 & & & \\ & & -1/3 & & \\ & & & 1/2 & \\ & & & & 1/2 \end{pmatrix}. \tag{4.20}$$

The $\mathbf{5}$ is easy to decompose:

$$\mathbf{5} = (\mathbf{3}, \mathbf{1})_{-1/3} \oplus (\mathbf{1}, \mathbf{2})_{1/2}$$

which is exactly $d_R \oplus e^c_L$. What's left is 10 dimensional. We know two 10-dimensional reps of $\mathsf{SU}(5)$, namely $R_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \Lambda^2 \mathbf{5} \equiv \mathbf{10}$ and $R_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \Lambda^3 \mathbf{5} = \Lambda^2 \bar{\mathbf{5}} \equiv \overline{\mathbf{10}}$. Let's decompose the first one under $\mathsf{SU}(5) \supset G_{\text{SM}}$ and see what we get:

$$\Lambda^2 \mathbf{5} = \left((\mathbf{3}, \mathbf{1})_{-1/3} \oplus (\mathbf{1}, \mathbf{2})_{1/2}\right) \otimes_{\text{AS}} \left((\mathbf{3}, \mathbf{1})_{-1/3} \oplus (\mathbf{1}, \mathbf{2})_{1/2}\right) \tag{4.21}$$

$$((\mathbf{3} \otimes \mathbf{3})_{\text{AS}}, 1)_{-2/3} \oplus (1, (\mathbf{2} \otimes \mathbf{2})_{\text{AS}})_1 \oplus (\mathbf{3}, \mathbf{2})_{-\frac{1}{3}+\frac{1}{2}=\frac{1}{6}} \tag{4.22}$$

$$= (\bar{\mathbf{3}}, 1)_{-2/3} \oplus (\mathbf{1}, \mathbf{1})_1 \oplus (\mathbf{3}, \mathbf{2})_{1/6} \tag{4.23}$$

$$= \left((\mathbf{3}, 1)_{2/3} \oplus (\mathbf{1}, \mathbf{1})_{-1} \oplus (\bar{\mathbf{3}}, \mathbf{2})_{-1/6}\right)^\star. \tag{4.24}$$

We conclude that a single generation of the standard model is

$$R_{\text{gen}} = \overline{\mathbf{10}} \oplus \mathbf{5}$$

of $\mathsf{SU}(5)$! This is a huge compression of information.

This has some shocking implications. In particular, the fact that quarks and leptons (like the electron) live in the same irrep of $\mathsf{SU}(5)$ means that the $\mathsf{SU}(5)$ interactions can violate baryon number. There will be processes like $qqq \to \ell$ by which the proton can decay into leptons. The fact that all the stuff is made of lots of protons and is still around is therefore a strong constraint on grand unification. The energy scale above which $\mathsf{SU}(5)$ is restored must be quite large, almost at the Planck energy.

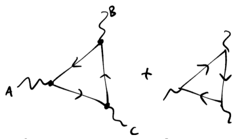
The same Higgs mechanism by which $\mathsf{SU}(2) \times \mathsf{U}(1)$ of the SM is broken down to $\mathsf{U}(1)_{\text{EM}}$ can break $\mathsf{SU}(5)$ down to G_{SM}. The constraint I just mentioned is a constraint on the energy scale where this happens. I will not explain this right now.

Instead, let's be even more ambitious. Recall the $\mathsf{SU}(5) \subset \mathsf{SO}(10)$ that we found in our discussion of spinor representations. And recall that the $\mathbf{16}_+$ spinor decomposed under this subgroup into

$$\mathbf{16}_- = \square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix} = \mathbf{5} \oplus \overline{\mathbf{10}} \oplus \mathbf{1}.$$

This single irrep is exactly R_{gen}, a generation of the standard model, plus a singlet: the right-handed neutrino.

A few more things, which it would be criminal not to mention:

- Something special about these particular representations of $\mathsf{G}_{\text{SM}}, \mathsf{SU}(5), \mathsf{SO}(10)$ (and the $\mathbf{27}$ of E_6, too) is the cancellation of anomalies. An important special case of this has to do with the vanishing of birdtrack diagrams of the following form:  where the loop includes a sum over all the representations of charged fields. You can see that this is a symmetric tensor $d_{ABC}(R)$ with 3 adjoint indices; its vanishing is a consistency condition of the gauge theory. If you compute it for R_{gen} of G_{SM} you'll get zero. In fact if we replace the external lines with currents for global symmetries, we again get an interesting set of invariants; their vanishing is no longer a consistency condition, but for the SM they also vanish. See *e.g.* here or here for the significance of this.

- Baryon number is violated by $\mathsf{SU}(5)$ interactions, but $\mathsf{U}(1)_{\text{B-L}}$, the transformation under which baryons and leptons have opposite charge, actually becomes part of $\mathsf{SO}(10)$. If we include the right-handed neutrino, you can check that it is traceless.

- Grand unification of the gauge groups also implies a unification of their coupling constants. The couplings of the three factors in G_{SM} are very different (hence the names Weak and Strong Interactions), but the running of the couplings under the renormalization group points to such a unification at some very high energy. (It works even better if there is some supersymmetry.)

- What breaks $\mathsf{SU}(5)$ down to G_{SM}? This can be accomplished by a Higgs field $\Phi_{\mathbf{24}}$ in the adjoint of $\mathsf{SU}(5)$ which takes a vacuum expectation value proportional to the generator of S in (4.20), $\langle\Phi_{\mathbf{24}}\rangle \propto \begin{pmatrix} 2\mathbb{1}_3 & 0 \\ 0 & -3\mathbb{1}_2 \end{pmatrix}$. It is possible to find a $\mathsf{SU}(5)$-invariant potential for such a field such that configurations of this form are the minima.

- When $\Phi_{\mathbf{24}}$ condenses, its kinetic terms give a mass to the gauge bosons of $\mathsf{SU}(5)$ not in G_{SM}. Since the adjoint decomposes as

$$\mathbf{24} = (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\bar{\mathbf{3}}, \mathbf{2})$$

  there are 12 of these, which mediate exotic processes like proton decay.

  $\mathsf{SO}(10)$ contains $1+n(n-1) = 21 = 45-24$ additional gauge bosons. Besides from subtraction, you can see this from the decomposition of the generators of $\mathsf{SO}(2n)$

in §3.9 into $\{H_a, E_{ab}, E_{ab}^\dagger, E'_{ab}, (E'_{ab})^\dagger, \sum_a H_a\}$, of which the first $n-1+n^2-n$ generate $\mathsf{SU}(n)$.

- Mass terms in the $\mathsf{SU}(5)$ GUT are of the form

$$(\mathbf{10}\cdot\mathbf{10}\cdot\mathbf{5}_\varphi) = \psi^{\mu\nu}\psi^{\rho\lambda}\varphi^\sigma\epsilon_{\mu\nu\rho\lambda\sigma} \overset{\varphi^\lambda=\delta^{\lambda 4}}{\to} u^c u$$

$$(\bar{\mathbf{5}}\cdot\mathbf{10}\cdot\bar{\mathbf{5}}_\varphi) = \psi_\mu\psi^{\mu\nu}\varphi^\star_\nu \overset{\varphi^\lambda=\delta^{\lambda 4}}{\to} d^c d + e^c e.$$

  These interactions conserve $\mathsf{U}(1)_{\mathrm{B-L}}$.

- Mass terms in the $\mathsf{SO}(10)$ GUT require a Higgs field in the $\mathbf{10}$, which decomposes into $\mathbf{5}\oplus\bar{\mathbf{5}}$ of $\mathsf{SU}(5)$. Then we can add a term of the form $(\mathbf{16}^+\cdot\mathbf{16}^+\cdot\mathbf{10}_\varphi)$ which contains an $\mathsf{SO}(10)$ singlet, since

$$\mathbf{16}^+\otimes\mathbf{16}^+ = \mathbf{10}\oplus\Lambda^3\mathbf{10}\oplus\left(\Lambda^5\mathbf{10}\right)_+ = \mathbf{10}\oplus\mathbf{120}\oplus\mathbf{126}$$

  (where $(\Lambda^5\mathbf{10})_+$ means the self-dual 5-form – it's an eigenstate of $\epsilon_{i_1\cdots i_{10}}$), as you may have shown on the homework. Notice that the 5-index antisymmetric rep contains a singlet of $\mathsf{SU}(5)$ which can give a mass to the right-handed neutrino once $\mathsf{SO}(10)$ is broken to $\mathsf{SU}(5)$.

- Here is a puzzle for you: a nonzero expectation value of what field (analogous to $\Phi_{\mathbf{24}}$ which breaks $\mathsf{SU}(5)$ to G_{SM}) breaks $\mathsf{SO}(10)$ to precisely $\mathsf{SU}(5)$?

  An adjoint higgs field with a vacuum expectation value proportional to $\sum_a H_a$, the total particle number in the fermion description of the spinor reps, will almost do it. In the notation of §3.9, this commutes with all the $\mathsf{SU}(5)$ generators H_a, E_{ab} and fails to commute with the E'_{ab} and $(E'_{ab})^\dagger$, and hence gives a mass to the associated gauge bosons. But it also leaves $\sum_a H_a$ – it breaks $\mathsf{SO}(10)$ to $\mathsf{U}(5)$, not $\mathsf{SU}(5)$.

  One answer is a 5-index antisymmetric tensor. If its (imaginary) self-dual part (the $\mathbf{126}_+$) gets an expectation value, this breaks $\mathsf{SO}(10)$ to $\mathsf{SU}(5)$, since the 5-index AS tensor is invariant under $\mathsf{SU}(5)$.

  Another, perhaps simpler, answer is just a Higgs in the $\mathbf{16}_- = \overline{\mathbf{10}}\oplus\mathbf{5}\oplus\mathbf{1}$ (the $\mathbf{16}_+$ would work just as well). You can see that the the subgroup of $\mathsf{SO}(2n)$ which preserves a single component of the spinor is exactly $\mathsf{SU}(n)$ from our description of the spinor rep: the singlet is the state with no particles, $|0\rangle = \left|\frac{1}{2}\cdots\frac{1}{2}\right\rangle$. The generators $E'_{ab}\propto c_a^\dagger c_a^\dagger$ take this state to a state with two particles, and so are broken. The $\mathsf{U}(1)\subset\mathsf{U}(n)$ elements generated by $\sum_a H_a$ act by a nontrivial phase; in contrast, the generators of the Cartan of $\mathsf{SU}(n)$ act trivially: $e^{\mathbf{i}\theta_a H_a}\left|\frac{1}{2}\cdots\frac{1}{2}\right\rangle = e^{\mathbf{i}\sum_a H_a/2}\left|\frac{1}{2}\cdots\frac{1}{2}\right\rangle = \left|\frac{1}{2}\cdots\frac{1}{2}\right\rangle$ since they are by definition orthogonal to $\sum_a H_a$.

These facts are important in string compactification: Calabi-Yau manifolds have holonomy $\mathsf{SU}(n)$. They have a non-vanishing holomorphic n-form (this is the singlet in the $(\Lambda^n \mathbf{2n})_+$) and they preserve a single spinor component, which means that compactification on such a manifold preserves some supersymmetry. See e.g. volume 2 of the book by Green, Schwarz and Witten.

[End of Lecture 20]