

Physics 215A QFT Fall 2026 Assignment 3 – Solutions

Due 12:30am Thursday, October 16, 2026

1. **Classical Maxwell theory.** [Peskin problem 2.1, lightly edited] Classical electromagnetism follows from the action

$$S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \right), \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

- (a) Derive two of the Maxwell equations (Gauss' Law and Faraday's Law) as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables:

$$0 = \frac{\delta S[A]}{\delta A_\mu(x)}.$$

Write the equations in the standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.

By the chain rule and product rule,

$$\begin{aligned} \frac{\delta}{\delta A_\mu(x)} \int d^4y \left(-\frac{1}{4} F_{\rho\sigma}(y) F^{\rho\sigma}(y) \right) &= -\frac{1}{2} \int d^4y \frac{\delta F_{\rho\sigma}(y)}{\delta A_\mu(x)} F^{\rho\sigma}(y) \\ \frac{\delta F_{\rho\sigma}(y)}{\delta A_\mu(x)} &= \partial_\rho \delta(x-y) \delta_{\mu\sigma} - (\rho \leftrightarrow \sigma) \end{aligned}$$

So the EOM is

$$0 = -\partial_\sigma F^{\mu\sigma} + j^\mu.$$

The components of this are the four equations

$$\vec{\nabla} \cdot \vec{E} = \rho, \quad \vec{\nabla} \times \vec{B} = -\partial_t \vec{E} + \vec{j}.$$

The other four Maxwell equations are in the constraint

$$0 = \epsilon^{\mu\rho\sigma\nu} \partial_\rho F_{\sigma\nu}$$

(which follows automatically from $F = dA$ by equality of mixed partials). Written in terms of E and B , the $\mu = t$ component gives $\vec{\nabla} \cdot \vec{B} = 0$ and the spatial components give $\vec{\nabla} \times \vec{E} = \partial_t \vec{B}$.

- (b) Construct the energy-momentum tensor for this theory, when $j^\mu = 0$. Note that the usual procedure

$$T_\nu^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta_\nu^\mu$$

does not result in a symmetric tensor. (It is also not gauge invariant.) To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} \equiv T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu,$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2} (E^2 + B^2), \quad \vec{S} = \vec{E} \times \vec{B}.$$

We find

$$T^{\mu\nu} = -F^{\mu\rho} \partial^\nu A_\rho + \frac{1}{4} \eta^{\mu\nu} F_{\cdot\cdot} F^{\cdot\cdot}. \quad (1)$$

The fact that it cannot be written in terms of only F is a bad sign (it is not gauge-invariant). The improved stress tensor is

$$\hat{T}^{\mu\nu} = F^{\mu\rho} F_\rho^\nu + \frac{1}{4} \eta^{\mu\nu} F_{\cdot\cdot} F^{\cdot\cdot}. \quad (2)$$

This is manifestly symmetric under exchange of $\mu \leftrightarrow \nu$, and gauge invariant because it only depends on F .

- (c) [Bonus problem] A better way to think about the energy-momentum tensor is to regard it as the response to a change in the background metric. (This is why it appears as a source in Einstein's equations, which is the equation of motion for the metric field.) To couple the Maxwell theory to a general background metric $g_{\mu\nu}$, we replace all the $\eta_{\mu\nu}$ s with $g_{\mu\nu}$ s:

$$S[A, g] = \int d^4x \sqrt{g} \left(-\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} + j^\mu A_\mu \right)$$

where the factor of $\sqrt{g} \equiv \sqrt{|\det g|}$ is required to make the integration measure coordinate-invariant, and $g^{\mu\nu}$ is the inverse metric: $g^{\mu\nu} g_{\nu\rho} = \delta_\rho^\mu$. Compare the resulting energy-momentum tensor

$$T_g^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S[A, g]}{\delta g_{\mu\nu}} \Big|_{g_{\mu\nu} = \eta_{\mu\nu}}.$$

with that of the previous part.

Notice that $T_g^{\mu\nu}$ is automatically symmetric and gauge invariant.

[Some useful identities are:

$$\frac{\delta g^{\mu\nu}(x)}{\delta g_{\rho\sigma}(y)} = -g^{\mu\rho}g^{\nu\sigma}\delta^D(x-y) \text{ and } \frac{\delta \det g(x)}{\delta g_{\mu\nu}(y)} = \delta^D(x-y) \det gg^{\mu\nu}.$$

For proofs of these statements see page 93 of [this document](#).]

The relative sign between $T^{\mu\nu}$ and $\frac{\delta S}{\delta g^{\mu\nu}}$ depends on the signature. Beware that most GR books use mostly-plus signature.

You might be bothered by the fact that the EM tensor is not gauge invariant if we include the coupling to the current. The resolution is that the gauge variation of the kinetic terms for the matter making up the current cancels this variation.

At the risk of using some ideas we haven't yet introduced in the class, suppose, for example, that we are speaking about QED. Then the $j^\mu A_\mu$ term is actually part of the kinetic term for the electron:

$$L_e = \bar{\psi}\gamma^\mu(\mathbf{i}\partial_\mu + A_\mu)\psi = \bar{\psi}\gamma^\mu\mathbf{i}\partial_\mu\psi + j^\mu A_\mu$$

which is gauge invariant (i.e. the lagrangian is gauge invariant, not just the action). The full stress tensor (the variation of the whole action $S_e + S_{\text{Maxwell}}$ with respect to $g_{\mu\nu}$) is therefore gauge invariant.

2. Maxwell's equations, quantumly.

- (a) Check that the oscillator algebra for the photon creation and annihilation operators

$$[\mathbf{a}_{ks}, \mathbf{a}_{k's}^\dagger] = \delta^3(k - k')\delta_{ss'}. \quad (3)$$

implies (using the mode expansion for $\mathbf{A}$) that

$$[\mathbf{A}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = -i\hbar \int \mathrm{d}^3k \, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \left(\delta_{ij} - \hat{k}_i \hat{k}_j \right)$$

(and also $[\mathbf{A}_i(\vec{r}), \mathbf{A}_j(\vec{r}')] = 0$ and $[\mathbf{E}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = 0$).

Conclude that it's not possible to simultaneously measure $E_x(\vec{r})$ and $B_y(\vec{r})$.

Plugging in the mode expansions, using the algebra of a s and $a^\dagger$ s, the factors

of $\sqrt{2\omega_k}$ and the phases cancel, and we get

$$\begin{aligned}
[\mathbf{A}_i(\vec{r}), \mathbf{E}_j(\vec{y})] &= \mathbf{i} \int_k \int_q \sum_{s,s'} \frac{1}{2} \sqrt{\frac{\omega_k}{\omega_q}} \left(e_{si}^*(k) e_{s'j}(k) e^{-\mathbf{i}kx + \mathbf{i}qy} [\mathbf{a}_{k,s}^\dagger, \mathbf{a}_{q,s'}] \right. \\
&\quad \left. - e_{si}(k) e_{s'j}^*(k) e^{\mathbf{i}kx - \mathbf{i}qy} [\mathbf{a}_{k,s}, \mathbf{a}_{q,s'}^\dagger] \right) \\
&= -\frac{\mathbf{i}}{2} \int \mathrm{d}^3k \sum_{s,s'} \left(e_{si}^*(k) e_{s'j}(k) e^{-\mathbf{i}k(x-y)} + e_{si}(k) e_{s'j}^*(k) e^{\mathbf{i}k(x-y)} \right) \\
&\quad - \frac{\mathbf{i}}{2} \int \mathrm{d}^3k \left(e^{-\mathbf{i}k(x-y)} + e^{\mathbf{i}k(x-y)} \right) \left(\delta_{ij} + \hat{k}_i \hat{k}_j \right). \tag{4}
\end{aligned}$$

The two terms are the same because we can change integration variables $k \rightarrow -k$.

The calculation of $[A, A]$ and $[E, E]$ is of the same form, except the terms cancel instead of add.

Once we have $[E, A]$, we can find $[E, B]$ by taking derivatives:

$$[E_i(r), B_j(r')] = \epsilon_{jkl} \frac{\partial}{\partial (x')^k} [E_i(r), A_l(r')].$$

The contribution from the $\hat{k}_i \hat{k}_l = \frac{k_i k_l}{k^2}$ term vanishes by $\epsilon_{jkl} k_k k_l = 0$.

The last statement about simultaneous measurement follows from

$$[E_x(r), B_y(r')] = \mathbf{i} \partial_z \delta(r - r') \neq 0.$$

- (b) Using the result of the previous part, check that the wave equation for $\mathbf{A}_i(x)$ follows from the Heisenberg equations of motion

$$\partial_t \vec{\mathbf{E}} = \frac{\mathbf{i}}{\hbar} [\mathbf{H}, \vec{\mathbf{E}}].$$

The key ingredient is

$$[H, a_{k,s}] = -\hbar \omega_k a_{k,s}, \quad [H, a_{k,s}^\dagger] = \hbar \omega_k a_{k,s}^\dagger.$$

So taking the commutator with H turns the expansion of E into that of A , except for an extra factor of ω_k^2 , which can be traded for $c^2 \vec{\nabla}^2$ when acting on $e^{\mathbf{i}\vec{k} \cdot \vec{x}}$.

Or, you can use the product rule for the commutator and the result of part a.

3. **Goldstone boson.** Here is a simple example of the Goldstone phenomenon, which I mentioned briefly in lecture. Consider again the complex scalar field from the previous assignment.

Suppose the potential is

$$V(\Phi^*\Phi) = g(\Phi^*\Phi - v^2)^2$$

where g, v are constants. The important features of V are that (1) it is only a function of $|\Phi|^2 = \Phi\Phi^*$, so that it preserves the particle-number symmetry generated by the operator $\mathbf{q}$ which was the hero of a previous homework problem, and (2) $V(x)$ is not minimized at $x = 0$.

Treat the system classically. Write the action $S[\Phi, \Phi^*]$ in polar coordinates in field space:

$$\Phi(x, t) = \rho e^{i\theta}$$

where both ρ, θ are functions of space and time.

- (a) Consider constant field configurations, and show that minimizing the potential fixes ρ but not the phase θ .

The potential is $g(\rho^2 - v^2)^2$, independent of θ . So $\theta \rightarrow \theta + \alpha$ is a symmetry.

- (b) Compute the mass² of the ρ field about its minimum, $m_\rho^2 = \frac{1}{2}\partial_\rho^2 V|_{\rho=v}$.
 $4gv^2$.

- (c) Now ignore the deviations of ρ from its minimum (it's heavy and slow and hard to excite), but continue to treat θ as a field. Plug the resulting expression

$$\Phi = v e^{i\theta(x,t)}$$

into the action. Show that θ is a massless scalar field.

$$S[\Phi = v e^{i\theta}] = \int d^d x dt \left(\frac{1}{2} \mu v^2 \left(\dot{\theta}^2 - \vec{\nabla} \theta \cdot \vec{\nabla} \theta \right) \right) \quad (5)$$

There's no mass term because that would violate the $\theta \rightarrow \theta + \alpha$ symmetry.

- (d) How does the $U(1)$ symmetry generated by $\mathbf{q}$ act on θ ?

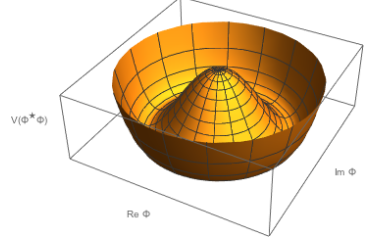
Since $\mathbf{q} : \Phi \rightarrow e^{i\alpha} \Phi = e^{i\alpha} \rho e^{i\theta}$, we can infer that $\mathbf{q} : \theta \rightarrow \theta + \alpha$. Alternatively, the action in terms of θ , (5) shows that $\pi_\theta = \mu \rho^2 \dot{\theta}$. Writing the charge in terms of θ gives

$$q = \int d^{d+1} x \mathbf{i} (\Phi^* \Pi^* - \Pi \Phi) = - \int \mu \rho^2 \dot{\theta} = - \int \pi_\theta.$$

Therefore

$$\delta \theta = \mathbf{i} \epsilon [q, \theta] = \epsilon.$$

This picture of the potential makes the story clear: the field θ is massless because it parametrizes rolling around in the circle of minima of the potential; the fact that these field values all have the same energy is enforced by the $U(1)$ symmetry.



4. Angular momentum as Noether charge.

Consider an arbitrary local scalar field theory with Lagrangian density $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ such that the action is invariant under rotations of space. A rotation acts on a scalar field by

$$\phi(x^i) \mapsto \phi(R_j^i x^j) \quad (6)$$

where R is a rotation matrix $R^T R = 1$. Consider an infinitesimal rotation $R_j^i = \delta_j^i + \omega_j^i$, for ω an arbitrary (small) real antisymmetric matrix specifying the plane and angle of rotation. Find the associated Noether currents in terms of ϕ .

Relate your answer to the form of the stress tensor. [Hint for interpreting the answer: a rotation is a space-dependent translation.]

The field changes according to

$$\phi(x) \mapsto \phi(Rx) = \phi(x + \omega x) = \phi(x) + \omega_j^i x^j \partial_i \phi \equiv \phi(x) + \Delta_\omega \phi. \quad (7)$$

So, by calculus, the Lagrangian density varies according to

$$\mathcal{L}(\phi(x), \partial \phi(x)) \mapsto \mathcal{L}(\phi(x + \omega x), \partial \phi(x + \omega x)) = \mathcal{L} + \omega_j^i x^j \frac{d}{dx^i} \mathcal{L} = \mathcal{L} + \frac{d}{dx^\mu} (\delta_i^\mu \omega_j^i x^j \mathcal{L}) \quad (8)$$

where at the last step we did a slightly sneaky thing: we can put the x inside the derivative because of the fact that $\omega_{ij} = -\omega_{ji}$ is an antisymmetric matrix, so the terms where the derivative hits the x vanish. Using the ‘prosaic method’, we have $\mathcal{L} \mapsto \mathcal{L} + \frac{d}{dx^\mu} \mathcal{J}^\mu$ with

$$\mathcal{J}^\mu = \delta_i^\mu \omega_j^i x^j \mathcal{L}. \quad (9)$$

Therefore, the collection of conserved currents is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta_\omega \phi - \mathcal{J}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \omega_j^i x^j \partial_i \phi - \mathcal{L} \delta_i^\mu \omega_j^i x^j. \quad (10)$$

Stripping of the factors of ω_j^i (and multiplying by two to make it look nicer, $j^\mu \equiv 2\omega^{ij} S_{ij}^\mu$), we get a collection of conserved currents with two antisymmetric indices:

$$S_{ij}^\mu = x_j \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_i \phi - \mathcal{L} \delta_i^\mu \right) - (j \leftrightarrow i). \quad (11)$$

Notice that it is manifestly antisymmetric in exchanging i and j because of the antisymmetry of ω .

Where have we seen this object before? Well, as you can see from the derivation, it is just

$$S_{ij}^\mu = x_j T_j^\mu - x_i T_i^\mu \quad (12)$$

where T is the stress tensor.

If you prefer, you may consider instead the full Lorentz group, which acts by

$$\phi(x^\mu) \mapsto \phi(\Lambda_\nu^\mu x^\nu) \quad (13)$$

where Λ is a $D \times D$ matrix preserving the Minkowski metric $\eta_{\mu\nu}$.

The same analysis applies to Lorentz transformations with the font changed from i, j to μ, ν .

Bonus problem: find the associated Noether charges in the case of a field theory of a vector such as electromagnetism, which transforms like $A_\mu(x) \mapsto (R^{-1})_\mu{}^\nu A_\nu(Rx)$.

A vector field transforms in a more complicated way:

$$\begin{aligned} A_\mu(x) \mapsto (R^{-1})_\mu{}^\nu A_\nu(Rx) &= (\delta - \omega)_\mu{}^\nu A_\nu(x + \omega x) = A_\mu(x) + \omega^\rho{}_\nu x^\nu \partial_\rho A_\mu(x) - \omega_\mu{}^\nu A_\nu(x) \\ &\equiv A_\mu(x) + (\Delta_\omega A)_\mu . \end{aligned} \quad (14)$$