

Physics 215A QFT Fall 2026 Assignment 2 – Solutions

Due 12:30pm Thursday, October 8, 2026

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

A comment about what to hand in: please feel free to skip straightforward steps in your final writeup to save yourself some typing.

1. **Brain-warmer on units.** Briefly show that we can convert between energy, momentum, inverse length, and inverse time by multiplying by various factors of $\hbar$ and c . Thus these dimensionful quantities are all directly comparable once we choose units where $\hbar = c = 1$. What are the length scale and time scale and mass scale associated with $1 \text{ GeV} = 10^9 \text{ eV}$? What are the units of the Newton constant of gravity, and what is its value expressed in terms of powers of GeV?

We have the following relations between units:

$$\text{energy} = \hbar \cdot \text{frequency} = \frac{\hbar}{\text{time}} = c^2 \cdot \text{mass}$$

(Actually ω is an angular frequency, so maybe there should be a factor of 2π .)

$$\text{length} = c \cdot \text{time}$$

$$\text{momentum} = c \cdot \text{mass} = \frac{\hbar}{\text{length}}$$

Using these we can eliminate everything in favor of mass, up to factors of $\hbar$ and c :

$$\begin{aligned} \text{energy} &= c^2 \cdot \text{mass} \\ \text{time} &= \frac{\hbar}{\text{energy}} = \frac{\hbar}{\text{mass} \cdot c^2} \end{aligned}$$

$$\begin{aligned} \text{length} &= \frac{\hbar}{c \cdot \text{mass}} \\ \text{momentum} &= c \cdot \text{mass} \end{aligned}$$

Using $\hbar = 10^{-34} \cdot 1.055 \text{ km}^2/\text{s}$ and $c = 10^8 \cdot 3.0 \text{ m/s}$, the length scale associated with an inverse GeV is then

$$\hbar c / \text{GeV} = \frac{(10^{-34} \cdot 1.055 \text{ Js})(10^8 \cdot 3 \text{ m/s})}{10^{-10} \cdot 1.6 \text{ J}} = 10^{-16} \cdot 1.97 \text{ m}$$

and the time scale is

$$\hbar/\text{GeV} = 10^{-25} \cdot 6.58s.$$

The mass scale is $\frac{\text{GeV}}{c^2} = 10^{-27} \cdot 1.78kg$.

Newton's constant is

$$G_N = 10^{-11} 6.67 \frac{m^3}{kg s^2},$$

an inverse mass-squared, so $[G_N] = -2$. Using the results above, in GeV Newton's constant is

$$G_N = 10^{-11} \cdot 6.67 \frac{\left(\frac{\hbar c}{10^{-16} \cdot 1.97 \text{GeV}}\right)^3}{\frac{\text{GeV}}{10^{-27} \cdot 1.78 c^2} \left(\frac{\hbar}{10^{-25} \cdot 6.58 \text{GeV}}\right)^2} = 10^{-39} \hbar c^5 6.72 \text{GeV}^{-2}.$$

2. **The correct way to think about equations of motion.** Suppose we have an action $S[\{\phi_r\}]$ that depends on a collection of fields $\phi_r(x)$ (say they live in D spacetime dimensions). The equation of motion is the condition for stationary phase in the path integral:

$$0 = \frac{\delta S}{\delta \phi_r(x)} \quad \text{for each } r. \quad (1)$$

Consider the special but important and familiar case when

- S is local: $S[\{\phi_r\}] = \int d^D x \mathcal{L}(\phi_r(x), \partial_\mu \phi_r(x), \partial_\mu \partial_\nu \phi_r(x), \dots)$ where the Lagrange density $\mathcal{L}$ depends only on the fields at x and their derivatives at x , and
- $\mathcal{L}$ depends only on ϕ and $\partial_\mu \phi$ (*i.e.* only first derivatives, and no higher derivatives of ϕ).

Show (using only the fact that the fields at different spacetime points are independent ($\frac{\delta \phi_r(x)}{\delta \phi_s(y)} = \delta_{rs} \delta^D(x - y)$), the chain rule, and integration by parts) that in this special case, the EOM (1) reduces to the Lagrange EOM that someone made you memorize:

$$0 = \frac{\partial \mathcal{L}}{\partial \phi_r} - \partial^\mu \frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi_r)} \quad \text{for each } r.$$

(Note that since we are interested here in the bulk equations of motion, we ignore boundary terms unless we are interested in field theory on a space with boundary. That is a worthy subject but an unnecessary complication for now.)

Under the given assumptions, the chain rule says that,

$$0 = \frac{\delta S}{\delta \phi_r(x)} \quad (2)$$

$$= \int d^D y \left(\frac{\partial \mathcal{L}}{\partial \phi_r(y)} \delta^D(x-y) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r(y))} \partial_\mu \delta^D(x-y) \right) \quad (3)$$

$$\stackrel{\text{IBP}}{=} \frac{\partial \mathcal{L}}{\partial \phi_r(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r(x))}. \quad (4)$$

3. **Momentum.** In this problem we consider a scalar field theory in d spatial dimensions. The scalar field is expanded in normal modes as usual:

$$\phi(\vec{x}) = \int d^d k \sqrt{\frac{\hbar}{2\omega_k}} \left(e^{i\vec{k}\cdot\vec{x}} a_k + e^{-i\vec{k}\cdot\vec{x}} a_k^\dagger \right),$$

where $\int d^d k \dots \equiv \int \frac{d^d k}{(2\pi)^d} \dots$. Consider the operator

$$\vec{P} \equiv \int d^d k \hbar \vec{k} a_k^\dagger a_k.$$

Since I wrote an integral for k , assume we are in the thermodynamic limit, and interpret the $\mathbf{a}$ s to be normalized as $[\mathbf{a}_k, \mathbf{a}_{k'}^\dagger] = (2\pi)^d \delta^d(k - k')$, a Dirac delta.

(a) Find $[\vec{P}, a_k^\dagger]$, and $[\vec{P}, a_k]$.

$$[\vec{P}, \mathbf{a}_k^\dagger] = \int d^d k' \hbar \vec{k}' [\mathbf{a}_{k'}^\dagger \mathbf{a}_{k'}, \mathbf{a}_k^\dagger] = \int d^d k' \hbar \vec{k}' \left(\mathbf{a}_k^\dagger (2\pi)^d \delta^d(k - k') \right) = \hbar \vec{k} \mathbf{a}_k^\dagger.$$

Similarly,

$$[\vec{P}, \mathbf{a}_k] = -\hbar \vec{k} \mathbf{a}_k.$$

(b) Show, using [3a](#) and the mode expansion of the scalar field, that

$$[\vec{P}, \phi(x)] = i\hbar \vec{\nabla} \phi(x).$$

The mode expansion for the field in infinite volume is

$$\phi(x) = \int d^d k \sqrt{\frac{\hbar}{2\mu\omega_k}} \left(e^{i\vec{k}\cdot\vec{x}} \mathbf{a}_k + e^{-i\vec{k}\cdot\vec{x}} \mathbf{a}_k^\dagger \right).$$

Therefore

$$[\vec{\mathbf{P}}, \phi(x)] = \int d^d k' \hbar \vec{k}' \int d^d k \sqrt{\frac{\hbar}{2\mu\omega_k}} [\mathbf{a}_{k'}^\dagger \mathbf{a}_{k'}, (e^{i\vec{k}' \cdot \vec{x}} \mathbf{a}_k + e^{-i\vec{k}' \cdot \vec{x}} \mathbf{a}_k^\dagger)] \quad (5)$$

$$= \int d^d k' \hbar \vec{k}' \int d^d k \sqrt{\frac{\hbar}{2\mu\omega_k}} \delta(k - k') (-e^{i\vec{k} \cdot \vec{x}} \mathbf{a}_k + e^{-i\vec{k} \cdot \vec{x}} \mathbf{a}_k^\dagger) \quad (6)$$

$$= \int d^d k \sqrt{\frac{\hbar}{2\mu\omega_k}} \hbar \vec{k} (-e^{i\vec{k} \cdot \vec{x}} \mathbf{a}_k + e^{-i\vec{k} \cdot \vec{x}} \mathbf{a}_k^\dagger) \quad (7)$$

$$= i\hbar \vec{\nabla} \phi(x). \quad (8)$$

(c) Conclude (using Taylor's theorem) that

$$e^{-i\vec{a} \cdot \vec{\mathbf{P}}/\hbar} \phi(x) e^{i\vec{a} \cdot \vec{\mathbf{P}}/\hbar} = \phi(x + a)$$

and that therefore $\vec{\mathbf{P}}$ generates translations. Therefore $\vec{\mathbf{P}}$ is the operator representing the momentum carried by the field (like the Poynting vector for the electromagnetic field).

Expanding the exponents on the LHS, this is the definition of the Taylor expansion. The previous part implies that¹

$$e^{-i\vec{a} \cdot \vec{\mathbf{P}}/\hbar} \phi(x) e^{i\vec{a} \cdot \vec{\mathbf{P}}/\hbar} = e^{\vec{a} \cdot \vec{\nabla}_x} \phi(x).$$

And Taylor's theorem can be written compactly as

$$e^{a\partial_x} f(x) = f(x + a).$$

An alternate route is to use the fact that

$$(1 - iaP/N)\phi(1 + iaP/N) = \phi(x) + \frac{a}{N} \partial_x \phi(x) + \mathcal{O}(N^{-2}) = \phi(x + a/N) + \mathcal{O}(N^{-2}).$$

Applying this equation N times and taking $N \rightarrow \infty$ gives our answer.

¹Here I am relying on the relation

$$e^{-ia \text{ad}_P} \phi(x) = e^{-iaP} \phi(x) e^{iaP}$$

where ad_P is a linear operator defined by $\text{ad}_P O \equiv [P, O]$. One way to prove this equation is just by Taylor expanding both sides in a and showing that they agree term-by-term. A slicker way is to notice that both sides satisfy the differential equation

$$\partial_a BHS(a) = \text{ad}_P BHS(a) \text{ with initial condition } BHS(a = 0) = \phi(x).$$

Since is a first order linear differential equation, it has a unique solution.

- (d) Find $\vec{\mathbf{P}} \left| \vec{k}_1, \vec{k}_2 \dots \vec{k}_n \right\rangle$, the action of this operator on a state of n phonons.

Conclude that $\hbar \vec{k}$ is the momentum of the phonon labelled by wavenumber $\vec{k}$, as promised in lecture.

Using the commutators from part (a) we have

$$\vec{\mathbf{P}} |k_1, k_2 \dots k_n\rangle = \sum_{i=1}^n \hbar \vec{k}_i |k_1, k_2 \dots k_n\rangle .$$

Since the eigenvalue of $\mathbf{P}$ is simply a sum of contributions from the individual particles, we can associate $\hbar k_i$ as the momentum of the phonon labelled $|k_i\rangle$.

- (e) [Bonus problem] Show that $\vec{\mathbf{P}}$ can be written as an integral of a local function of $\phi(x)$ and its derivatives.

One way to do this is to invert the relation between ϕ and $a, a^\dagger$. Another way is to guess: since it is quadratic in a and $a^\dagger$, it should be quadratic in ϕ ; acting with $\mathbf{i}\partial_x$ on ϕ produces a factor of k . So, we guess

$$\vec{\mathbf{P}} = - \int d^x \phi(x) \vec{\nabla} \phi(x) . \quad (9)$$

You can check this is correct by plugging in the mode expansions of ϕ and π , and using $\int d^d x e^{\mathbf{i}\vec{k}\cdot x} = (2\pi)^d \delta^d(k)$ to do one of the two k integrals. In doing so it is important to notice that

$$\int d^d k k^i a_k a_{-k} = 0 \quad (10)$$

since the integrand is odd under $k \rightarrow -k$.

We'll see that this is indeed the form we get from Noether's theorem.

4. Complex scalar field and antiparticles.

[This problem is related to Peskin problem 2.2.] So far we've discussed scalar field theory with one *real* scalar field. The particles created by such a field are their own antiparticles.

To understand this statement better, consider a scalar field theory in $d + 1$ dimensions with *two* real fields ϕ_1, ϕ_2 . Organize them into one complex field $\Phi \equiv \frac{1}{\sqrt{2}} (\phi_1 + \mathbf{i}\phi_2)$, with $\Phi^* = \frac{1}{\sqrt{2}} (\phi_1 - \mathbf{i}\phi_2)$, and let

$$S[\Phi, \Phi^*] = \int d^d x dt \left(\frac{1}{2} \mu \partial_t \Phi \partial_t \Phi^* - \frac{1}{2} \mu v^2 \vec{\nabla} \Phi \cdot \vec{\nabla} \Phi^* - V(\Phi^* \Phi) \right)$$

for some potential function $V(x)$.

(a) Show that

$$S[\Phi, \Phi^*] = \int \left(\sum_{i=1,2} \left(A (\partial_t \phi_i)^2 - B \vec{\nabla} \phi_i \cdot \vec{\nabla} \phi_i \right) - V((\phi_1^2 + \phi_2^2)/2) \right),$$

and where A, B are constants you must determine. If $V(q^2) = \frac{1}{2} \tilde{m}^2 q^2$, notice that the action is just the sum of two copies of the action of the theory we considered previously.

Essentially,

$$S[\Phi, \Phi^*] = \int \left(\sum_{i=1,2} \left(\frac{1}{2} (\partial_t \phi_i)^2 - \frac{1}{2} \vec{\nabla} \phi_i \cdot \vec{\nabla} \phi_i \right) - V((\phi_1^2 + \phi_2^2)/2) \right).$$

Including the factors, $A = \mu/4, B = \mu v^2/4$.

(b) Show by doing the Legendre transformation that the associated hamiltonian is

$$\mathbf{H} = \int d^d x \left(C \Pi \Pi^* + D \vec{\nabla} \Phi \cdot \vec{\nabla} \Phi^* + V(\Phi \Phi^*) \right)$$

where C, D are constants you must determine, and the canonical momenta are

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \frac{1}{2} \mu \dot{\Phi}^*, \quad \Pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^*} = \frac{1}{2} \mu \dot{\Phi}$$

with the Lagrangian density $\mathcal{L}$ defined by $S = \int dt d^d x \mathcal{L}$.

$$C = \frac{2}{\mu}, D = \frac{1}{2} \mu v^2.$$

Notice the slight weirdness that if π_α are the momenta conjugate to ϕ_α , $\alpha = 1, 2$, then $\Pi \propto \pi_1 - i\pi_2$.

- (c) This theory has a continuous symmetry under which $\Phi \rightarrow e^{i\alpha} \Phi, \Phi^* \rightarrow e^{-i\alpha} \Phi^*$ with α a real constant. Show that the action S does not change if I make this replacement. ²
- (d) The existence of a continuous symmetry means a conserved charge – a hermitian operator that commutes with the Hamiltonian, which generates the symmetry (this is the quantum version of the Noether theorem). Show that

$$\mathbf{q} \equiv \int d^d x \, i (\Phi^* \Pi^* - \Pi \Phi)$$

²This is called a U(1) symmetry: it is a unitary rotation (hence ‘U’) on a one-dimensional (hence ‘(1)’) complex vector. Notice that on the real components ϕ_1, ϕ_2 it acts as a two-dimensional rotation:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

The name for this group is SO(2). So U(1) is the same as SO(2).

generates this transformation, in the sense that the change in the field under a transformation with infinitesimal α is

$$\delta\Phi = \mathbf{i}\alpha\Phi = -\mathbf{i}\alpha[\mathbf{q}, \Phi], \quad \text{and} \quad \delta\Phi^* = -\mathbf{i}\alpha\Phi^* = -\mathbf{i}\alpha[\mathbf{q}, \Phi^*].$$

Show that $[\mathbf{q}, \mathbf{H}] = 0$.

We could appeal to Noether's theorem, since $\mathbf{q}$ is the conserved charge associated with a symmetry transformation. But the exercise is to show it directly using the equal-time commutators $[\Phi(x), \Pi(y)] = i\delta(x - y)$ and its complex conjugate equation $[\Phi^*(x), \Pi^*(y)] = i\delta(x - y)$. All other combinations of Φ s and Π s commute. The two terms of $\mathbf{H}$ must separately commute, so let's study them individually.

$$[\mathbf{q}, \int_y |\Pi|^2] = \mathbf{i} \int_x \int_y [(\Phi^* \Pi^* - \Pi \Phi)_x, \Pi_y^* \Pi_y] \quad (11)$$

$$= \mathbf{i} \int_x \int_y ([\Phi_x^*, \Pi^* \Pi_y] \Pi_x^* - \Pi_x [\Phi_x, \Pi^* \Pi_y]) \quad (12)$$

$$= \mathbf{i} \int_x \int_y ([\Phi_x^*, \Pi^*] \Pi_x^* \Pi_y - \Pi_x \Pi_y^* [\Phi_x, \Pi_y]) \quad (13)$$

$$= \mathbf{i} \int_x (\mathbf{i} \Pi^* \Pi - \mathbf{i} \Pi^* \Pi) = 0 \quad (14)$$

where we repeatedly used the identity $[A, BC] = B[A, C] + [A, B]C$.

Similarly, to find the commutator with $V(|\Phi|^2)$, we can use the chain rule $[A, f(B)] = f'(B)[A, B]$, which is true whenever $[A, B]$ commutes with B . The key ingredient then is $[\Pi(x), \Phi^* \Phi_y] = -\mathbf{i} \Phi^*(x) \delta(x - y)$. So

$$[\mathbf{q}, V(|\Phi|^2(y))] = \mathbf{i} \int_x (\Phi_x^* [\Pi_x^*, V_y] - [\Pi_x, V_y] \Phi_x) \quad (15)$$

$$= \mathbf{i} \int_x (\Phi_x^* V'(-\mathbf{i} \Phi \delta(x - y)) - (-\mathbf{i} \Phi^* \delta(x - y) V') \Phi_x) = |\Phi|^2 V'(1 - 1) = 0. \quad (16)$$

Stepping back a bit, we can see that each of these cancellations happened because the terms in H are invariant under the symmetry transformation – they are each a product of charged operators of net charge zero.

- (e) For the case where $V(\Phi\Phi^*) = \frac{1}{2}\tilde{m}^2\Phi\Phi^*$ the hamiltonian is quadratic. Diagonalize it in terms of *two sets* of creation operators and annihilation operators. Work in the continuum. You should find something of the form

$$\Phi(x) = A \sum_k \frac{1}{\sqrt{\omega_k}} \left(e^{\mathbf{i}kx} \mathbf{a}_k + e^{-\mathbf{i}kx} \mathbf{b}_k^\dagger \right) \quad (17)$$

(and a similar expression for Π) for some constant A that you should determine (you may wish to do the next part first).

In the version of the problem set I posted earlier, the normalization of Φ was not correct, assuming that a, b were the canonical raising and lowering operators. If you used it too literally you should have run into factor-of-two trouble on the next part.

We can quantize $\phi_{1,2}$ just as we did for a single real scalar field in lecture. Call their mode operators $\mathbf{a}_{1,2}$ respectively. Then we can add them to get Φ :

$$\Phi = \frac{1}{\sqrt{2}} (\phi_1 + \mathbf{i}\phi_2) = \sum_k \sqrt{\frac{\hbar}{2\mu}} \sum_k \frac{1}{\sqrt{\omega_k}} \left(e^{\mathbf{i}kx} \left(\underbrace{\mathbf{a}_{1k} + \mathbf{i}a_{2k}}_{\equiv \sqrt{2}\mathbf{a}} \right) + e^{-\mathbf{i}kx} \left(\underbrace{\mathbf{a}_{1k}^\dagger + \mathbf{i}a_{2k}^\dagger}_{\equiv \sqrt{2}\mathbf{b}_k^\dagger} \right) \right).$$

Note that the extra $\sqrt{2}$ s are required so that the mode algebra $[\mathbf{a}_{ik}, \mathbf{a}_{jk'}^\dagger] = \delta_{ij}\delta_{kk'}$ implies the correct commutators $[\Phi(x), \Pi(y)] = \mathbf{i}\delta(x-y)$.

To figure out the constant A we should do the check in the next part, which gives the following normalization:

$$\begin{aligned} \Phi &= \sqrt{\frac{\hbar}{\mu}} \sum_k \frac{1}{\sqrt{\omega_k}} \left(e^{\mathbf{i}kx} \mathbf{a}_k + e^{-\mathbf{i}kx} \mathbf{b}_k^\dagger \right) \\ \Phi^\star &= \sqrt{\frac{\hbar}{\mu}} \sum_k \frac{1}{\sqrt{\omega_k}} \left(e^{\mathbf{i}kx} \mathbf{b}_k + e^{-\mathbf{i}kx} \mathbf{a}_k^\dagger \right) \\ \Pi &= \frac{1}{\mathbf{i}} \sqrt{\frac{\hbar\mu}{4}} \sum_k \sqrt{\omega_k} \left(e^{\mathbf{i}kx} \mathbf{b}_k - e^{-\mathbf{i}kx} \mathbf{a}_k^\dagger \right) \\ \Pi^\star &= \frac{1}{\mathbf{i}} \sqrt{\frac{\hbar\mu}{4}} \sum_k \sqrt{\omega_k} \left(e^{\mathbf{i}kx} \mathbf{a}_k - e^{-\mathbf{i}kx} \mathbf{b}_k^\dagger \right) \end{aligned}$$

Let's check the Hamiltonian:

$$\begin{aligned} \mathbf{H} &= \int d^d x \left(\frac{2}{\mu} \Pi^\star \Pi + \frac{1}{2} \mu v^2 \vec{\nabla} \Phi \cdot \vec{\nabla} \Phi^\star + \frac{1}{2} m^2 \Phi^\star \Phi \right) \\ &= \frac{1}{2} \int d^d k \left[\left(\omega_k + \frac{v^2 k^2}{\omega_k} + \frac{\tilde{m}^2}{\mu \omega_k} \right) (a_k^\dagger a_k + b_k b_k^\dagger) + \left(\omega_k - \frac{v^2 k^2}{\omega_k} - \frac{\tilde{m}^2}{\mu \omega_k} \right) (a_k b_{-k} + h.c.) \right] \\ &= \int d^d k \hbar \omega_k (a_k^\dagger a_k + b_k b_k^\dagger). \end{aligned}$$

At the last step we used $\omega_k = \sqrt{v^2 k^2 + \tilde{m}^2/\mu}$ to cancel the $a_k b_{-k} + h.c.$ terms. This is the same frequency we get from the equations of motion.

(f) Write the canonical commutators

$$[\Phi(x), \Pi(x')] = i\hbar\delta^d(x - x'), \quad [\Phi(x), \Pi^\star(x')] = 0$$

(and the hermitian conjugate expressions) in terms of $\mathbf{a}$ and $\mathbf{b}$.

With the correct normalization in (17), we can write

$$a_k = \sqrt{\frac{\mu\omega_k}{2\hbar}} \int d^d x e^{-ikx} (\Phi(x) - \frac{\mathbf{i}}{\mu\omega_k} \Pi(x)) \quad (18)$$

which implies by complex conjugation

$$a_k^\dagger = \sqrt{\frac{\mu\omega_k}{2\hbar}} \int d^d x e^{ikx} (\Phi^\dagger(x) + \frac{\mathbf{i}}{\mu\omega_k} \Pi^\dagger(x)) . \quad (19)$$

A similar expression holds for b_k :

$$b_k^\dagger = \sqrt{\frac{\mu\omega_k}{2\hbar}} \int d^d x e^{-ikx} (\Phi(x) + \frac{\mathbf{i}}{\mu\omega_k} \Pi(x)) \quad (20)$$

which implies by complex conjugation

$$b_k = \sqrt{\frac{\mu\omega_k}{2\hbar}} \int d^d x e^{ikx} (\Phi^\dagger(x) - \frac{\mathbf{i}}{\mu\omega_k} \Pi^\dagger(x)) . \quad (21)$$

Then the canonical commutators give $[\mathbf{a}_k, \mathbf{a}_{k'}^\dagger] = (2\pi)^d \delta^d(k - k')$, $[\mathbf{a}, \mathbf{b}] = 0$, $[\mathbf{b}_k, \mathbf{b}_{k'}^\dagger] = (2\pi)^d \delta^d(k - k')$ etc (with appropriate momentum labels implied).

(g) Rewrite $\mathbf{q}$ in terms of the mode operators.

$$\mathbf{q} = \sum_k (\mathbf{a}_k^\dagger \mathbf{a}_k - \mathbf{b}_k^\dagger \mathbf{b}_k) = \sum_k (N_k^a - N_k^b) .$$

(h) Evaluate the charge of each type of particle created by $\mathbf{a}_k^\dagger$ and $\mathbf{b}_k^\dagger$ (*i.e.* find $[\mathbf{q}, \mathbf{a}^\dagger]$).

I claim that the particle created by $\mathbf{a}^\dagger$ is the antiparticle of that created by $\mathbf{b}^\dagger$ in the sense that they have opposite quantum numbers. This means that we can add terms to the Hamiltonian by which they can annihilate each other, without breaking any symmetries. What might such a term look like? Write something local in position space, *i.e.* of the form $\Delta H = \int d^d x V(\Phi, \Phi^\dagger)$.

The particles have opposite charge ± 1 in the sense that

$$[\mathbf{q}, \mathbf{a}] = \mathbf{a}, \quad [\mathbf{q}, \mathbf{b}] = -\mathbf{b}$$

so that $\mathbf{a}$ adds charge 1 to an eigenstate of $\mathbf{q}$, while $\mathbf{b}$ subtracts 1. Note that this is necessary so that Φ in (17) has definite charge 1.

There is a sense in which adding the term $\int d^D x \delta m^2 \Phi^\dagger(x) \Phi(x)$ accomplishes this, since

$$\int d^D x \Phi^\dagger(x) \Phi(x) = \int d^D k \omega_k \left((a_k b_{-k} + b_k a_{-k}) + \{a_k^\dagger, a_k\} + \{b_k^\dagger, b_k\} \right).$$

The term $a_k b_{-k}$ annihilates a pair of particles. However, it is a bit of a fake, since adding this term just modifies the vacuum. (That is, the groundstate of the new Hamiltonian is just annihilated by a different combination of oscillators.)

For example, the term $\int d^D x \lambda (\Phi^\dagger \Phi)^2$ certainly leads to processes where the particles and antiparticles annihilate, and it preserves the symmetry.