

Physics 215A QFT Fall 2026 Assignment 1 – Solutions

Due 12:30 pm Thursday, October 1, 2026

- Please hand in your homework electronically via Canvas. The preferred option is to typeset your homework. It is easy to do and you need to learn to do it anyway as a practicing scientist. A LaTeX template file with some relevant examples can be found [here](#), and the tex file for the assignment itself is on the homework page. If you need help getting set up or have any other questions please email me. I am happy to give TeX advice.
- To hand in your homework, please submit a pdf file through the course's Canvas website, under the assignment labelled hw01.

Thanks in advance for following these guidelines. Please ask me by email if you have any trouble.

1. **Survey.** What area(s) of physics are you working on or hoping to work on? (QFT is a big subject and knowing this will help guide some choices I have to make.)
2. **Brain-warmer.** Consider the Fourier transformation

$$q_n = \frac{1}{\sqrt{N}} \sum_k e^{-ikna} q_k \equiv \sum_k U_{nk} q_k.$$

In the sum, k runs over the set $\{\frac{2\pi}{Na}j, j = 0..N-1\}$. Check that this transformation from real space (modes labelled by n) to Fourier space (modes labelled by k) is encoded in an $N \times N$ unitary matrix, U (that is, $UU^\dagger = \mathbb{1} = U^\dagger U$).

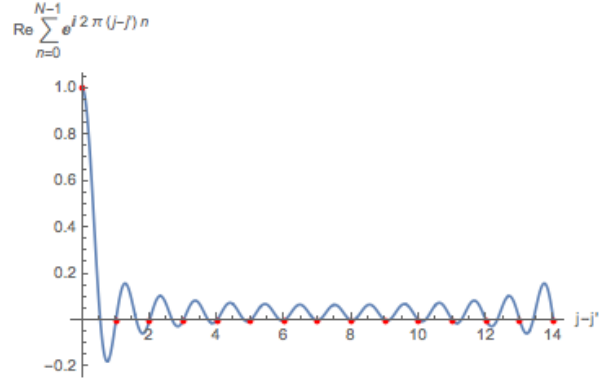
More precisely,

$$q_k = \frac{1}{\sqrt{N}} \sum_{n=1}^N e^{ikx_n} q_n \equiv \sum_n U_{kn} q_n, \quad k = n \frac{2\pi}{Na}, \quad n = 1 \dots N.$$

We can check that U is indeed a unitary matrix (I am writing $k = \frac{2\pi}{Na}j, k' = \frac{2\pi}{Na}j'$, where $j, j' \in 0..N-1$ are integers) :

$$\sum_n U_{kn} U_{nk'}^\dagger = \sum_n U_{kn} U_{k'n}^* = \frac{1}{N} \sum_n e^{ikx_n} e^{-ik'x_n} = \frac{1}{N} \frac{1 - e^{2\pi i(j-j')}}{1 - e^{2\pi i(j-j')/N}} = \delta_{jj'}.$$

At right is what that function looks like for real argument; notice that it vanishes at all the integers other than 0.



This expression makes it seem more confusing than necessary: if $k = k'$ then every term is 1 and there are N terms; if $k \neq k'$ (and they are both allowed values of the wavenumber), then it is the sum over all N th roots of unity, which vanishes.

It's also worth noting that $U_{nk}^\dagger = U_{kn}^* = U_{-kn}$, so unitarity also implies

$$\sum_n U_{kn} U_{nk'}^\dagger = \sum_n U_{kn} U_{-k'n} = \delta_{kk'}.$$

The inverse fourier transform is

$$q_n = \frac{1}{\sqrt{N}} \sum_{k>0}^{2\pi/a} e^{-ikx_n} q_k = \sum_k U_{nk}^\dagger q_k = \sum_k U_{-kn} q_k.$$

3. More balls and springs.

- (a) In lecture, we studied a one-dimensional lattice of balls and springs. Now consider a system in two dimensions with Lagrangian

$$L = \frac{m}{2} \sum_{x,y=1}^N \left(\dot{\phi}_{xy}^2 - c_1^2 \left(\frac{\phi_{xy} - \phi_{x+1,y}}{a} \right)^2 - c_2^2 \left(\frac{\phi_{xy} - \phi_{x,y+1}}{a} \right)^2 \right) \quad (1)$$

where ϕ_{xy} is a real variable. (This does not quite describe a two-dimensional grid of balls and springs (like a mattress), since ϕ is a scalar quantity.) Assume periodic boundary conditions in both x, y . Find the equations of motion, and find the normal modes (hint: the problem is translation invariant in two dimensions and linear). What is the dispersion relation? What is its long-wavelength, continuum limit? Interpret the constants c_1, c_2 and tell me what is special about the case $c_1 = c_2$.

The eom is

$$0 = \frac{\delta}{\delta \phi_{xy}(t)} \int dt L/m = -\ddot{\phi}_{xy} - \frac{c_1^2}{a^2} (2\phi_{xy} - \phi_{x+1,y} - \phi_{x-1,y}) - \frac{c_2^2}{a^2} (2\phi_{xy} - \phi_{x,y+1} - \phi_{x,y-1}).$$

Translation invariant and linear means solvable by Fourier transform. Here we can Fourier transform in both x and y :

$$\phi_{xy} = \frac{1}{\sqrt{N^2}} \sum_{k_x, k_y} \phi_{\vec{k}} e^{i\vec{k} \cdot \vec{x}}$$

with $\vec{k} = (k_x, k_y)$. I've assumed that the number of sites is N in both directions. Note that reality of ϕ_{xy} implies $\phi_k = \phi_{-k}^*$. Plugging this in to the eom gives

$$0 = \left(-\partial_t^2 - \sum_{\alpha=x,y} \frac{c_\alpha^2}{a^2} (2 - 2 \cos a k_\alpha) \right) \phi_k.$$

So the dispersion relation is

$$\omega^2 = \sum_{\alpha=x,y} \frac{c_\alpha^2}{a^2} (2 - 2 \cos a k_\alpha).$$

In the continuum (meaning long-wavelength) limit,

$$\frac{1}{a^2} (2 - 2 \cos a k_\alpha) = k_\alpha^2 + \mathcal{O}(ak)^4.$$

So this limit is well-described by the continuum action

$$S = \frac{m}{2a^2} \int dt dx dy \left(\dot{\phi}^2 - \sum_{\alpha} c_\alpha^2 (\partial_\alpha \phi)^2 \right).$$

$c_{1,2}$ are the speeds of propagation in the x and y directions. When they are equal, the system has rotation symmetry (and in fact full Lorentz symmetry) at low energies. That is, at wavenumbers where we can ignore the terms of order k^4 , there is a continuous rotation symmetry; there is an exact microscopic but discrete rotation symmetry, under rotation by $\pi/2$.

- (b) **Dimensional reduction.** Now consider a generalization where we have different numbers of lattice sites in the two directions:

$$L = \frac{m}{2} \sum_{x=1}^{N_x} \sum_{y=1}^{N_y} \left(\dot{\phi}_{xy}^2 - c_1^2 \left(\frac{\phi_{xy} - \phi_{x+1,y}}{a} \right)^2 - c_2^2 \left(\frac{\phi_{xy} - \phi_{x,y+1}}{a} \right)^2 \right). \quad (2)$$

Consider a special case with a small number of sites in the y -direction, such as $N_y = 2$, and interpret this as a chain in the x direction, with more than one kind of excitation that propagates in the x -direction. What is the spectrum?

We already have the answer: the dispersion relation is

$$\omega^2 = \sum_{\alpha=x,y} \frac{c_\alpha^2}{a^2} (2 - 2 \cos a k_\alpha).$$

But if we interpret k_y as merely a label on modes propagating in the x -direction, then the terms involving k_y are just like mass terms in a relativistic dispersion relation:

$$\omega^2 = m(k_y)^2 + c_x k_x^2 + \mathcal{O}(a k_x)^4$$

with

$$m(k_y)^2 = \frac{c_y^2}{a^2} (2 - 2 \cos a k_y).$$

This is called a Kaluza-Klein tower. Notice that one of the modes (the one with $k_y = 0$) is massless.

Some context for the name ‘Kaluza-Klein tower’: it is possible that there are extra dimensions of space, in addition to the three we observe. For example, imagine that all the fields of the Standard Model, for which the scalar field in this problem is a proxy, lived on $\mathbb{R}^3 \times S^1$, instead of on $\mathbb{R}^3$. That is, they depend on an extra coordinate y , in which they are periodic: $\phi(\vec{x}, y) = \phi(\vec{x}, y + L)$. Here $L = N_y a$ is the radius of the extra dimension. Now imagine, as is actually the case, that we are only capable of measuring the spectrum of excitations above the vacuum below some largest energy (determined by how powerfully we can smash particles together with our accelerators). If this largest accessible energy were less than $m(k_y = \frac{2\pi}{L})$, we would not be able to distinguish the spectrum from that of the world without the extra dimension (which would have only the $k_y = 0$ (massless) mode – we would never know. Maybe this is the case. It is not so important for this argument that the extra dimension is a circle. Any compact manifold has a gap in the spectrum of the Laplacian, which is what determines the spectrum of masses.

4. Heisenberg picture fields.

Here we will try to understand in what sense the field momentum of a free scalar field is $\pi \sim \dot{\phi}$, and we will explain the factor of $i\omega$ by which π and ϕ differ.

I usually think in what is called *Schrödinger picture*, where we evolve the states in time

$$|\psi(t)\rangle = \mathbf{U}(t)^\dagger |\psi(0)\rangle = e^{-i\mathbf{H}t/\hbar} |\psi(0)\rangle$$

and leave the operators alone. It is sometimes useful to define time-dependent operators by implementing the change of basis associated with $\mathbf{U}$ on the operators¹:

$$\mathbf{A}(t) \equiv \mathbf{U}(t)\mathbf{A}\mathbf{U}(t)^\dagger = e^{+i\mathbf{H}t/\hbar}\mathbf{A}e^{-i\mathbf{H}t/\hbar}.$$

This is called *Heisenberg picture*.

First consider a simple harmonic oscillator,

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{q}^2 = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$$

with

$$\begin{aligned}\mathbf{q} &= \sqrt{\frac{\hbar}{2m\omega}}(\mathbf{a} + \mathbf{a}^\dagger) = \sqrt{\frac{\hbar}{2m\omega}}2\text{Re}(\mathbf{a}). \\ \mathbf{p} &= -i\sqrt{\frac{\hbar m\omega}{2}}(\mathbf{a} - \mathbf{a}^\dagger) = \sqrt{\frac{m\hbar\omega}{2}}2\text{Im}(\mathbf{a}).\end{aligned}$$

(a) Using the algebra satisfied by $\mathbf{H}$ and $\mathbf{a}$, show that

$$\mathbf{q}(t) \equiv e^{+i\mathbf{H}t}\mathbf{q}e^{-i\mathbf{H}t} = \sqrt{\frac{\hbar}{2m\omega}}2\text{Re}(e^{-i\omega t}\mathbf{a}).$$

[Hint for one approach: $O(t) = e^{i\mathbf{H}t}Oe^{-i\mathbf{H}t}$ is the unique solution to the differential equation $\partial_t O(t) = i[\mathbf{H}, O(t)]$ with $O(0) = O$. What is $[\mathbf{H}, \mathbf{a}]$?

Because $[H, a] = -\omega a$, and $[H, a^\dagger] = \omega a^\dagger$, we have

$$e^{i\mathbf{H}t}\mathbf{a}e^{-i\mathbf{H}t} = e^{-i\omega t}\mathbf{a}, \quad \text{and} \quad e^{i\mathbf{H}t}\mathbf{a}^\dagger e^{-i\mathbf{H}t} = e^{+i\omega t}\mathbf{a}^\dagger. \quad (3)$$

Now add them together to reconstruct q .

(b) Using the expression above, show that

$$\mathbf{p}(t) \equiv e^{+i\mathbf{H}t}\mathbf{p}e^{-i\mathbf{H}t} = m\partial_t\mathbf{q}(t)$$

in agreement with what you would want from the Lagrangian formulation and from classical mechanics.

On the one hand, we can use (3) to evolve p . On the other hand we can differentiate $q(t)$ with respect to t . They give the same answer, yay.

Another way to do this is to use the Heisenberg equations of motion: $\mathbf{p}(t) = \frac{im}{\hbar}[\mathbf{H}, \mathbf{q}]$.

¹Recall that the signs are designed so that matrix elements are the same in either picture:

$$\underbrace{(\langle\phi(0)|\mathbf{U}(t))}_{= \langle\phi(t)|} \underbrace{\mathbf{A}(\mathbf{U}(t)^\dagger|\psi(0)\rangle)}_{|\psi(t)\rangle} = \langle\phi(0)|\underbrace{(\mathbf{U}(t)\mathbf{A}\mathbf{U}(t)^\dagger)}_{\mathbf{A}(t)}|\psi(0)\rangle$$

and the two pictures differ just by moving around parentheses.

The above was pretty simple, I hope. Now we consider a scalar quantum field theory, in say $d + 1 = 1 + 1$ dimensions:

$$\mathbf{H} = \int d^d x \left(\frac{\pi(x)^2}{2\mu} + \frac{1}{2} \mu v_s^2 (\vec{\nabla} \phi \cdot \vec{\nabla} \phi) + \frac{1}{2} \tilde{m}^2 \phi^2 \right) = \sum_k \hbar \omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2} \right).$$

$$\phi(x) = \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k L^d}} \left(e^{i\vec{k} \cdot \vec{x}} \mathbf{a}_k + e^{-i\vec{k} \cdot \vec{x}} \mathbf{a}_k^\dagger \right),$$

$$\pi(x) = \frac{1}{i} \sum_k \sqrt{\frac{\hbar\mu\omega_k}{2L^d}} \left(e^{i\vec{k} \cdot \vec{x}} \mathbf{a}_k - e^{-i\vec{k} \cdot \vec{x}} \mathbf{a}_k^\dagger \right), \quad (4)$$

- (c) Find ω_k . (Note that I've added a mass term $\tilde{m}^2 \phi^2$, relative to the model we studied in lecture. This is why I use μ instead of m for the object which looks like the inertial mass. Also note that $\tilde{m}$ does not have dimensions of mass.)

One way to identify the natural frequency of the normal mode of wavenumber $\vec{k}$ is to plug in fourier transforms $\phi(x) = \sum_k \phi_k e^{i\vec{k} \cdot \vec{x}}$ and similarly for π , to obtain

$$H = \sum_k \left(\frac{\pi_k \pi_{-k}}{2\mu} + \underbrace{\left(\mu v_s^2 \vec{k} \cdot \vec{k} + \tilde{m}^2 \right)}_{=\mu\omega_k^2} \phi_k \phi_{-k} \right).$$

We recognize this as the same form as the expression we got for the coupled chain of springs and identify ω_k^2 accordingly.

Alternatively, we could simply plug the expressions for ϕ and π into H . This is a little more painful, but instructive: Each of the three terms in H gives a sum of four kinds of terms, of the form:

$$\sum_{k,k'} \underbrace{\int dx e^{\pm i(k \pm k')x}}_{=2\pi\delta(k \pm k')} f(k, k').$$

The delta function leaves behind a single sum over k , with terms of the form $a_k^\dagger a_k + h.c.$ and $a_k a_{-k} + h.c.$.

Now choose ω_k so that the terms which involve both k and $-k$ (i.e. $a_k a_{-k}$ and its hermitian conjugate) cancel out between π^2 and $\vec{\nabla} \phi^2 + \tilde{m}^2 \phi^2$, leaving behind just a term of the form $\sum_k \omega (a_k^\dagger a_k + a_k a_k^\dagger)$. This gives the same answer for ω .

- (d) Do a Legendre transformation to construct the action,

$$S[\phi] = \int dt d^d x \mathcal{L}(\phi, \dot{\phi}) = \int dt L.$$

The Heisenberg equation relates the canonical momentum and the time derivative of the field: $\mathbf{i}\dot{\phi} = [\phi, H] = \frac{\pi}{\mu}$, so

$$L = \int dx \left(\pi \dot{\phi} - H \right) = \frac{1}{2} \int dx \left(\mu \dot{\phi}^2 - \mu v_s^2 \vec{\nabla} \phi \cdot \vec{\nabla} \phi - \tilde{m}^2 \phi^2 \right).$$

(e) Show that

$$\phi(t, x) \equiv e^{+\mathbf{iH}t} \phi(x) e^{-\mathbf{iH}t} = \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k L^d}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x} - \mathbf{i}\omega_k t} \mathbf{a}_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x} + \mathbf{i}\omega_k t} \mathbf{a}_k^\dagger \right)$$

Mode by mode, this is the same calculation as above, and the same for the next part.

(f) Using the previous result, show that

$$\pi(t, x) \equiv e^{+\mathbf{iH}t} \pi(x) e^{-\mathbf{iH}t} = \mu \partial_t \phi(t, x)$$

so that all is right with the world, in the sense that this agrees with (4).