Perturbative QFT is Asymptotic; is Divergent; is Problematic in Principle

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A very brief review is given on what it means for a series to be asymptotic followed by a basic example in non-relativistic quantum mechanics. Finally, Freeman J. Dyson's mostly qualitative argument for why QED should be expected to be divergent in general, is given. In such a case, the theory is either incomplete or new mathematical methods are needed to deal with these divergent expansions to give a more complete understanding of what the theory is saying. The Borel transform is discussed as an effort to define a sum for such a factorially divergent series.

INTRODUCTION

Under the current paradigm of quantum field theory, we often have to do without exact solutions for measurable quantities. The effective theories that describe interactions must contain nonlinear terms in the Lagrangian in order to couple different Fourier modes and different fields[1] (i.e. to get interesting stuff to happen). These terms that make interesting things happen, also make it difficult to solve for measurable quantities. We're often forced to expand an expression for these quantities into a power-series. The problem is, however, that the series is often asymptotic and thus diverges. Given a particular asymptotic expansion in QFT, the precision of the predicted value of that observable is therefore limited, in principle, as opposed to a practical limitation on computing the infinite order term. For practical purposes, we can ignore this divergence because the answer is right, up to some decimal and by the natural ordering of terms in a power-series, we're never at odds with in which order the terms should be added in the expansion.

To emphasize, there is no issue with QFT's practical predictive power, but nonetheless F. J. Dyson reasonably took issue with the in-principle-problem of a divergent series describing physical phenomena. It is not a mathematically well defined theory. In an interview[2], F. J. Dyson discusses his "ambition of making quantum electrodynamics into a completely solvable theory...in which all the renormalizations would be done...and everything could be done, in principle, as accurately as you want." He later realized that these divergences in QED were likely fundamental to the theory of perturbative QED itself. Ultimately, he came to the conclusion that either different mathematical methods were needed or a new physical theory is needed (which may also include the need for new math).

In this paper, we discuss the former scenario. In particular, we discuss the Borel summation which is a summation method used for divergent series. But first, we talk about some of the basics about asymptotic expansions and how they arise in non-relativistic quantum mechanics.

ASYMPTOTIC EXPANSIONS: A VERY BRIEF REMINDER

If one cannot determine an integral in some analytic form, another way to tackle the integral is to expand it somehow. A common technique is to expand the integrand in some power-series for some small parameter and compute the integral term by term in the expansion[3]. Sometimes the expansion of the integrand works in a particular domain but the integral itself may be over a region in which the domain is not valid. Often this will generate an asymptotic expansion. One example for an asymptotic expansion is the following

$$I(x) = 4x^3 e^{x^4} \int_x^\infty e^{-t^4} dt \tag{1}$$

for $x \to \infty$. One can expand this to yield

$$I(x) = 4x^3 e^{x^4} \int_x^\infty e^{-t^4} dt = 1 - \frac{3}{4x^4} + \frac{3 \cdot 7}{(4x^4)^2} + \dots \tag{2}$$

This is a power series in the parameter $\lambda = \frac{1}{4x^4}$. Because there is a factorial coefficient for each term in the expansion, there will be some x-dependent order after which, each successive term of the series will increase and eventually diverge. For the above expansion, that order is roughly $n_c \approx x^4$. Thus, as the expansion parameter λ goes to zero, or as $x \to \infty$, one can take more and more terms in the expansion as a good approximation to the actual function $I(x)$.

AN EXAMPLE FROM NON-RELATIVISTIC QUANTUM MECHANICS

Asymptotic integrals do not just show up in high energy QFT, but also ordinary quantum mechanics. Per-

[6] It should be noted, however, that there exist quantum field theories that are exactly solvable, but from my understanding, they are usually relegated to lower dimensions. You can see section 22.2 of Peskin & Schroeder for more information.

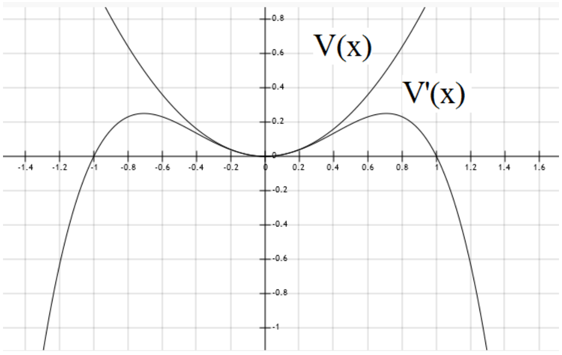

FIG. 1: A plot of the potential $V(x) = x^2$ and $V'(x) = x^2 - x^4$. Because $V'(x = \pm\infty) = -\infty$ the states for the perturbed Hamiltonian are mostly unbound states with the possibility of only quasi-bound states.

turbation theory in quantum mechanics is used to determine how the eigenfunctions change as a result of some perturbation of the Hamiltonian. In non-degenerate perturbation theory, one can follow the states as the perturbation of the form λH_1 is switched on. The assumption of perturbation theory is that the actual states that are being expanded are analytic functions of the parameter λ in a circle of finite radius about $\lambda = 0$ in the complex plane. So there is a requirement that it should be possible to analytically continue to states of differing λ.

Let us now look at a quick example regarding the perturbation to the SHO hamiltonian. Under the perturbation, the full hamiltonian will be of the form,

$$H = H_0 + \lambda H_1 = \frac{p^2}{2m} + \frac{1}{2}kx^2 + \frac{1}{4}\lambda x^4 \qquad (3)$$

for positive k.

The form of the unperturbed and perturbed potential is shown in Fig. 1. For $\lambda > 0$, both H and H_0 tend to ∞ as $q \to \infty$ and so the states are always bound states with a discrete spectrum in both cases. For $\lambda < 0$, H_0 has only bound states, and H has no true bound states, except for a set of quasi-bound states that are ultimately delocalized. No matter how small $|\lambda|$ gets, the $\frac{1}{4}\lambda x^4$ term will always overpower $\frac{1}{2}kx^2$ for $|x| > \sqrt{\frac{2k}{|\lambda|}}$ and all the states will have a finite probability of escaping the potential due to the tunneling process.

Perturbation theory in quantum mechanics expands the new states in terms of the old ones. How can we get spatially extended states from a set of states spatially localized about an origin? We can't! For the harmonic oscillator, all the states need to go to 0 at $x = \infty$, therefore all states constructed from these states must have this property as well. So the boundary condition won't let you. The analyticity requirement is not met, and therefore the expansion is not entirely valid. Ultimately, the expansion is asymptotic, but lends itself as a good approximation up to some order of the expansion for the quasi-bound states well below the potential maximum. Also, the energies of the perturbed states will take the form of $E_a(\lambda) = A_a(\lambda) + S_a(\lambda)$ where $A_a(\lambda)$ is a part that is analytic in λ and $S_a(\lambda)$ is a nonperturbative part that is exponentially small as $\lambda \to 0$.

FREEMAN J. DYSON'S ARGUMENT OF DIVERGENCE IN QED

Dyson gives a simple physical argument that indicates the power series expansions in QED will be divergent even after renormalization. Also, the divergence is nontrivial, as opposed to the trivial divergences that occur for the scattering amplitude of a free particle where the particle has a possibility of being captured into a permanently bound state.

Suppose we have the form of a physical quantity given as a power series in the charge coupling in QED. The power series is obtained by integrating the equations of motion of the theory. If the expansion is such that,

$$F(e^2) = a_0 + a_2e^2 + a_4e^4 + ... < \infty \qquad (4)$$

for some nonzero value of e^2, then $F(e^2)$ is analytic at $e = 0$, this must mean that the physical quantity is an analytic function of e for $e = 0$. Therefore, if it is an analytic function, we can allow e^2 to extend in the complex plane and it must have a finite radius of convergence about $e^2 = 0$; so $e^2 \to -e^2$ or $F(e^2) \to F(-e^2)$ should also be convergent for sufficiently small values of e^2.

We can interpret the physical quantity $F(-e^2)$ as arising in a world where like charges attract each other. In the classical limit of large particle number and large distances, the potential between charges is just the classical Coulomb potential with a sign reversal. One can imagine a "pathological" state with a large number of electron-positron pairs with the electrons separated from the positrons in different regions of space such that the negative Coulomb potential is larger than the rest energy and kinetic energy of the particles themselves.

The point is that for any physical state of the system, there is a large potential barrier separating that physical state with these "pathological" states, in some sense similar to the quartic potential perturbation of the harmonic oscillator where the quasi-bound states have a finite overlap with states that can escape to regions of arbitrarily large negative potential. In that case, any quasi-bound state is unstable against the spontaneous tunneling beyond the potential maxima. In the case of the alternate QED where e^2 has gone to $-e^2$, any physical state is unstable against the spontaneous generation of many particles. Once in such a pathological state, the system will generate even more particles and the vacuum will "disintegrate" by this spontaneous polarization. Such a theory will not have equations of motions that can be put into an analytic form and thus, the expansion of physical quantities will not be analytic for $F(-e^2)$ for arbitrary values of e^2. The expansion, therefore has a 0 radius of convergence so we should exepect an asymptotic series for $F(e^2)$ that will not be convergent.

There are two conclusions one could draw from the fact that the series is divergent. In the case that $F(e^2)$ really is well defined, we will likely need new mathematical methods to determine a better means of figuring out what it is besides standard perturbation theory. If the coefficients of the expansion contain all the information QED has to offer, then we will likely need both different mathematics and different physical theories to describe such phenomena and QED would not be a closed theory.

BOREL SUMMATION

Even if we are given a divergent series, there are ways in which we can assign a value to its sum by defining the Borel transform. In particular, if the series is factorially divergent, we can define the Borel transform as

$$F(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^{n+1} \Rightarrow B\{F\}(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \qquad (5)$$

where $F(\lambda)$ is some initial expansion in powers of λ of a physical quantity in QED or QCD.

Then, the Borel integral can be defined as

$$F \ = \int_0^{\infty} dt e^{\frac{-t}{\lambda}} B\{F\}(t). \qquad (6)$$

If the Borel integral exists, it gives the defined sum for the originally divergent series.

All that is left to show is that the Borel sum equals the original function $F(\lambda)$ non-perturbatively.By the Watson-Nevanlinnna-Sokal theorem, equality is guaranteed in the case that F satisfies certain analyticiy requirements. For renormalizable theories, it turns out that these conditions are too strong.

If we assume a particular form for the a_n such that

$$a_n = K c^n \Gamma(n+1+b) \qquad (7)$$

then except when b is a negative integer, the Borel transformation is

$$B\{F\}(t) = \frac{K\Gamma(1+b)}{(1-at)^{1+n}} \qquad (8)$$

for $b = -m$ where m is a positive integer, the transform is given as

$$B\{F\}(t) = \frac{(-1)^m}{\Gamma(m)}(1-ct)^{m-1} ln(1-ct) + \text{polynomial in t.} \qquad (9)$$

So when c is positive, the series does not alternate in sign and from (8) we see that that yields singularities for positive t in the Borel plane. So therefore, the Borel integral does not exist as defined, and we cannot define a sum for these divergent series. Such non-sign-alternating series are expected in QED and QCD.

The Borel transform is still useful as a generating function for the series coefficients a_n. The behavior of the divergence in the original series expansion is encoded in these singularities. Singularities closer to the origin at $t = 0$ of the Borel plane, correspond to faster divergences, as that corresponds to higher values of c. There various sources of these type of singularities. For example, if a series expansion, in a renormalizable theory, contains Nth order contributions that are proprtional to $N!$, they are called renormalons.

We can also define the integral by taking the limit as the singularities are above or blow the positive t axis. In this case, the sum has an imaginary part given by

$$ImF\ (\lambda) = \pm \frac{\pi K}{c} e^{-\frac{1}{c\lambda}} (c\lambda)^{-b} \qquad (10)$$

and the ambiguity of the sum is defined as the difference in the two values obtained by having a singularity slightly above or below the t-axis.

These ambiguities are exponentially small in the expansion parameter, and in this sense they are non-perturbative. One can then add terms with the form of these ambiguities to the original series expansion. The ambiguities in the defining sum cancel leading to an improved approximation of the function $F(\lambda)$.

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