Physics 215A: Quantum Field Theory
Fall 2026

Lecturer: McGreevy
These lecture notes live here. Please email corrections to mcgreevy at physics dot ucsd dot edu.

Last updated: 2026/09/24, 14:23:32

Contents

0.1 Introductory remarks

Quantum field theory (QFT) is the quantum mechanics of *extensive degrees of freedom*. What I mean by this is that at each point of space, there's some stuff that can wiggle. Such a collection of stuff is called a field, and QFT really is just the quantum theory of fields.

It's not surprising that QFT is so useful, since this situation happens all over the place. Some examples of 'stuff' are: the atoms in a solid, or the electrons in those atoms, or the spins of those electrons. A less obvious, but more visible, example is the electromagnetic field, even in vacuum. More examples are provided by other excitations of the vacuum, and it will be our job here to understand those very electrons and atoms that make up a solid in these terms. The vacuum has other less-long-lasting excitations which are described by the Standard Model of particle physics, an important and overwhelmingly successful example of a QFT.

Some examples of QFT are Lorentz invariant ('relativistic'). That's a nice simplification when it happens. Indeed this seems to happen in particle physics. We're going to focus on this case for most of this quarter. Still I would like to emphasize: though some of the most successful applications of QFT are in the domain of high energy particle physics, this is not a class on that subject, and I will look for opportunities to emphasize the universality of QFT.

A consequence of combining quantum mechanics with special relativity is that the number of particles isn't fixed. That is: there are processes where the number of particles changes in time. Later on (in (3.1)), we'll understand in what sense it's a necessary consequence of Lorentz symmetry. The converse is false: particle production can and does happen all the time in non-relativistic situations. This business of particle production is a crucial point of departure and motivation for QFT, worth emphasizing, so let me stop and emphasize it.

Few-particle QM. In classes with the title 'Quantum Mechanics', we generally study quantum systems where the Hilbert space $\mathcal{H}_1$ holds states of a single particle (or sometimes a fixed small number of them), which are *rays* in $\mathcal{H}_1$.

The observables of such a system are represented by hermitian operators acting on $\mathcal{H}_1$. For example, the particle has a position $\vec{\mathbf{x}}$ and a momentum $\vec{\mathbf{p}}$ each of which is a d-vector of operators (for a particle in d space dimensions). The particle could be, for example, an electron (in which case it also has an inherent two-valuedness called spin) or a photon (in which case it also has an inherent two-valuedness called polarization).

Time evolution is generated by a Hamiltonian $\mathbf{H}$ which is made from the position and momentum (and whatever internal degrees of freedom the particle has), in the sense that $\mathbf{i}\hbar\partial_t |\psi\rangle = \mathbf{H} |\psi\rangle$. Finally, the fourth (most ersatz) axiom of QM regards measurement: when measuring an observable $\mathbf{A}$ in a state $|\psi\rangle \in \mathcal{H}$, we should decompose the state in the eigenbasis of $\mathbf{A}$: $\mathbf{A}|a\rangle = a|a\rangle$, $|\psi\rangle = \sum_a |a\rangle\langle a|\psi\rangle$; the probability to get the answer a is $|\langle a|\psi\rangle|^2$.

By the way: The components of the state vector in the position basis $\langle \vec{x}|\psi\rangle = \psi(\vec{x})$ is a function of space, the wavefunction. This looks like a field. It is not what we mean by a field in QFT. Meaningless phrases like 'second quantization' may conspire to try to confuse you about this.

An important point I want to convey (as in the first sentence above) is that a QFT is an ordinary quantum-mechanical system, obeying these same axioms, but with a different Hilbert space, involving a lot more degrees of freedom (dofs).

Now suppose you want to describe quantumly the emission of a photon from an atom with an excited electron. Surely this is something for which we need QM. But it's not something that can happen within $\mathcal{H}_1$, since the number of particles changes during the process. How do you do it?

In the first section of this course we'll follow an organic route to discovering an answer to this question. This will have the advantage of making it manifest that the four axioms of QM just reviewed are still true in QFT. It will de-emphasize the role of Lorentz symmetry; in fact it will explicitly break it. Actually, Lorentz symmetry will emerge on its own!

'Divergences'. Another intrinsic and famous feature of QFT discernible from the definition I gave above is its flirtation with infinity. I said that there is 'stuff at each point of space': how much stuff is that? Well, there are two senses in which 'the number of points of space' is infinite: (1) space can go on forever (the infrared (IR)), and (2) in the continuum, in between any two points of space there are more points (the ultraviolet (UV)). The former may be familiar from statistical mechanics, where it is associated with the thermodynamic limit, which is where the interesting things happen. This limit is not the origin of the famous 'divergences' in QFT.

Every QFT is an Effective Field Theory. In fact, the origin of the famous 'divergences' is hubris, of trying to use a tool where it is not meant to be used. One lesson I want to emphasize is that every QFT has a regime of usefulness. A QFT which is not meant to describe everything in the world down to arbitrarily short distances is called an Effective Field Theory (EFT). Some QFTs have the decency to tell us when they are wrong (they are called 'non-renormalizable').

Weak coupling versus strong coupling. Before getting into it, I want to advertise the existence of an important dichotomy. This quarter, we're going to study a number of important and physically-relevant quantum field theories quite successfully. This success can be attributed to the fact that these theories are *weakly-coupled*. As we'll see, this means that we can understand them by starting from a large (very large) collection of springs, and doing perturbation theory. To quote from Brian Skinner's great description of his own experience of learning this aspect of quantum field theory: "So is the universe made of tiny springs...?" Yes, younger Brian, that is exactly what we are saying.

Except: not every quantum field theory is *usefully* described by this machinery. If the tiny springs are coupled to each other too strongly, then this starting point isn't so helpful for describing their groundstate and low-lying excitations. Among the excitations of our vacuum, those that participate in the accurately-named Strong Interactions have this property; so do lots of other quantum field theories that arise in other ways, such as in condensed matter physics. What I mean by "this starting point isn't so helpful" is that the groundstate (the vacuum) may not be adiabatically related to that of the free springs. The multiplicity of phases of matter is reflected in the phases of QFT. Understanding this will be a job for the later quarters of the QFT sequence. Understanding the simple cases well is a necessary first step.

Sources and acknowledgement. The material in these notes is collected from many places, among which I should mention in particular the following:

Peskin and Schroeder, *An introduction to quantum field theory* (Wiley)

Zee, *Quantum Field Theory in a Nutshell* (Princeton, 2d Edition)

Le Bellac, *Quantum Physics* (Cambridge)

Schwartz, *Quantum field theory and the standard model* (Cambridge)

David Tong's lecture notes

Many other bits of wisdom come from the Berkeley QFT course of Prof. Lawrence Hall.

I also recommend the newer book:

Fradkin, *Quantum Field Theory: An Integrated Approach* (Princeton)

which takes a similar perspective to mine.

0.2 Conventions

Following most QFT books, I am going to use the $+---$ signature convention for the Minkowski metric. I am used to the other convention, where time is the weird one, so I'll need your help checking my signs. More explicitly, denoting a small spacetime displacement as $dx^\mu \equiv (dt, d\vec{x})^\mu$, the Lorentz-invariant distance is:

$$ds^2 = +dt^2 - d\vec{x} \cdot d\vec{x} = \eta_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} .$$

(spacelike is negative). We will also write $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\partial_t, \vec{\nabla}_x\right)^\mu$, and $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$. I'll use $\mu, \nu...$ for Lorentz indices, and $i, k, ...$ for spatial indices.

$D = d+1$ is the number of dimensions of spacetime. The convention that repeated indices are summed is always in effect unless otherwise indicated.

$\equiv$ means 'equals by definition'. $A \overset{!}{=} B$ means we are demanding that $A = B$. $A \overset{?}{=} B$ means A probably doesn't equal B.

A consequence of the fact that english and math are written from left to right is that time goes to the left.

A useful generalization of the shorthand $\hbar \equiv \frac{h}{2\pi}$ is

$$đk \equiv \frac{\mathrm{d}k}{2\pi}.$$

I will also write $\not{\delta}^d(q) \equiv (2\pi)^d \delta^{(d)}(q)$. I will try to be consistent about writing Fourier transforms as

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{ikx} \tilde{f}(k) \equiv \int đ^d k \; e^{ikx} \tilde{f}(k) \equiv f(x).$$

WLOG $\equiv$ without loss of generality. IFF $\equiv$ if and only if.
RHS $\equiv$ right-hand side. LHS $\equiv$ left-hand side. BHS $\equiv$ both-hand side.
IBP $\equiv$ integration by parts. WLOG $\equiv$ without loss of generality.
$+\mathcal{O}(x^n) \equiv$ plus terms which go like x^n (and higher powers) when x is small.
$+h.c. \equiv$ plus hermitian conjugate.

We work in units where $\hbar$ and the speed of light, c, and Boltzmann's constant, k_B, are equal to one unless otherwise noted. When I say 'Peskin' I usually mean 'Peskin & Schroeder'.

Please email me if you find typos or errors or violations of the rules above.

1 From particles to fields to particles again

In modern physics, when I hear the word 'particle' I reach for my Fourier analyzer...
– J. Ziman, Electrons and Phonons.

Here is a way to discover QFT starting with some prosaic ingredients. Besides the advantages mentioned above, it will allow us to check that we are on the right track with well-understood experiments.

1.1 Quantum sound: Phonons

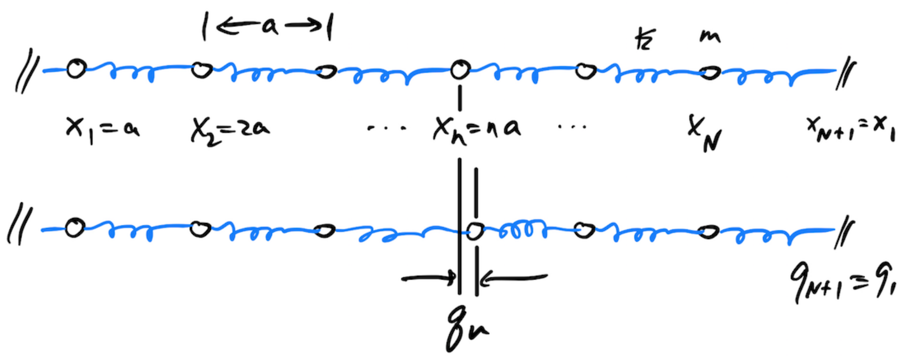
Let's think about a crystalline solid. The specific heat of solids (how much do you have to heat it up to change its internal energy by a given amount) was a mystery before QM. The first decent (QM) model was due to Einstein, where he supposed that the position of each atom in a solid is a (independent) quantum harmonic oscillator with frequency ω. This correctly predicts that the specific heat decreases as the temperature is lowered, but is very crude. Obviously the atoms interact: that's why they arrange themselves in a nice crystal pattern, and that's why there are sound waves, as we will see. By treating the elasticity of the solid quantum mechanically, we are going to discover quantum field theory. One immediate benefit of this will be a framework for quantum mechanics where particles can be created and annihilated.

What holds the atoms in a solid in place? Their interactions with their neighbors. Here we use two pieces of physics input: these interactions are translation invariant, and so only depend on the differences of positions $V(q_i - q_j)$, and they decay with distance, so we won't go too far wrong only considering interactions between neighbors.

As a more accurate toy model of a one-dimensional crystalline solid, let's consider a linear chain of particles of mass m, each connected to its neighbors by springs with spring constant κ. When in equilibrium, the masses form a regular one-dimensional crystal lattice (equally spaced mass points). Now let q_n denote the displacement of the nth mass from its equilibrium position x_n and let p_n be the corresponding momentum. Assume there are N masses and (for simplicity) impose periodic boundary conditions: $q_{n+N} = q_n$. The equilibrium positions themselves are

$$x_n = na, n = 1, 2...N$$

where a is the lattice spacing.



The Hamiltonian for the collection of particles is:

$$\mathbf{H} = \sum_{n=1}^{N} \left(\frac{\mathbf{p}_n^2}{2m} + \frac{1}{2}\kappa \left(\mathbf{q}_n - \mathbf{q}_{n-1}\right)^2 \right) + \lambda \mathbf{q}^4. \tag{1.1}$$

Notice that this system is an ordinary QM system, made of *particles*. In particular, the whole story below will take place within the fixed Hilbert space of the positions of the N particles.

I've included a heuristic token anharmonic term $\lambda \mathbf{q}^4$ to remind us that we are leaving stuff out; for example we might worry whether we could use this model to describe *melting*. A more precise form would be something like $\sum_{j=2} \lambda_j \sum_n \left(\frac{\mathbf{q}_n - \mathbf{q}_{n-1}}{a}\right)^j$, which describes the next terms in the Taylor expansion of some potential $V(q_n - q_{n-1})$ about the equilibrium configuration. If the springs are stiff, then the positions will not want to depart from their equilibrium values ($q_n - q_{n-1} \ll a$) and, even if the λ_j are all comparable, it might be a good approximation to ignore such terms. This partly explains the ubiquity of harmonic oscillators in discussions of physics[1]. By the way, the fact that a solid will suddenly melt if you heat it up or reduce the pressure sufficiently is an indication of what I was saying earlier about the multiplicity of phases of matter reflecting the multiplicity of phenomena in QFT – in a regime where the solid melts, our starting point here in terms of harmonic oscillators will emphatically not be a good description!

Now we'll just set $\lambda = 0$. (It will be a little while before we turn back on the *interactions* resulting from nonzero λ; bear with me.) This Hamiltonian above describes a collection of coupled harmonic oscillators, with a matrix of spring constants κ_{ab}, that is, the potential is of the form $V = \kappa_{ab}\mathbf{q}_a\mathbf{q}_b$. If we diagonalize the matrix of spring constants (which is a real symmetric matrix and so has real eigenvalues), we will have a description in terms of decoupled oscillators, called *normal modes*.

[1]The rest of the explanation is that it's the only system we can solve.

Since our system has (discrete) translation invariance (*i.e.* replacing (q_n, p_n) with $(q_{n+1}, p_{n+1}$ doesn't change the Hamiltonian), these modes are labelled by a wavenumber k[2], i.e. the matrix of spring constants κ_{ab} is diagonalized by Fourier transformation:

$$q_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{-\mathbf{i}kx_n} q_n, \quad p_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{-\mathbf{i}kx_n} p_n, \tag{1.3}$$

(Notice that in the previous expressions I didn't use boldface; that's because this step is really just classical physics. Note the awkward (but in field theory, inevitable) fact that we'll have (field) momentum operators $\mathbf{p}_k$ labelled by a wavenumber aka momentum.) The reason the Fourier kernel $e^{\mathbf{i}kx}$ appears here is that it diagonalizes the translation operator $\mathbf{T}$ defined by $\mathbf{T}f(x) = f(x+a)$:

$$\mathbf{T}e^{\mathbf{i}kx} \equiv e^{\mathbf{i}k(x+a)} = e^{\mathbf{i}ka} e^{\mathbf{i}kx}. \tag{1.4}$$

In case this expression seems at all mysterious, note that for our discrete, finite chain, $\mathbf{T}$ is represented by the matrix

$$\mathbf{T}_{nm} = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & 0 & 0 & 1 & \\ & & & \ddots & \\ 1 & 0 & & & 0 \end{pmatrix}_{nm} . \tag{1.5}$$

It might also help to introduce an auxiliary quantum system to think about this linear algebra problem: label the (position) basis states by $|n\rangle, n = 0..N-1$ (with the argument of the ket understood mod N). Then the translation operator is $\mathbf{T} = \sum_n |n-1\rangle\langle n|$. (1.4) says that its eigenvectors are $|k\rangle \equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{\mathbf{i}kan} |n\rangle$, as you can also check explicitly.

Regulators: Here we have our first meaningful encounter with the two ways in which divergences arise in field theory. Here, because N is finite, k takes discrete values $(1 = e^{\mathbf{i}kNa})$; this is a long-wavelength "IR" regulator. Because of the lattice structure, k is periodic (only $e^{\mathbf{i}kan}, n \in \mathbb{Z}$ appears): $k \equiv k + 2\pi/a$; this is a short-distance "UV" regulator. The range of k can be taken to be

$$0 \le k \le \frac{2\pi(N-1)}{Na} .$$

[2] The inverse transformation is:

$$q_n = \frac{1}{\sqrt{N}} \sum_{k>0}^{2\pi/a} e^{\mathbf{i}kx_n} q_k, \quad p_n = \frac{1}{\sqrt{N}} \sum_{k>0}^{2\pi/a} e^{\mathbf{i}kx_n} p_k. \tag{1.2}$$

Because of the periodicity in k, we can equivalently label the set of wavenumbers by[3]:

$$0 < k \leq \frac{2\pi}{a} \quad \text{or} \quad -\frac{\pi}{a} < k \leq \frac{\pi}{a}. \tag{1.6}$$

That is, because of the lattice spacing, there is a largest possible k, and a smallest possible wavelength (which is just the lattice spacing a itself).

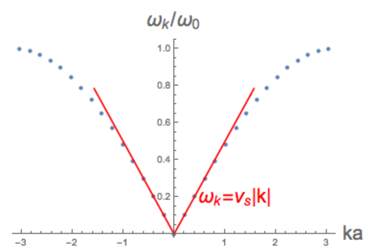
Summary: Because the system is in a box (periodic), k-space is discrete. Because the system is on a lattice, k-space is periodic. There are N oscillator modes altogether. The transformation from real space (modes labelled by n) to Fourier space (modes labelled by k) is an $N \times N$ unitary matrix.

The whole hamiltonian is a bunch of decoupled oscillators, labelled by these funny wave numbers[4]:

$$\mathbf{H} = \sum_k \left(\frac{\mathbf{p}_k \mathbf{p}_{-k}}{2m} + \frac{1}{2} m \omega_k^2 \mathbf{q}_k \mathbf{q}_{-k} \right) \tag{1.7}$$

where the frequency of the mode labelled k is

$$\omega_k \equiv 2\sqrt{\frac{\kappa}{m}} \sin \frac{|k|a}{2}. \tag{1.8}$$



[End of Lecture 1]

Why might we care about this frequency in (1.8)? For one thing, consider the Heisenberg (or Hamilton) equation of motion for the deviation of one spring:

$$\mathbf{i}\partial_t \mathbf{q}_n = [\mathbf{q}_n, \mathbf{H}] = \mathbf{i}\frac{\mathbf{p}_n}{m}, \quad \mathbf{i}\partial_t \mathbf{p}_n = [\mathbf{p}_n, \mathbf{H}].$$

Combining these gives:

$$m\ddot{q}_n = -\kappa\left((q_n - q_{n-1}) + (q_n - q_{n+1})\right) = -\kappa\left(2q_n - q_{n-1} - q_{n+1}\right).$$

In terms of the Fourier-mode operators:

$$m\ddot{\mathbf{q}}_k = -\kappa\left(2 - 2\cos ka\right)\mathbf{q}_k \ .$$

---

[3]This range of independent values of the wavenumber in a lattice model is called the *Brillouin zone*. There is some convention for choosing a fundamental domain which prefers the last one but I haven't found a reason to care about this.

[4]Note that $q_n = q_n^\star$ implies that $q_k = q_{-k}^\star$ (and similarly for ps), so the expression (1.7) is nice and hermitian.

Plugging in a Fourier ansatz in time $q_k(t) = \sum_\omega e^{-\mathbf{i}\omega t} q_{k,\omega}$ turns this into a completely algebraic equation (*i.e.* no derivatives) which says $\omega^2 = \omega_k^2 = \left(\frac{4\kappa}{m}\right) \sin^2 \frac{|k|a}{2}$ for the allowed modes. We see that (the classical version of) this system describes *waves*:

$$0 = \left(\omega^2 - \omega_k^2\right) q_{k,\omega} \overset{k \ll 2\pi/a}{\simeq} \left(\omega^2 - v_s^2 k^2\right) q_{k,\omega}. \tag{1.9}$$

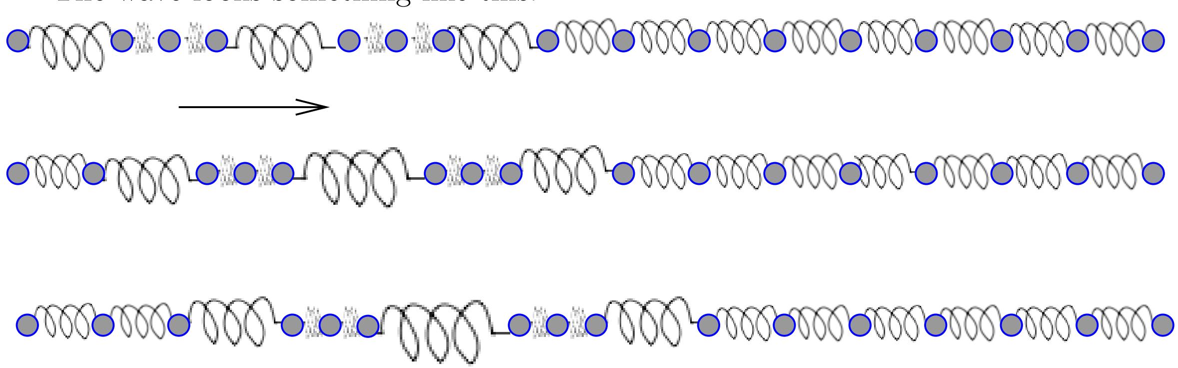
The result for small k is the Fourier transform of the wave equation:

$$\left(\partial_t^2 - v_s^2 \partial_x^2\right) q(x,t) = 0 \ . \tag{1.10}$$

v_s is the speed of propagation of the waves, in this case the speed of sound. Comparing to the dispersion relation (1.8), we have found

$$v_s = \frac{\partial \omega_k}{\partial k}|_{k\to 0} = a\sqrt{\frac{\kappa}{m}}.$$

The wave looks something like this:



These are sound waves. So the story I am telling is a quantization of sound waves. Soon we will understand quantization of electromagnetic (EM) waves, too. (Note however that it is bad practice to use 'quantize' as a verb meaning some operation we do on a classical system. Really the arrow goes the other way: classical mechanics is a crude approximation to quantum mechanics. There can even be different quantum systems with the same classical limit.)

Another way to derive the equations of motion is by varying the action:

$$S[q] = \int dt L, \qquad L = \sum_n p_n \dot{q}_n - H = \frac{1}{2}\sum_n m\dot{q}_n^2 - \frac{1}{2}\kappa_{n\ell} q_n q_\ell. \tag{1.11}$$

$$0 = \frac{\delta S}{\delta q_n(t)} = -m\ddot{q}_n - \kappa_{n\ell} q_\ell.$$

By the way, you may have learned some other ways to think about the Lagrange equations of motion. Forget them and use functional derivatives. The only thing you need to know is

$$\frac{\delta q_\ell(s)}{\delta q_n(t)} = \delta_{n\ell}\delta(t-s) \tag{1.12}$$

which says different degrees of freedom are independent. This is just the continuum limit in time of the statement

$$\frac{\partial q_{\ell\alpha}}{\partial q_{n\beta}} = \delta_{\ell n}\delta_{\alpha\beta}$$

that the values of the function at different times and places are independent. Here $q_{\ell\alpha} \equiv q_\ell(t_\alpha)$ and these are just ordinary partial derivatives.

We can put back the terms in the long-wavelength expansion (1.9) that we left out in the wave equation in (1.9). If the next correction is of the form $\omega_k^2 = v_s^2 k^2 + gk^4 + \cdots$, then the corresponding wave equation is instead

$$\left(\partial_t^2 - v_s^2\partial_x^2 + g\partial_x^4 + \cdots\right) q(x,t) = 0. \tag{1.13}$$

Notice that by dimensional analysis, since the $\partial_x^4 q$ has two more derivatives than the $\partial_x^2 q$ term, g/v_s^2 must have units of length^2. What length scale is it that suppresses the higher-derivative terms? The (tiny!) lattice spacing. More about the long-wavelength expansion below.

What action produces this wave equation as its equation of motion?

$$S[q] = \int dt \int dx \left(\frac{1}{2}(\partial_t q)^2 - \frac{1}{2}v_s^2(\partial_x q)^2 + \frac{g}{2}(\partial_x^2 q)^2 + \cdots\right). \tag{1.14}$$

Here the independent variable that we must vary to get the equations of motion[5] ($0 = \frac{\delta S[q]}{\delta q(x,t)}$, which is (1.13)) is $q(x,t)$, a *field*. Thus, we have seen field theory emerge from this lattice model of particles connected by springs.

Here is a slight generalization, by way of recap. Consider the slightly more general class of (still quadratic) Hamiltonians

$$\mathbf{H} = \sum_n \frac{\mathbf{p}_n^2}{2m_n} + \frac{1}{2}\sum_{nm} \kappa_{nm} q_n q_m.$$

[5]To derive (1.13) from this action, you'll need to use the basic fact $\frac{\delta q(x,t)}{\delta q(y,t)} = \delta(x-y)\delta(t-s)$ (plus ordinary calculus); this is the continuum limit in space of the relation (1.12).

We allowed the masses to vary, and we made a whole matrix of spring constants. Notice that only the symmetric part $\frac{1}{2}(\kappa_{nm} + \kappa_{mn})$ of this matrix appears in the Hamiltonian. Also κ must be real so that $\mathbf{H}$ is hermitian. To simplify our lives we can redefine variables

$$Q_n \equiv \sqrt{m_n} q_n, \quad P_n = p_n / \sqrt{m_n}, \quad V_{nm} \equiv \frac{\kappa_{nm}}{\sqrt{m_n m_m}}$$

in terms of which

$$\mathbf{H} = \frac{1}{2} \left(\sum_n P_n^2 + \sum_{nm} V_{nm} Q_n Q_m \right).$$

Notice that $[q_n, p_m] = \mathbf{i}\delta_{nm} \Leftrightarrow [Q_n, P_n] = \mathbf{i}\delta_{nm}$.

Now since V_{nm} is a symmetric matrix, and hence hermitian, it can be diagonalized. That is, we can find an $N \times N$ unitary matrix U so that $UVU^\dagger$ is a diagonal matrix:

$$\sum_{nm} U_{\alpha n} V_{nm} \left(U^\dagger\right)_{m\beta} = \delta_{\alpha\beta} \omega_\alpha^2 \tag{1.15}$$

where I assumed that all the eigenvalues of V are positive – otherwise the system would be unstable. U is unitary means:

$$\sum_n U_{\alpha n} \left(U^\dagger\right)_{n\beta} = \delta_{\alpha\beta}, \quad \sum_\alpha \left(U^\dagger\right)_{n\beta} U_{\beta m} = \delta_{nm}. \tag{1.16}$$

To take advantage of this, we make the change of variables to the *normal modes*

$$\tilde{Q}_\alpha = \sum_n U_{\alpha n} Q_n.$$

Multiplying the BHS of this equation by $U^\dagger$, we have the inverse relation

$$Q_n = \sum_\alpha \left(U^\dagger\right)_{n\alpha} \tilde{Q}_\alpha.$$

Notice that $Q_n = Q_n^\dagger$ is hermitian. This means

$$Q_n = Q_n^\dagger = \sum_n \left(U^\dagger\right)^\star_{n\alpha} \tilde{Q}_\alpha^\dagger = \sum_n \left(U^T\right)_{n\alpha} \tilde{Q}_\alpha^\dagger = \sum_n \tilde{Q}_\alpha^\dagger U_{\alpha n}.$$

Similarly, we define $\tilde{P}_\alpha = \sum_n U_{\alpha n} P_n$.

Now let's look at what this does to the terms in $\mathbf{H}$ (this first equation is sometimes called Parsival's Theorem):

$$\sum_n P_n^2 = \sum_n P_n^\dagger P_m = \sum_{\alpha\beta} \underbrace{\sum_n U_{\alpha n} \left(U^\dagger\right)_{n\beta}}_{\overset{(1.16)}{=} \delta_{\alpha\beta}} \tilde{P}_\alpha^\dagger \tilde{P}_\beta = \sum_\alpha \tilde{P}_\alpha^\dagger \tilde{P}_\alpha.$$

$$\sum_{nm} V_{nm} Q_n Q_m = \sum_{nm} V_{nm} Q_n^\dagger Q_m = \sum_{\alpha\beta} \underbrace{\sum_{nm} U_{\alpha n} V_{nm} \left(U^\dagger\right)_{m\beta}}_{\stackrel{(1.15)}{=} \delta_{\alpha\beta}\omega_\alpha^2} \tilde{Q}_\alpha^\dagger \tilde{Q}_\beta = \sum_\alpha \omega_\alpha^2 \tilde{Q}_\alpha^\dagger \tilde{Q}_\alpha.$$

The special case we considered earlier, $\kappa_{nm} = \left(\frac{1}{2}(\mathcal{T} + \mathcal{T}^T) - \mathbb{1}\right)_{nm}$, where each mass is only connected to its two neighbors, is special in three ways:

1. First, it is *local* in the sense that only nearby springs couple to each other, so κ_{nm} is only nonzero when n and m are close together.

2. Second, it has *discrete translation invariance*, that is, it is invariant under $(Q_n, P_n) \to (Q_{n+1}, P_{n+1})$. This is just because κ_{nm} does not depend explicitly on n and m. Because of the latter property, the normal modes are plane waves, $U_{kn} = e^{\mathbf{i}kna}/\sqrt{N}$ which has the consequences that $\left(U^\dagger\right)_{kn} = U_{-k,n}$ and hence that $Q_k^\dagger = Q_{-k}$.

3. Finally, it has a certain additional symmetry that also has a right to be called translation invariance from the microscopic point of view. This is the symmetry under $q_n \to q_n + \epsilon$, which shifts each mass by the same, arbitrary amount. This doesn't stretch the springs at all, and so is a symmetry of H. More precisely, this symmetry arises because the potential depends on q_n only through differences, $V = V(q_n - q_m)$.

   Notice that the transformation on the field q_n is non-linear, in the sense that the field shifts under the symmetry (as opposed to a linear transformation like $q \to Rq$). This is related to the fact that the transformation takes one minimum-energy configuration to another; in this sense, the symmetry is said to be *spontaneously broken*. An important general consequence of a non-linearly realized continuous symmetry is the presence of a massless mode, a Goldstone mode. We've already seen that there is such a massless mode in this system. This is an example of Goldstone's theorem. It's a bit mysterious in this example, but will become clearer later.

QM. So far the fact that quantumly $[\mathbf{q}_n, \mathbf{p}_{n'}] = \mathbf{i}\hbar\delta_{nn'}\mathbb{1}$ hasn't really mattered in our analysis (go back and check – we could have derived the wave equation classically)[6]. For the Fourier modes, this implies the commutator

$$[\mathbf{q}_k, \mathbf{p}_{k'}] = \sum_{n,n'} \mathbf{U}_{kn}\mathbf{U}_{k'n'}[\mathbf{q}_n, \mathbf{p}_{n'}] = \mathbf{i}\hbar\mathbb{1}\sum_n \mathbf{U}_{kn}\mathbf{U}_{k'n} = \mathbf{i}\hbar\delta_{k,-k'}\mathbb{1}.$$

[6] In case you are rusty, or forget the numerical factors like I do, here is a concise summary of the

(In the previous expression I called $\mathbf{U}_{kn} = \frac{1}{\sqrt{N}} e^{-\mathbf{i}kx_n}$ the unitary matrix realizing the discrete Fourier kernel.)

To make the final step to decouple the modes with k and $-k$, introduce the annihilation and creation operators[7]

$$\textbf{For } k \neq 0\textbf{:} \qquad \mathbf{q}_k = \sqrt{\frac{\hbar}{2m\omega_k}}\left(\mathbf{a}_k + \mathbf{a}^\dagger_{-k}\right), \quad \mathbf{p}_k = \frac{1}{\mathbf{i}}\sqrt{\frac{\hbar m\omega_k}{2}}\left(\mathbf{a}_k - \mathbf{a}^\dagger_{-k}\right). \tag{1.17}$$

They satisfy

$$[\mathbf{a}_k, \mathbf{a}^\dagger_{k'}] = \delta_{kk'}\mathbb{1}.$$

In terms of these, the hamiltonian is

$$\mathbf{H} = \sum_k \hbar\omega_k \left(\mathbf{a}^\dagger_k \mathbf{a}_k + \frac{1}{2}\right) + \frac{p_0^2}{2m} \tag{1.18}$$

operator solution of the quantum harmonic oscillator:

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{q}^2 = \frac{\hbar\omega}{2}\left(\mathbf{P}^2 + \mathbf{Q}^2\right) = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$$

with

$$\mathbf{a} \equiv \frac{1}{\sqrt{2}}\left(\mathbf{Q} + \mathbf{i}\mathbf{P}\right), \quad \mathbf{a}^\dagger \equiv \frac{1}{\sqrt{2}}\left(\mathbf{Q} - \mathbf{i}\mathbf{P}\right).$$

Here I've defined these new operators to hide the annoying factors:

$$\mathbf{Q} \equiv \left(\frac{m\omega}{\hbar}\right)^{1/2}\mathbf{q}, \quad \mathbf{P} \equiv \left(\frac{1}{m\hbar\omega}\right)^{1/2}\mathbf{p}.$$

$$[\mathbf{q}, \mathbf{p}] = \mathbf{i}\hbar\mathbb{1} \quad \Longleftrightarrow \quad [\mathbf{Q}, \mathbf{P}] = \mathbf{i}\hbar\mathbb{1} \quad \Longleftrightarrow \quad [\mathbf{a}, \mathbf{a}^\dagger] = \mathbb{1}.$$

The *number operator* $\mathbf{N} \equiv \mathbf{a}^\dagger\mathbf{a}$ satisfies

$$[\mathbf{N}, \mathbf{a}] = -\mathbf{a}, \quad [\mathbf{N}, \mathbf{a}^\dagger] = +\mathbf{a}^\dagger \quad .$$

So $\mathbf{a}$ and $\mathbf{a}^\dagger$ are lowering and raising operators for the number operator. The eigenvalues of the number operator have to be positive, since

$$0 \leq \| \, \mathbf{a}\left|n\right\rangle \, \|^2 = \left\langle n\right| \mathbf{a}^\dagger\mathbf{a}\left|n\right\rangle = \left\langle n\right| \mathbf{N}\left|n\right\rangle = n\left\langle n|n\right\rangle$$

which means that for $n = 0$ we have $\mathbf{a}\left|n = 0\right\rangle = 0$. If it isn't zero (*i.e.* if $n \geq 1$), $\mathbf{a}\left|n\right\rangle$ is also an eigenvector of $\mathbf{N}$ with eigenvalue $n - 1$. It has to stop somewhere! So the eigenstates of $\mathbf{N}$ (and hence of $\mathbf{H} = \hbar\omega\left(\mathbf{N} + \frac{1}{2}\right)$ are

$$\left|0\right\rangle, \quad \left|1\right\rangle \equiv \mathbf{a}^\dagger\left|0\right\rangle, \quad ..., \left|n\right\rangle = c_n\left(\mathbf{a}^\dagger\right)^n\left|0\right\rangle ...$$

where we must choose c_n to normalize these states. The answer that gives $\left\langle n|n\right\rangle = 1$ is $c_n = \frac{1}{\sqrt{n!}}$.

[7]You might notice that when $k = 0, \omega_k = 0$. This last step applies to the modes with $\omega_k \neq 0$, hence $k \neq 0$. The 'zero-mode' must be treated specially. It is neglected in many treatments of this topic but actually has a lot of physics in it. If you are curious see this discussion, page 11.

– it is a sum of decoupled oscillators, and a free particle describing the center-of-mass motion of the whole solid.

The discovery of Fock space. The groundstate is

$$|0\rangle \equiv |0\rangle_{\text{osc}} \otimes |p_0 = 0\rangle\,, \qquad \text{where} \quad \mathbf{a}_k\,|0\rangle_{\text{osc}} = 0,\ \forall\ k, \text{and } \mathbf{p}_0\,|p_0\rangle = p_0\,|p_0\rangle\ .$$

We know this is the groundstate because H is a constant plus a bunch of squares, and the squares all annihilate this state.

An excitation above the groundstate by the creation operator with wavenumber k

$$\mathbf{a}_k^\dagger\,|0\rangle \propto |\text{one phonon with momentum } \hbar k\rangle \tag{1.19}$$

has energy $\hbar\omega_k$ above the groundstate. (This is an eigenstate because $[N_k, \mathbf{a}_k^\dagger] = \mathbf{a}_k^\dagger$, where $N_k = \mathbf{a}_k^\dagger \mathbf{a}_k$.) The excitation is called a *phonon* with momentum $\hbar k$.[8] This is what in undergrad QM we would have called "$|k\rangle$"; we can make a state with one phonon in a position eigenstate by taking superpositions:

$$|\text{one phonon at position } x\rangle = \sum_k e^{\mathbf{i}kx}\,|\text{one phonon with momentum } \hbar k\rangle \sim \sum_k e^{\mathbf{i}kx}\mathbf{a}_k^\dagger\,|0\rangle\,.$$

This is the state that in single-particle QM we would have called "$|x\rangle$".[9]

The number operator (of the SHO with label k) $\mathbf{N}_k \equiv \mathbf{a}_k^\dagger \mathbf{a}_k$ counts the number of phonons with momentum k. The ground state is the state with no phonons – for this reason we could also call it the 'vacuum'. We can also make a state with two phonons:

$$|k, k'\rangle = \mathbf{a}_k^\dagger \mathbf{a}_{k'}^\dagger\,|0\rangle$$

whose energy above the groundstate is

$$E - E_0 = \omega_k + \omega_{k'}. \tag{1.20}$$

Note that all these states have non-negative energy.

So this construction allows us to describe situations where the number of particles $\mathbf{N} = \sum_k \mathbf{N}_k$ can vary! That is, we can now describe dynamical processes in which the number of particles changes. Let me emphasize: In QM, we would describe the Hilbert space of two (distinguishable) particles as a tensor product of the Hilbert space of each.

[8] I put 'proportional to' rather than 'equal' in (1.19) because there can be a k-dependent normalization factor. We'll see soon that Lorentz symmetry prefers a particular normalization here which we will adopt.

[9] For now, the fact that $|k\rangle$ and $|x\rangle$ are related by a Fourier transform will have to serve for evidence that $\hbar k$ is the momentum of the particle we've just discovered. Later we will show that indeed the state $\mathbf{a}_k^\dagger\,|0\rangle$ has momentum $\hbar k$ above the groundstate.

How can we act with an operator that enlarges the hilbert space?? We just figured out how to do it.

We can specify basis states for this Hilbert space

$$\left(\mathbf{a}^\dagger_{k_1}\right)^{n_{k_1}}\left(\mathbf{a}^\dagger_{k_2}\right)^{n_{k_2}}\cdots|0\rangle = |\{n_{k_1}, n_{k_2}, ...\}\rangle$$

by a collection of *occupation numbers* n_k, eigenvalues of the number operator for each normal mode (and the center-of-mass momentum p_0).

Notice that in this description it is manifest that a phonon has no identity. We only keep track of how many of them there are and what is their wavenumber. They cannot be distinguished. Also notice that we can have as many we want in the same mode – n_k can be any non-negative integer. These are an example of *bosons*. (Further evidence for this is that $\mathbf{a}^\dagger_k\mathbf{a}^\dagger_{k'} = \mathbf{a}^\dagger_{k'}\mathbf{a}^\dagger_k$.)

Lessons from the hard work of the first two lectures:

- Starting from a collection of particles, we chained them together, and made a field; treating this system quantumly, we found a new set of particles. The new particles (the normal modes) are *collective* excitations: their properties can be very different from those of the constituent particles. For example, the constituent particles have dispersion relation $E(p) = \frac{p^2}{2m}$, but the phonons have dispersion relation $E(p) = \hbar\omega_p \overset{p\ll h/a}{\sim} v_s|p|$.

  And the constituent particles are distinguishable by their locations, but phonons are indistinguishable from each other. The state of such indistinguishable particles is determined merely by specifying a collection of positions (or momenta) and of spin states – we don't need to say which is which (and in fact, we cannot).

- Notice that there are some energies where there aren't any phonon states. In particular, the function (1.8) has a maximum. More generally, in a system with discrete translation invariance, there are *bands* of allowed energies. In the continuum limit (roughly $a \to 0$ with the correct quantities held fixed), to which we devolve soon, this maximum (which occurs at $k = \frac{\pi}{a}$) goes off to the sky.

- **Lorentz invariance can emerge.** The dispersion relation for the long-wavelength ($ka \ll 1$) sound mode was $\omega^2 = v^2\vec{k}^2$. This is the fourier transform of

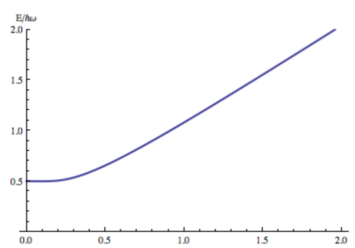
$$(\partial_t^2 - v^2\vec{\nabla}^2)\phi(x) = \partial_\mu\partial^\mu\phi(x) = 0, \tag{1.21}$$

  a wave equation with Lorentz symmetry (if v is the speed appearing in the Minkowski metric). To get (1.21), we had to ignore the $\mathcal{O}(a^4k^4)$ terms in the long-wavelength expansion of the dispersion relation, $2(1-\cos(ka))$. The lattice breaks Lorentz symmetry, but its effects go away for $ka \ll 1$. This point might make you think that the Lorentz symmetry which is so precious to us in particle physics could emerge in a similar way, but with a much smaller a than the lattice spacing in solids. There are strong constraints on how small this can be (*e.g.* this well-appreciated paper) so it is very useful to treat it as a fundamental principle.

- When $N < \infty$, everything is completely finite. And yet we see that at low energies the system is well-described by a quantum field theory. Even in the thermodynamic limit, $N \to \infty$ with fixed a, there is a maximum possible ω_k (and a maximum possible k) for each phonon. This is a well-regulated QFT. Observe the tension between the non-symmetric, safe lattice and the symmetric, dangerous continuum.

Phonons are real: heat capacity of (insulating) solids. The simplest demonstration that phonons are real is the dramatic decrease at low temperatures of the heat capacity of insulating solids. At high temperatures, the equipartition theorem of classical thermodynamics correctly predicts that the energy of the solid from the lattice vibrations should be T times the number of atoms, so the heat capacity, $C_V = \partial_T E$ should be independent of T.

At low temperatures $T < \Theta_D$, this is wrong. Θ_D is the temperature scale associated with the frequencies of the lattice vibrations (say the maximum of the curve ω_k above). The resolution lies in the thermal energy of a quantum harmonic oscillator for $T < \omega$, the energy goes to a constant $\frac{1}{2}\hbar\omega$, independent of T:



So the heat capacity (the slope of this curve) goes to zero as $T \to 0$[10].

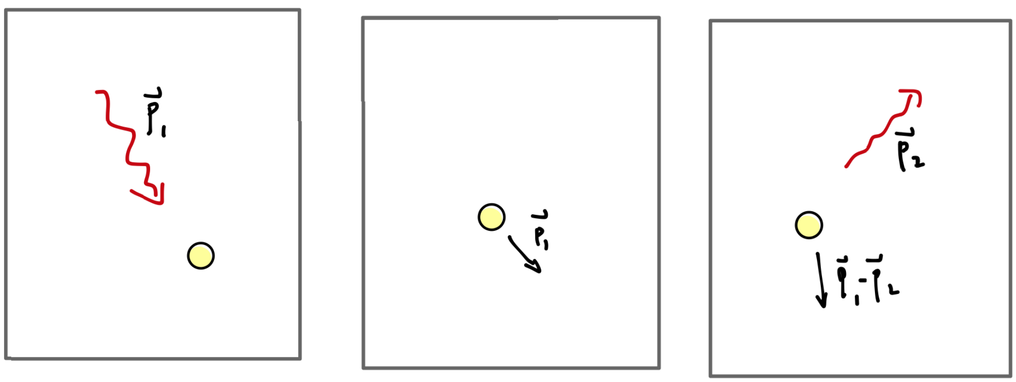
Phonons are real: the Mössbauer effect. Here is another dramatic consequence of the quantization of the lattice vibrations of solids, known as the *Mössbauer effect*, first described in words. The nuclei of the atoms in a solid have various energy levels; when hit with a γ-ray photon, these nuclei can experience transitions from the groundstate to some excited energy level. If an excited nucleus somewhere in the lattice gets hit by a very energetic photon (a γ-ray) of some very specific energy $E_\gamma = \Delta E \equiv E_{\text{excited}} - E_0$, the nucleus can absorb and re-emit that photon. The resulting sharp resonant absorption lines at $E_\gamma = \Delta E$ are indeed observed.

[10]More precisely, the energy in thermal equilibrium of our collection of springs is

$$\sum_k \frac{\omega_k}{e^{\omega_k/T}-1} \overset{L\to\infty, T\ll\Theta_D}{=} \int đ^d k \frac{v_s k}{e^{v_s k/T}-1} = \frac{T^{d+1}}{v_s^{d-1}} \int đ^d q \frac{q}{e^q - 1}, \quad q \equiv v_s k/T.$$

So the heat capacity is $C_V \sim T^d \to 0$ as $T \to 0$.

This sounds simple, but here is a mystery about this, depicted in the comic strip below: Consider a nucleus alone in space in the excited state, after it gets hit by a photon. The photon carried a momentum $p_\gamma = E_\gamma/c$. Momentum is conserved, and it must be made up by some *recoil* of the absorbing nucleus. This means that not all of the energy of the photon can go into exciting the nucleus. When it emits a photon again, it needn't do so in the same direction. This means that the nucleus remains in motion with momentum $\Delta\vec{p} = \vec{p}_1 - \vec{p}_2$. But if some of its energy $\Delta E = E_{\text{excited}} - E_0$ goes to kinetic energy of recoil, not all of that energy can go to the final photon, and the emitted photon energy will be *less* than E_γ by $E_{\text{recoil}} = \frac{\Delta p^2}{2M}$. This can be as big as $E^{\text{max}}_{\text{recoil}} = \frac{(2\vec{p})^2}{2M} = \frac{(2E\gamma/c)^2}{2M}$ (in the case of scattering by angle π). So instead of a sharp absorption line, it seems that we should see a broad bump of width $\frac{(E\gamma/c)^2}{M}$. But we *do* see a sharp line when scattering off a solid!



The solution of the puzzle is phonons: for a nucleus in a lattice to recoil requires that the springs are stretched – it must excite a lattice vibration, it must create some phonons. But there is a nonzero probability for it to create *zero* phonons. In this case, the momentum conservation is made up by an acceleration of *the whole solid*, which is very massive (its mass is NM where N is the number of nuclei!), and therefore does not recoil very much at all (it loses only energy $\frac{p_\gamma^2}{2NM}$)[11].

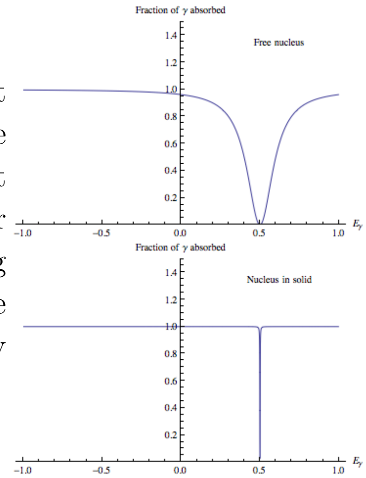
[11] Let's see this in terms of our solution in terms of Fourier modes. Periodic boundary conditions and demanding that the particles are identical says that $|\{q_n\}\rangle \equiv |\{q_n + a\}\rangle$ are the same state. In terms of the $k = 0$ mode, this is

$$q_0 = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} q_n e^{-\mathbf{i}0x_n} \equiv \frac{1}{\sqrt{N}} \left(\sum_{n=1}^{N} q_n + Na \right), \quad i.e.\ q_0 \equiv q_0 + \sqrt{N}a.$$

This means that the wavefunction for the zeromode must satisfy

$$e^{\mathbf{i}p_0 q_0} = e^{\mathbf{i}p_0(q_0 + \sqrt{N}a)} \implies p_0 \in \frac{2\pi\mathbb{Z}}{\sqrt{N}a}$$

This allows for very sharp resonance lines. In turn, this effect has allowed for some very high-precision measurements. The different widths in these cartoon absorption spectra don't do justice to the relative factor of N. An essentially similar effect makes it possible to get precise peaks from scattering of X-rays off of a solid (Bragg scattering) – there is a finite amplitude for the scattering to occur without exciting any phonons.



This is actually a remarkable thing: although solids seem ordinary to us because we encounter them frequently, the rigidity of solids is a quantum mechanical emergent phenomenon. You can *elastically* scatter photons off of a solid only because the atoms making up the solid participate in this collective behavior wherein the whole solid acts like a single quantum object!

Interactions. We now pause the celebration to point out some limitations of this analysis. Many aspects of the above discussion are special to the fact that our Hamiltonian was quadratic in the canonical operators q, p. Certainly our ability to completely solve the system is. In particular, the additivity of the energy of the excitations (1.20) is an artifact of this limit where the Hamiltonian is quadratic.

Notice that the number of phonons *of each momentum* $\mathbf{N}_k \equiv \mathbf{a}_k^\dagger \mathbf{a}_k$ is conserved for each k (*i.e.* commutes with $\mathbf{H}$). This is a lot of symmetry! But if we add generic cubic and quartic terms in $\mathbf{q}$ (or if we couple our atoms to the photon field) even the total number of phonons $\sum_k \mathbf{N}_k$ will no longer be a conserved quantity[12]. So a description of such particles that forced us to fix their number wouldn't be so great.

For example, think about expanding a general cubic term $\sum_{nml} \lambda_{nml} q_n q_m q_l$ in os-

---

and the first excited state has energy

$$\frac{p_0^2}{2m}|_{p_0=\frac{2\pi}{\sqrt{N}a}} = \frac{1}{2}\frac{1}{Nm}\left(\frac{2\pi}{a}\right)^2.$$

The factor of $\frac{1}{Nm}$ is the suppression by the mass of the whole solid. (Recall that $N \sim 10^{24}$.)

[12] Note that it *is* possible to make a non-quadratic action for conserved particles, but this requires adding more degrees of freedom – the required $\mathsf{U}(1)$ symmetry must act something like

$$(q^1, q^2) \to (\cos\theta q^1, \sin\theta q^2).$$

We can reorganize this as a complex field $\Phi = q^1 + \mathbf{i}q^2$ on which the symmetry acts by $\Phi \to e^{\mathbf{i}\theta}\Phi$.

cillators. It has terms like

$$a^\dagger_{k_1} a^\dagger_{k_2} a^\dagger_{k_3}, \quad a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3}, \quad a^\dagger_{k_1} a_{k_2} a_{k_3}, \quad a_{k_1} a_{k_2} a_{k_3}$$

all of which change the number of phonons. A quartic term will also contain terms like $a^\dagger a^\dagger a a$ which preserve the number of phonons, but describe an interaction between them, where they exchange momentum.

More generally, I would like to remind you again that not every QFT can be usefully considered as nearly a bunch of harmonic oscillators. Finding ways to think about the ones that can't is an important job for later.

Towards scalar field theory. It is worthwhile to put together the final relation between the 'position operator' and the phonon annihilation and creation operators[13]:

$$\mathbf{q}_n = \sqrt{\frac{\hbar}{2\mu}} \sum_k \frac{1}{\sqrt{\omega_k}} \left(e^{\mathbf{i} k x_n} \mathbf{a}_k + e^{-\mathbf{i} k x_n} \mathbf{a}_k^\dagger \right) + \frac{1}{\sqrt{N}} \mathbf{q}_0 \tag{1.22}$$

and the corresponding relation for its canonical conjugate momentum

$$\mathbf{p}_n = \frac{\mu}{\mathbf{i}} \sqrt{\frac{\hbar}{2\mu}} \sum_k \sqrt{\omega_k} \left(e^{\mathbf{i} k x_n} \mathbf{a}_k - e^{-\mathbf{i} k x_n} \mathbf{a}_k^\dagger \right) + \frac{1}{\sqrt{N}} \mathbf{p}_0 .$$

The items in red are the ways in which p and q differ; they can all be understood from the relation $p = \mu \dot{q}$ as you will see on the homework. These expressions are formally identical to the formulae we'll find below expressing a scalar field in terms of creation and annihilation operators (such as Peskin eqns. (2.25) and (2.26)). The stray factors of μ arise because we didn't 'canonically normalize' our fields and absorb the ms into the field, *e.g.* defining $\phi \equiv \sqrt{\mu} q$ would get rid of them. The other difference is because we still have IR and UV regulators in place, so we have sums instead of integrals.

Path integral reminder in a box. [A useful reference is Zee §I.2] If we use the path integral description, some of these things (in particular the continuum, sound-wave limit) are more obvious-seeming.

Let's remind ourselves how the path integral formulation of QM works for a particle in one dimension with $\mathbf{H} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q})$. The basic statement is the following formula for the propagator – the amplitude to propagate from position eigenstate $|q_0\rangle$ to position eigenstate $|q\rangle$ during a time interval t is

$$\langle q | e^{-\mathbf{i}\mathbf{H}t} | q_0 \rangle = \int_{q(0)=q_0}^{q(t)=q} [dq] e^{\mathbf{i} \int_0^t ds \, \left(\frac{1}{2} \dot{q}^2 - V(q) \right)} .$$

[13] Note that relative to (1.17), we relabeled the summation variable $k \to -k$ in the second term.

Here $[dq] \equiv \mathcal{N} \prod_{l=1}^{M_t} dq(t_l)$ – the path integral measure is defined by a limiting procedure ($M_t \equiv \frac{t}{\Delta t} \to \infty, \Delta t \to 0, t$ fixed), and $\mathcal{N}$ is a normalization factor that always drops out of physical quantities so I don't need to tell you what it is.

Recall that the key step in the derivation of this statement is the evaluation of the propagator for an infinitesimal time step:

$$\langle q_2| e^{-\mathbf{i}\mathbf{H}\Delta t} |q_1\rangle = \langle q_2| e^{-\mathbf{i}\Delta t \frac{\mathbf{p}^2}{2m}} e^{-\mathbf{i}\Delta t V(\mathbf{q})} |q_2\rangle + \mathcal{O}(\Delta t^2) \ .$$

An integral expression for this involving only numbers (no operators, no states) can be obtained by inserting resolutions of the identity

$$\mathbb{1} = \mathbb{1}^2 = \left(\int đp|p\rangle\langle p|\right)\left(\int \mathrm{d}q|q\rangle\langle q|\right) = \int đp \int dq|p\rangle\langle q|e^{ipq}$$

in between the two exponentials. For a more extensive reminder, please see §2.4 of this document.

Two quick but invaluable applications of the path integral:

1. The path integral *explains* the origin of the variational principle and the special role of configurations that solve the equations of motion. They are points of stationary phase in the path integral, where

$$0 = \frac{\delta S}{\delta q(t)}. \tag{1.23}$$

   In case you are not familiar with functional derivatives: From the definition of the path integral as a limit it should be clear what is meant by this expression. If we didn't take the limit, the stationary phase condition is that S is extremized in every direction $q_i = q(t_i)$:

$$0 = \frac{\partial S}{\partial q_i}.$$

   The functional derivative in (1.23) means the same thing but looks nicer because we can pretend everything is smooth rather than discrete. Since q_i and q_j are independent variables,

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij} \qquad \text{and therefore} \qquad \frac{\delta q(t)}{\delta q(s)} = \delta(t-s).$$

   With the help of the chain rule, this is the only functional derivative you need to know how to take[14].

   Part of this statement is a discovery of the true role of $\hbar$: its job is to make up the units so that the exponent in $e^{\mathbf{i}S[q]/\hbar}$ is dimensionless.

[14]If you are not yet comfortable with the machinery of functional derivatives, please work through pages 2-28 through 2-30 of this document right now.

2. Above I wrote a formula for the real-time propagator. Euclidean path integrals are also very useful, because they compute ground-state expectation values. Here's why:

   The vacuum can be prepared by starting in an arbitrary state and acting with e^{-TH} for some large T, and then normalizing (as usual when discussing path integrals, it's best to not worry about the normalization and only ask questions that don't depend on it),

   $$|0\rangle = \mathcal{N} e^{-\mathbf{H}T} |\text{any}\rangle .$$

   To see this, just expand in the energy eigenbasis. This 'imaginary time evolution operator' $e^{-\mathbf{H}T}$ has a path integral representation just like the real time operator, by nearly the same calculation

   $$e^{-\mathbf{H}T} = \int [Dq] e^{-\int_{-T}^{0} d\tau L(q(\tau),\dot{q}(\tau))} |q(0)\rangle\langle q(-T)|.$$

   Doing the same thing to prepare $\langle 0|$, making a sandwich of $f(\mathbf{q})$ and taking $T \to \infty$ we can forget about the arbitrary states at the end, and we arrive at

   $$\langle f(\mathbf{q})\rangle \equiv \frac{\langle 0| f(\mathbf{q}) |0\rangle}{\langle 0|0\rangle} = \frac{1}{Z}\int [Dq] e^{-S_E} f(q(0)) = \frac{1}{Z}\int \prod_i^{M_t} dq_i \; e^{-\sum_j L(q_j)} f(q_0) \tag{1.24}$$

   where $Z \equiv \int \prod_i dq_i \; e^{-\sum_j L(q_j)}$ (here i, j are discrete time labels) and $S_E[q]$ is the euclidean action. Relative to the real time action, there is an important sign[15]:

   $$S_E[q] = \int d\tau \left((\partial_\tau q)^2 + V(q) \right).$$

   This sign makes good sense – it means that configurations with large V are *suppressed* in the euclidean path integral.

   This trick of using imaginary time evolution to prepare the groundstate is a crucial one that we will use all the time below[16].

   In the special case where L is quadratic in q and $\dot{q}$, this can be written as

   $$Z = \int \prod_i dq_i \; e^{-q_i D_{ij} q_j}$$

[15] By the way, I use square brackets for $S[q]$ to remind myself that S is a function*al* of q, a machine that eats functions and spits out numbers – for each function $q(\tau)$, $S[q]$ is a number. But in fact my main point here is that for our purposes there is not such a big difference between a functional and a function of many variables (the values of the function at a discrete collection of points).

[16] In lecture, I mentioned that we don't know how to do imaginary time evolution in the laboratory. Here is one recent attempt to do it using measurements. A reason to believe that it shouldn't be possible to do this *efficiently* is that it can be used to solve NP complete problems (by encoding their solution into the groundstate of a Hamiltonian).

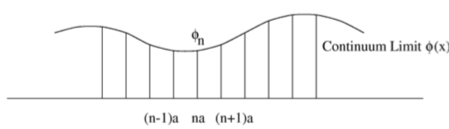
where D_{ij} is the (real, symmetric) matrix which discretizes the action. Repeated indices are summed. This is a gaussian integral. This is the path integral point of view on why quadratic terms are special.

This discussion of path integrals has been for a single degree of freedom q. But as we've seen, we can make a field theory by coupling together a collection of such degrees of freedom q_n. The path integral measure for a field theory is just the product of the measures for each of these q_n – it's just a matter of sticking more labels on the integration variables.

1.2 Scalar field theory

Scalar field theory in one dimension. [Zee §1.3] The (real-time) path integral for our collection of oscillators is

$$Z = \int [dq_1 \cdots dq_N] e^{\mathbf{i}S[q]}$$

with $S[q] = \int dt \left(\sum_n \frac{1}{2} m_n \dot{q}_n^2 - V(\{q\})\right) \equiv \int dt L(q, \dot{q})$. The potential is $V(\{q\}) = \sum_n \frac{1}{2}\kappa \left(q_{n+1} - q_n\right)^2$. Now suppose we have poor eyesight and can't resolve the individual atoms in the chain; rather we're only interested in the *long-wavelength* (small-wavenumber) physics. Sometimes this is described as taking a continuum limit, $a \to 0, N \to \infty$. Basically the only thing we need is to think of $q_n = q(x = na)$ as defining a smooth function: 

[Note that the continuum field is often called $\phi(x)$ instead of $q(x)$ for some reason. At least the letters $q(x)$ and $\phi(x)$ look similar.]

By Taylor expanding $q(x_{n-1})$ about $q(x_n)$, we have

$$\left(q_n - q_{n-1}\right)^2 \simeq a^2 \left(\partial_x q\right)^2 |_{x=na}, \quad a \sum_n f(q_n) \simeq \int dx f(q(x)).$$

The path integral becomes:

$$Z = \int [dq] e^{\mathbf{i}S[q]}$$

with $[dq]$ now representing an integral over all configurations $q(t, x)$ (*defined* by this limit) and

$$S[q] = \int dt \int dx \frac{1}{2} \left(\mu \left(\partial_t q\right)^2 - \mu v_s^2 \left(\partial_x q\right)^2 - rq^2 - uq^4 - \ldots\right) \equiv \int dt \int dx \mathcal{L} \quad (1.25)$$

where I've introduced some parameters μ, v_s, r, u determined from $m, \kappa, a...$ in some ways that we needn't worry about, except to say that they are finite in the continuum limit. The $\cdots$ includes terms like $a^4 \left(\partial_x q\right)^4$ that are small when $k \ll \frac{1}{a}$, so we ignore them. $\mathcal{L}$ is the Lagrangian *density* whose integral over space is the Lagrangian $L = \int dx \mathcal{L}$.

The equation of motion is the stationary phase condition,

$$0 = \frac{\delta S}{\delta q(x,t)} = -\mu \ddot{q} + \mu v_s^2 \partial_x^2 q - rq - 2uq^3 - ...$$

In this expression I have written a functional derivative; with our lattice regulator, it is simply a(n extremely useful) shorthand notation for the collection of partial derivatives $\frac{\partial}{\partial q_n}$.

From the phonon problem, we automatically found $r = u = 0$, and the equation of motion is just the wave equation (1.10). This happened because of the symmetry $q_n \to q_n + \epsilon$. This is the operation that *translates* the whole crystal. It guarantees low-energy phonons near $k = 0$ because it means $q(x)$ can only appear in S via its derivatives. (This is a general property of *Goldstone modes*; more on this later.)

The following will be quite useful for our subsequent discussion of quantum light. We can construct a hamiltonian from this action by defining a canonical field-momentum $\pi(x) = \frac{\partial \mathcal{L}}{\partial_t q} = \mu \partial_t q$ and doing the Legendre transformation:

$$H = \sum_n \left(p_n \dot{q}_n - L_n\right) = \int dx \left(\pi(x)\dot{q}(x) - \mathcal{L}\right) = \int dx \left(\frac{\pi(x)^2}{2\mu} + \mu v_s^2 \left(\partial_x q(x)\right)^2 + rq^2 + uq^4 + ...\right). \tag{1.26}$$

(Here in the first step I wrote $L \equiv \sum_n L_n$ to emphasize locality.) I emphasize that the position along the chain x here is just a *label* on the fields, not a degree of freedom or a quantum operator. This dramatic shift of perspective is why I am spending time on this step. (Note that I suppress the dependence of all the fields on t just so it doesn't get ugly, not because it isn't there.)

The field q is called a *scalar field* because it doesn't have any indices decorating it. This is to be distinguished from the Maxwell field, which is a vector field, and which we'll discuss next. (Note that vibrations of a crystal in *three* dimensions actually do involve vector indices! We omit this complication.)

The lattice spacing a and the size of the box Na in the discussion above are playing very specific roles in *regularizing* our 1-dimensional scalar field theory. The lattice spacing a implies a maximum wavenumber or shortest wavelength and so is called an "ultraviolet (UV) cutoff", because the UV is the short-wavelength end of the visible light spectrum. The size of the box Na implies a maximum wavelength mode which fits in the box and so is called an "infrared (IR) cutoff".

If we also take the infinite volume limit, then the sums over k become integrals. In this limit we can make the replacement

$$\frac{1}{L^d}\sum_k \rightsquigarrow \int đ^d k, \quad L^d \delta_{kk'} \rightsquigarrow (2\pi)^d \delta^{(d)}(k-k').$$

A check of the normalization factors comes from combining these two rules

$$\forall k',\ 1 = \sum_k \delta_{k,k'} = \int đ^d k (2\pi)^d \delta^{(d)}(k-k').$$

Continuum (free) scalar field theory in $d+1$ dimensions. These continuum expressions are easy to generalize to scalar field theory in any number of dimensions. Let's do this directly in infinite volume and set $\mu = 1$ by rescaling fields (*i.e.* $\phi \equiv \sqrt{\mu} q$). The action is

$$S[\phi] = \int d^d x dt \, \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2} v_s^2 \vec{\nabla}\phi \cdot \vec{\nabla}\phi - V(\phi)\right). \tag{1.27}$$

This is almost what we would have found for the long-wavelength ($ka \ll 1$) description of a d-dimensional lattice of masses on springs, like a mattress (except that there would have been one ϕ for each direction in which the atoms can wiggle). The equation of motion is

$$0 = \frac{\delta S[\phi]}{\delta \phi(x)} = -\partial_t^2 \phi + v_s^2 \nabla^2 \phi - V'(\phi). \tag{1.28}$$

For the harmonic case $V(\phi) = \frac{1}{2} m^2 \phi^2$ we know what we're doing, and (1.28) is called the Klein-Gordon equation,

$$0 = \left(\partial_\mu \partial^\mu + m^2\right)\phi. \tag{1.29}$$

(Notice that I've set $v_s = c = 1$ here, and this is where we have committed to a choice of signature convention; take a look at the conventions page §0.2.). In relativistic notation, the Lagrangian density is just $\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right)$. This describes free continuum real massive relativistic scalar quantum field theory. (Match the adjectives to the associated features of the lagrangian; collect them all!)

Let's solve it by canonical methods. The canonical momentum[17] is $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$ and the Hamiltonian (which we can instantly promote to a quantum operator by using boldface symbols) is then

$$\mathbf{H} = \int d^d x \left(\frac{\boldsymbol{\pi}(x)^2}{2} + \frac{1}{2} v_s^2 \left(\vec{\nabla}\boldsymbol{\phi} \cdot \vec{\nabla}\boldsymbol{\phi}\right) + \frac{1}{2} m^2 \boldsymbol{\phi}^2\right).$$

[17]More correctly, π is the canonical momentum *density*, since $\mathcal{L}$ is the Lagrangian density not the Lagrangian. But everyone always forgets the word 'density' in both cases.

Note that all these terms are positive.

A translation invariant linear problem is solved by Fourier transforms:

$$\boldsymbol{\phi}(x) = \int đ^d k \; e^{\mathbf{i}\vec{k}\cdot\vec{x}} \boldsymbol{\phi}_k, \text{ and } \boldsymbol{\pi}(x) = \int đ^d k \; e^{\mathbf{i}\vec{k}\cdot\vec{x}} \boldsymbol{\pi}_k. \tag{1.30}$$

The hamiltonian is then

$$\mathbf{H} = \int đ^d k \left(\frac{1}{2} \boldsymbol{\pi}_k \boldsymbol{\pi}_{-k} + \frac{1}{2} \left(v_s^2 \vec{k}^2 + m^2 \right) \boldsymbol{\phi}_k \boldsymbol{\phi}_{-k} \right)$$

where $\vec{k}^2 \equiv (-\mathbf{i}\vec{k}) \cdot (\mathbf{i}\vec{k}) = \vec{k} \cdot \vec{k}$. Just as in (1.7), this is merely a sum of decoupled oscillators, except for the coupling between wavenumbers k and $-k$. Comparing with (1.7), we can read off the normal mode frequencies, aka the dispersion relation:

$$\omega_k^2 = v_s^2 \vec{k}^2 + m^2.$$

$\omega = \pm \omega_k$ is precisely the condition for the Fourier mode $e^{\mathbf{i}\vec{k}\cdot\vec{x} - \mathbf{i}\omega t}$ to solve the Klein-Gordon equation (1.29).

We can decouple the modes with wavenumber k and $-k$ as above by introducing the ladder operators[18]

$$\boldsymbol{\phi}_k \equiv \sqrt{\frac{\hbar}{2\omega_k}} \left(\mathbf{a}_k + \mathbf{a}_{-k}^\dagger \right), \quad \boldsymbol{\pi}_k \equiv \frac{1}{\mathbf{i}} \sqrt{\frac{\hbar \omega_k}{2}} \left(\mathbf{a}_k - \mathbf{a}_{-k}^\dagger \right), \qquad [\mathbf{a}_k, \mathbf{a}_{k'}^\dagger] = (2\pi)^d \delta^{(d)}(k - k'). \tag{1.31}$$

Their commutator follows from $[\phi(x), \pi(y)] = \mathbf{i}\delta^{(d)}(x - y)$[19]. In terms of the ladder operators,

$$\mathbf{H} = \int đ^d k \; \hbar \omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2} L^d \right).$$

Combining the previous two steps (1.30) and (1.31), the field operators are

$$\begin{aligned} \boldsymbol{\phi}(\vec{x}) &= \int đ^d k \; \sqrt{\frac{\hbar}{2\omega_k}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x}} \mathbf{a}_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x}} \mathbf{a}_k^\dagger \right), \\ \boldsymbol{\pi}(\vec{x}) &= \frac{1}{\mathbf{i}} \int đ^d k \; \sqrt{\frac{\hbar \omega_k}{2}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x}} \mathbf{a}_k - e^{-\mathbf{i}\vec{k}\cdot\vec{x}} \mathbf{a}_k^\dagger \right). \end{aligned} \tag{1.32}$$

The mode expansions (1.32) contain a great deal of information. First notice that $\boldsymbol{\phi}$ is manifestly hermitian. Next, notice that from $\boldsymbol{\phi}(\vec{x}) \equiv \boldsymbol{\phi}(\vec{x}, 0)$ by itself we cannot disentangle $\mathbf{a}_k$ and $\mathbf{a}_k^\dagger$, since only the combination $\mathbf{a}_k + \mathbf{a}_{-k}^\dagger$ multiplies $e^{\mathbf{i}\vec{k}\cdot\vec{x}}$. The

[18]Beware that the mode operators $\mathbf{a}_k$ defined here differ by powers of $2\pi/L$ from the finite-volume objects in the previous discussion. These agree with Peskin's conventions.

[19]I emphasize that this is really the same equation as our starting point for each ball on springs: $[\mathbf{q}_n, \mathbf{p}_{n'}] = \mathbf{i}\hbar \mathbb{1} \delta_{nn'}$.

momentum $\boldsymbol{\pi}$ contains the other linear combination. However, if we evolve the field operator in time using the Heisenberg equation (as you did on the HW), we find

$$\phi(\vec{x},t) \equiv e^{\mathbf{i}\mathbf{H}t}\phi(\vec{x})e^{-\mathbf{i}\mathbf{H}t} = \int đ^d k \sqrt{\frac{\hbar}{2\omega_k}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x}-\mathbf{i}\omega_{\vec{k}}t}\mathbf{a}_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x}+\mathbf{i}\omega_{\vec{k}}t}\mathbf{a}_k^\dagger \right). \tag{1.33}$$

Indeed we can check that the relation $\boldsymbol{\pi} = \dot{\boldsymbol{\phi}}$ holds.

Negative frequencies. Notice that the dependence on spacetime is via a sum of terms of the form:

$$e^{\mathbf{i}\vec{k}\cdot\vec{x}-\mathbf{i}\omega_{\vec{k}}t} = e^{-\mathbf{i}k_\mu x^\mu}|_{k^0=\omega_{\vec{k}}}$$

and their complex conjugates. These are precisely all the solutions to the wave equation (1.29). For each $\vec{k}$, there are two solutions to the second order ODE $(\partial_t^2 + k^2 + m^2)\phi_k(t) = 0$, namely $e^{\mathbf{i}\omega t}$ with $\omega = \pm\omega_k$, one with positive frequency and one with negative frequency. You might have worried that solutions with both signs of the frequency mean that the world might explode or something (like it would if we tried to replace the Schrödinger equation for the wavefunction with a Klein-Gordon equation). This danger is evaded in a beautiful way: the coefficient of the positive frequency solution with wavenumber $\vec{k}$ is the destruction operator for the mode; the associated negative frequency term comes with the creation operator for the same mode, as a consequence of reality of the field $\phi = \phi^\dagger$. (Some words about antimatter might be appropriate here, but it will be clearer later when we talk about examples of particles that are not their own antiparticles.)

'Momentum'. Finally, in a relativistic system, anything we can say about time should also be true of space, up to some signs. So the fact that we were able to generate the time dependence by conjugation with the unitary operator $e^{\mathbf{i}\mathbf{H}t}$ (as in (1.33)) says that we should be able to generate the space dependence by conjugating by a unitary operator of the form $e^{-\mathbf{i}\vec{\mathbf{P}}\cdot\vec{x}}$. Here $\vec{\mathbf{P}}$ is the last in a long list of objects in this section with a claim to the name 'momentum'. It is the conserved charge associated with spatial translation symmetry, the generator of spatial translations. Its form in terms of the fields will be revealed below when we speak about Noether's theorem. For now, let me emphasize that it is distinct from the objects $p_n, \pi(x)$ (which were 'momenta' in the sense of canonical momenta of various excitations) and also from the wavenumbers $\vec{k}$ of various modes, which (when multiplied by $\hbar$) are indeed spatial momenta of single particles. (This statement gives us an expectation for what is the total momentum of a state of a collection of particles which we will check below in §1.4.) In terms of the momentum operator, then, we can write

$$\boldsymbol{\phi}(x^\mu) = e^{\mathbf{i}\mathbf{P}_\mu x^\mu}\boldsymbol{\phi}(0)e^{-\mathbf{i}\mathbf{P}_\mu x^\mu}$$

with $\mathbf{P}_\mu \equiv (\mathbf{H}, \vec{\mathbf{P}})_\mu$.