Non-abelian bosonization

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In this paper Witten's proceudre of non-abelian bosonization in 1+1 dimensions reviewed, CFT algebra of currents briefly described as well as applications to half-integer spin chain systems and Hubbard model at half-filling.

INTRODUCTION

Sometimes it's good to have a choice between languages in terms of which talking about 1+1D systems-phenomenas might become looking easier and semiclasical if translated from fermi to bose language.

So the first section is a "vocabulary" of doing those translations whereas the second might serve as an introductory "phrasebook".

NON-ABELIAN BOSONISATION

A non-abelian bosonisation introduceed by Witten in 1983 [1] allows to translate *any* fermi theory into *local* bose theory while having *all* of the original symmetries conserved. And of course, the equivalence can be proliferated to interacting terms and perturbations can be nicely converted too. This was a major improvement compared to the usual bosonisation procedure that would allow to transform only local fermi fields to local bose fields and wouldn't respect symmetries, making newly-made off-diagonal bose currents some complicated non-local expressions.

Let's take look at Lagrangian for free massles Dirac fermions in 1+1D:

$$\mathcal{L} = \bar{\psi}_i^{\dagger} i\gamma_\mu \partial^\mu \psi_i(x) \tag{1}$$

It is scale-invariant and has a fixed point under RG, which are important points for future purposes. We might stick in some arbitrary number of indices in fields, suppose $2N$ of them. Then it has two gauge symmetries- U(1)×U(1) and SU(N)×SU(N) [or just a big one- O($2N$) × O($2N$), because O($2N$) $\simeq$ U(1) × SU(N)] and corresponding currents:

$$J_R = \psi_R^{i\dagger}\psi^i, \qquad J_L = \psi_L^{i\dagger}\psi^i \tag{2}$$

$$J_R^a = \psi_R^{i\dagger} t^a_{ij}\psi^j, \qquad J_L^a = \psi_L^{i\dagger} t^a_{ij}\psi^j \tag{3}$$

Applying anticommutation relations for fermi fields one could come up with relations

$$[J_R^a(x), J_R^a(y)] = if^{abc}J_R^c(x)\delta(x-y) + i\frac{k}{4\pi}\delta^{ab}\delta'(x-y)$$

$$[J_L^a(x), J_L^a(y)] = if^{abc}J_L^c(x)\delta(x-y) + i\frac{k}{4\pi}\delta^{ab}\delta'(x-y)$$

where f^{abc} are structure constants of the t^as- generators of the SU(N). Currents form a Kak-Moody current CFT algebra, which in our case appears to be "level-1" as $k = 1$ (which should be integer for well-behaveness).

Now, conservation laws:

$$\partial_- J_R^{ij} = 0, \qquad \partial_+ J_L^{ij} = 0 \tag{4}$$

where the so-called light-cone components used with $x_\pm = (x_0 \pm x_1)/\sqrt{2}$. Here is a crucial moment- a set of bosonic fields with the same symmetries and current algebra needed. First, let's try to satisfy the conservation relations:

$$J_R(x) = \frac{i}{2\pi}g^{-1}(x)\partial_+ g(x), \qquad J_L(x) = -\frac{i}{2\pi}(\partial_- g(x))g^{-1}(x) \tag{5}$$

where $g \in$ SU(N). We are heading towards bosonised field theory that is a non-linear sigma model whose fields taking values on a compact Lie group [SU(N)] for every-point of space-time. A non-linear sigma model field taking values on a group manifold is known as the principal chiral field. Such matrix-valued field has doubled $G_L \times G_R$ symmetry $[g(x) \to h_L g(x) h_R^1]$ which generate two chiral currents. What Lagrangian will govern g? It is tempting to say

$$\mathfrak{L} = \frac{1}{4\lambda^2} Tr\partial_\mu g \partial_\mu g^{-1} \tag{6}$$

which is the only renormalizable and chirally-invariant choice in 1+1D. But since the RG beta-function is non-vanishing, it's not scale-invariant and since beta-function is positive [for G=SU(N)], it's asymptotically free which means it has a non-vanishing mass gap and cannot be equivalent to the massless fermions theory.

However there is another renormalizable term which is not so manifestly chirally invariant though. Let's consider the theory in Euclidian 2D space-time that is isomorphic to S^2 and thus field configurations are maps

$S^2 \to G$. We could deal with them as with boundary conditions on a one-extra-dimensional field $\bar{g}(y)$, smoothly extended into a bulk [who is a ball B, the interior of S^2], which is allowed since the maps are trivial and have a trivial homotopy group. Arising Wess-Zumino piece of action is given by

$$\Gamma[g] = \frac{1}{24\pi}\int_B d^3y\epsilon^{ijk}Tr\bar{g}^{-1}\partial_i\bar{g}\bar{g}^{-1}\partial_j\bar{g}\bar{g}^{-1}\partial_k\bar{g} \qquad (7)$$

which is defined only modulo 2π, $\Gamma \to \Gamma + 2\pi$ due to topologically inequivalent ways to extend a configuration $g(x)$ from S^2 to the ball B. The correct non-linear sigma model has the action:

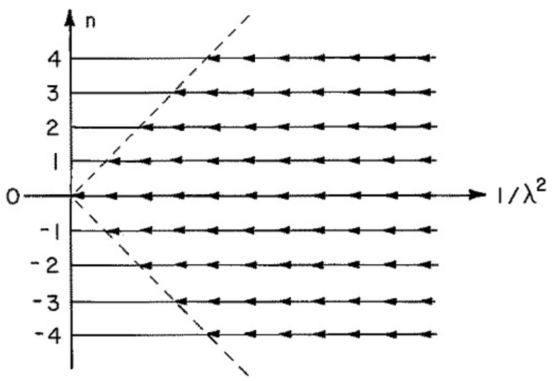
$$\mathcal{L} = \frac{1}{4\lambda^2}Tr\partial_\mu g\partial_\mu g^{-1} + n\Gamma[g] \qquad (8)$$

where coupling constant $n \in Z$ for single-valuedness of the path-integral. The action is known as the Wess-Zumino-Witten (WZW) model of level n [and secretly it is equal $n = k$ to a level of associated Kac-Moody algebra]. It is both renormalizable and scale invariant [win!] and is exactly solvable with CFT (Knizhnik and Zamolodchikov, 1984) [2].

To make sure the WZW model to be equivalent to a theory of free fermions, we should find a fixed point for some λ and n. A one-loop computation [1] yields for G = SU(N)

$$\beta(n,\lambda) = \frac{N}{4\pi}\lambda^2\left[1 - \left(\frac{\lambda^2 n}{4\pi}\right)\right] \qquad (9)$$

providing the following picture of charge flow:



This result predicts a stable fixed point at a critical value of the coupling constant $\lambda_c^2 = 4\pi/k$ and it has been proved [2] that this is indeed exact.

Upon constraction of currents for WZW model, their Kac-Moody "level-n" algebra structure reveals [what a coincidence!] and from agreement with fermion cousins it follows that $n = k = 1$. Finally, a theory of N free Dirac fermions is equivalent to an SU$(N)_1$ "level-1" WZW model at its fixed point and a U(1) free boson what is being reflected in two terms of a full action:

$$S = \int d^2x\frac{1}{2}(\partial_\mu\phi)^2 + S_{WZW}^{k=1}[g] \qquad (10)$$

APPLICATIONS OF NON-ABELIAN BOSONISATION

Now, applications to spin chains and 1D Hubbard model [which is roughly the same as a chain + randomness of distribution of particles on sites + arbitrariness of filling factor + scattering abilities]. In the weak-coupling regime those are equivalent to a theory of N = 2 Dirac fermions. The full action undergoes decomposition into charge and spin sectors with U(1) and SU$(2)_1$ [it's spin 1/2] symmetries respectively.

$$S = \int d^2x\frac{1}{2}(\partial_\mu\phi_c)^2 + S_{WZW}^{k=1}[g] \qquad (11)$$

The profit is that we can now study RG properties, mass/gap generation of the two sectors kind of separately from each other with the spin sector at low energies described by the SU$(2)_1$ WZW model fixed point. This is a valid description for 1D Hubbard model at half-filling and at the same time 1D spin-1/2, and even more generally, all half-integer quantum antiferromagnets at low energies. Due to the charge gap, charge degrees of freedom can be effectively decoupled and projected out, so we don't care about them.

For a much more detailed discussion please turn to Chapter 7 of Fradkin [3].

[1] E. Witten, "Nonabelian Bosonization in Two-Dimensions," Commun. Math. Phys. **92**, 455 (1984).
[2] V. G. Knizhnik and A. B. Zamolodchikov, "Current Algebra and Wess-Zumino Model in Two-Dimensions," Nucl. Phys. B **247**, 83 (1984).
[3] E. Fradkin, "Field theories in CMT," second edition, 2013.