Physics 215C: Quantum Field Theory

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0.1 Introductory remarks

I will begin with some comments about my goals for this course.

The main goal is to make a study of coarse-graining in quantum systems with extensive degrees of freedom. For silly historical reasons, this is called the renormalization group (RG) in QFT. By 'extensive degrees of freedom' I mean that we are going to study models which, if we like, we can sprinkle over vast tracts of land, like *sod* (see Fig. 1). And also like sod, each little patch of degrees of freedom only interacts with its neighboring patches: this property of sod and of QFT is called *locality*. [1]

Figure 1: Sod.

By 'coarse-graining' I mean ignoring things we don't care about, or rather only paying attention to them to the extent that they affect the things we do care about.[2] In my experience, learning to do this is approximately synonymous with *understanding*.

In the course of doing this, I would like to try to convey the Wilsonian perspective on the RG, which (among many other victories) provides an explanation of the *totalitarian principle of physics* that anything that can happen must happen. [3]

And I have a collection of subsidiary goals:

- I would like to convince you that "non-renormalizable" does not mean "not worth your attention," and explain the incredibly useful notion of an Effective Field Theory.

[1] More precisely, in quantum mechanics, we specify the degrees of freedom by their Hilbert space; by an extensive system, I mean one in which the Hilbert space is of the form $\mathcal{H} = \otimes_{\text{patches of space}} \mathcal{H}_{\text{patch}}$ and the interactions are lcoal $\mathbf{H} = \sum_{\text{patches}} \mathbf{H}(\text{nearby patches})$.

[2] To continue the sod example in 2+1 dimensions, a person laying the sod in the picture above cares that the sod doesn't fall apart, and rolls nicely onto the ground (as long as we don't do high-energy probes like bending it violently or trying to lay it down too quickly). These *long-wavelength* properties of *rigidity* and *elasticity* are collective, emergent properties of the microscopic constituents (sod molecules) – we can describe the dynamics involved in covering the Earth with sod (never mind whether this is a good idea in a desert climate) without knowing the microscopic theory of the sod molecules (I think they might be called 'grass'). Our job is to think about the relationship between the microscopic model (grassodynamics) and its macroscopic counterpart (in this case, suburban landscaping).

[3] More precisely, this means that the Hamiltonian should contain all terms consistent with symmetries, organized according to a derivative expansion in a way we will understand.

- There is more to QFT than perturbation theory about free fields in a Fock vacuum. In particular, we will spend some time thinking about non-perturbative physics, effects of topology, solitons. Topology is one tool for making precise statements without perturbation theory (the basic idea: if we know something is an integer, it is easy to get many digits of precision!).

- I will try to resist making too many comments on the particle-physics-centric nature of the QFT curriculum. QFT is also quite central in many aspects of condensed matter physics, and we will learn about this. From the point of view of someone interested in QFT, high energy particle physics has the severe drawback that it offers only one example! (OK, for some purposes you can think about QCD and the electroweak theory separately...)

- There is more to QFT than the S-matrix. In a particle-physics QFT course you learn that the purpose in life of correlation functions or green's functions or off-shell amplitudes is that they have *poles* (at $p^\mu p_\mu - m^2 = 0$) whose residues are the S-matrix elements, which are what you measure (or better, are the distribution you sample) when you scatter the particles which are the quanta of the fields of the QFT.

  I want to make two extended points about this:

  1. In many physical contexts where QFT is relevant, you can actually measure the off-shell stuff. This is yet another reason why including condensed matter in our field of view will deepen our understanding of QFT.

  2. The Green's functions don't always have simple poles! There are lots of interesting field theories where the Green's functions instead have power-law singularities, like $G(p) \sim \frac{1}{p^{2\Delta}}$. If you fourier transform this, you don't get an exponentially-localized packet. The elementary excitations created by a field whose two point function does this *are not particles*. (Any conformal field theory (CFT) is an example of this.) The theory of particles (and their dance of creation and annihilation and so on) is a proper subset of QFT.

Here is a confession, related to several of the points above: The following comment in the book *Advanced Quantum Mechanics* by Sakurai had a big effect on my education in physics: *... we see a number of sophisticated, yet uneducated, theoreticians who are conversant in the LSZ formalism of the Heisenberg field operators, but do not know why an excited atom radiates, or are ignorant of the quantum-theoretic derivation of Rayleigh's law that accounts for the blueness of the sky.*

I read this comment during my first year of graduate school and it could not have applied more aptly to me. I have been trying to correct the defects in my own education which this exemplifies ever since.

I bet most of you know more about the color of the sky than I did when I was your age, but we will come back to this question. (If necessary, we will also come back to the radiation from excited atoms.)

So I intend that there will be two themes of this course: coarse-graining and topology. Both of these concepts are important in both hep-th and in cond-mat. As for what these goals mean for what topics we will actually discuss, this depends somewhat on the results of pset 00. Topics which I hope to discuss include:

- theory of renormalization (things can look different depending on how closely you look; this is how we should organize our understanding of extensive quantum systems)
- effective field theory (how to do physics without a theory of everything)
- effects of topology in QFT (this includes anomalies, topological solitons and defects, topological terms in the action)
- deep mysteries of gauge theory.

I welcome your suggestions regarding what physics we should study.

We begin with some parables from quantum mechanics.

0.2 Conventions

You will have noticed above that I already had to commit to a signature convention for the metric tensor. I will try to follow Zee and use $+---$. I am used to the other signature convention, where time is the weird one.

We work in units where $\hbar$ and c are equal to one unless otherwise noted.

The convention that repeated indices are summed is always in effect.

A useful generalization of the shorthand $\hbar \equiv \frac{h}{2\pi}$ is

$$đp \equiv \frac{\mathrm{d}p}{2\pi}.$$

I will try to be consistent about writing fourier transforms as

$$\int \frac{\mathrm{d}^4 p}{(2\pi\hbar)^4} e^{ipx/\hbar} \tilde{f}(p) \equiv \int đ^4 p \; e^{ipx/\hbar} \tilde{f}(p) \equiv f(x).$$

RHS $\equiv$ right-hand side.
LHS $\equiv$ left-hand side.
BHS $\equiv$ both-hand side.

I reserve the right to add to this page as the notes evolve.

Please tell me if you find typos or errors or violations of the rules above.

1 Ideas from quantum mechanics, I

1.1 Broken scale invariance

Reading assignment: Zee chapter III.

Here we will study a simple quantum mechanical example (that is: an example with a finite number of degrees of freedom) which exhibits many interesting features that can happen in strongly interacting quantum field theory – asymptotic freedom, dimensional transmutation. Because the model is simple, we can understand these phenomena without resort to perturbation theory. I learned this example from Marty Halpern.

Consider the following ('bare') action:

$$S[x] = \int dt \left(\frac{1}{2}\dot{\vec{x}}^2 + g_0 \delta^{(2)}(\vec{x}) \right) \equiv \int dt \left(\frac{1}{2}\dot{\vec{x}}^2 - V(\vec{x}) \right)$$

where $\vec{x} = (x, y)$ are two coordinates of a quantum particle, and the potential involves $\delta^{(2)}(\vec{x}) \equiv \delta(x)\delta(y)$, a Dirac delta function. (Notice that I have absorbed the inertial mass m in $\frac{1}{2}mv^2$ into a redefinition of the variable x, $x \to \sqrt{m}x$.)

First, let's do dimensional analysis (always a good idea). Since $\hbar = c = 1$, all dimensionful quantites are some power of a length. Let $[X]$ denote the number of powers of length in the units of the quantity X; that is, if $X \sim (\text{length})^{\nu(X)}$ then we have $[X] = \nu(X)$, a number. We have:

$$[t] = [\text{length}/c] = 1 \implies [dt] = 1.$$

The action appears in exponents and is therefore dimensionless (it has units of $\hbar$), so we had better have:

$$0 = [S] = [\hbar]$$

and this applies to each term in the action. We begin with the kinetic term:

$$0 = [\int dt \dot{\vec{x}}^2] \implies$$

$$[\dot{\vec{x}}^2] = -1 \implies [\dot{\vec{x}}] = -\frac{1}{2} \implies [\vec{x}] = \frac{1}{2}.$$

Since $1 = \int dx \delta(x)$, we have $0 = [dx] + [\delta(x)]$ and

$$[\delta^D(\vec{x})] = -[x]D = -\frac{D}{2}, \quad \text{and in particular } [\delta^2(\vec{x})] = -1.$$

This implies that the naive ("engineering") dimensions of the coupling constant g_0 are $[g_0] = 0$ – it is dimensionless. Classically, the theory does not have a special length scale; it is scale invariant.

The Hamiltonian associated with the Lagrangian above is

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + V(\vec{x}).$$

Now we treat this as a quantum system. Acting in the position basis, the quantum Hamiltonian operator is

$$\mathbf{H} = -\frac{\hbar^2}{2}\left(\partial_x^2 + \partial_y^2\right) - g_0\delta^{(2)}(\vec{x})$$

So in the Schrödinger equation $\mathbf{H}\psi = \left(-\frac{\hbar^2}{2}\nabla^2 + V(\vec{x})\right)\psi = E\psi$, the second term on the LHS is

$$V(\vec{x})\psi(\vec{x}) = -g_0\delta^{(2)}(\vec{x})\psi(0).$$

To make it look more like we are doing QFT, let's solve it in momentum space:

$$\psi(\vec{x}) \equiv \int \frac{d^2p}{(2\pi\hbar)^2}\, e^{\mathbf{i}\vec{p}\cdot\vec{x}/\hbar}\varphi(\vec{p})$$

The delta function is

$$\delta^{(2)}(x) = \int \frac{d^2p}{(2\pi\hbar)^2}\, e^{\mathbf{i}\vec{p}\cdot\vec{x}/\hbar}.$$

So the Schrödinger equation says

$$\begin{aligned}\left(-\frac{1}{2}\nabla^2 - E\right)\psi(x) &= -V(x)\psi(x)\\ \int \text{đ}^2p e^{ip\cdot x}\left(\frac{p^2}{2} - E\right)\varphi(p) &= +g_0\delta^2(x)\psi(0)\\ &= +g_0\left(\int \text{đ}^2p e^{\mathbf{i}p\cdot x}\right)\psi(0) \qquad (1.1)\end{aligned}$$

which (integrating the both-hand side of (1.1) over x: $\int d^2x e^{\mathbf{i}p\cdot x}$ ((1.1))) says

$$\left(\frac{\vec{p}^2}{2} - E\right)\varphi(\vec{p}) = +g_0\underbrace{\int \frac{d^2p'}{(2\pi\hbar)^2}\varphi(\vec{p}')}_{=\psi(0)}$$

There are two cases to consider:

- $\psi(\vec{x}=0) = \int đ^2 p \varphi(\vec{p}) = 0$. Then this is a free theory, with the constraint that $\psi(0) = 0$,

$$\left(\frac{\vec{p}^2}{2} - E\right)\varphi(\vec{p}) = 0$$

i.e. plane waves which vanish at the origin, *e.g.* $\psi \propto \sin \frac{p_x x}{\hbar} e^{\pm \mathbf{i} p_y y/\hbar}$. These scattering solutions don't see the delta-function potential at all.

- $\psi(0) \equiv \alpha \neq 0$, some constant to be determined. This means $\vec{p}^2/2 - E \neq 0$, so we can divide by it :

$$\varphi(\vec{p}) = \frac{g_0}{\frac{\vec{p}^2}{2} - E}\left(\int đ^2 p \varphi(\vec{p})\right) = \frac{g_0}{\frac{\vec{p}^2}{2} - E}\alpha.$$

The integral on the RHS is a little problematic if $E > 0$, since then there is some value of p where $p^2 = 2E$. Avoid this singularity by going to the boundstate region: $E = -\epsilon_B < 0$. So:

$$\varphi(\vec{p}) = \frac{g_0}{\frac{\vec{p}^2}{2} + \epsilon_B}\alpha.$$

What happens if we integrate this $\int đ^2 p$ to check self-consistency – the LHS should give α again:

$$0 \stackrel{!}{=} \underbrace{\int đ^2 p \varphi(\vec{p})}_{=\psi(0)=\alpha\neq 0}\left(1 - \int đ^2 p \frac{g_0}{\frac{\vec{p}^2}{2} + \epsilon_B}\right)$$

$$\Longrightarrow \qquad \boxed{\int đ^2 p \frac{g_0}{\frac{\vec{p}^2}{2} + \epsilon_B} = 1}$$

is the condition on the energy ϵ_B of possible boundstates.

But there's a problem: the integral on the LHS behaves at large p like

$$\int \frac{d^2 p}{p^2} = \infty \ .$$

At this point in an undergrad QM class, you would give up on this model. In QFT we don't have that luxury, because this happens all over the place. Here's what we do instead:

We cut off the integral at some large $p = \Lambda$:

$$\int^{\Lambda} \frac{d^2 p}{p^2} \sim \log \Lambda \ .$$

This our first example of the general principle that a classically scale invariant system will exhibit logarithmic divergences. It's the only kind allowed by dimensional analysis.

More precisely:

$$\int^{\Lambda} \frac{d^2p}{\frac{p^2}{2}+\epsilon_B} = 2\pi \int_0^{\Lambda} \frac{pdp}{\frac{p^2}{2}+\epsilon_B} = 2\pi \log\left(1+\frac{\Lambda^2}{2\epsilon_B}\right) \ .$$

So in our cutoff theory, the boundstate condition is:

$$1 = g_0 \int^{\Lambda} \frac{\text{đ}^2p}{\frac{p^2}{2}+\epsilon_B} = \frac{g_0}{2\pi\hbar^2} \log\left(1+\frac{\Lambda^2}{2\epsilon_B}\right) \ .$$

A solution only exists for $g_0 > 0$. This makes sense since only then is the potential attractive (recall that $V = -g_0\delta$).

Now here's a trivial step that offers a dramatic new vista: solve for ϵ_B.

$$\epsilon_B = \frac{\Lambda^2}{2} \frac{1}{e^{\frac{2\pi\hbar^2}{g_0}} - 1} \ . \tag{1.2}$$

As we remove the cutoff ($\Lambda \to \infty$), we see that $E = -\epsilon_B \to -\infty$, the boundstate becomes more and more bound – the potential is too attractive.

Suppose we insist that the boundstate energy ϵ_B is a fixed thing – imagine we've *measured* it to be 200 MeV[4]. Then, given some cutoff Λ, we should solve for $g_0(\Lambda)$ to get the boundstate energy we require:

$$g_0(\Lambda) = \frac{2\pi\hbar^2}{\log\left(1+\frac{\Lambda^2}{2\epsilon_B}\right)}.$$

This is the crucial step: this silly symbol g_0 which appeared in our action doesn't mean anything to anyone (see Zee's dialogue with the S.E.). *We are allowing $g_0 \equiv$ the bare coupling to be cutoff-dependent.*

Instead of a dimensionless coupling g_0, the useful theory contains an arbitrary *dimensionful* coupling constant (here ϵ_B). This phenomenon is called *dimensional transmutation* (d.t.). The cutoff is supposed to go away in observables, which depend on ϵ_B instead.

In QCD we expect that in an identical way, an arbitrary scale Λ_{QCD} will enter into physical quantities. (If QCD were the theory of the whole world, we would work in units where it was one.) This can be taken to be the rest mass of some mesons – boundstates of quarks. Unlike this example, in QCD there are many boundstates, but their energies are dimensionless multiplies of the one dimensionful scale, Λ_{QCD}. Nature chooses $\Lambda_{QCD} \simeq 200$ MeV.

[This d.t. phenomenon was maybe first seen in a perturbative field theory in S. Coleman, E. Weinberg, *Phys Rev* **D7** (1973) 1898. We'll come back to their example.]

[4]Spoiler alert: I picked this value of energy to stress the analogy with QCD.

There's more. Go back to (1.2):

$$\epsilon_B = \frac{\Lambda^2}{2}\frac{1}{e^{\frac{2\pi\hbar^2}{g_0}}-1} \neq \sum_{n=0}^{\infty} g_0^n f_n(\Lambda)$$

it is *not* analytic (*i.e.* a power series) in $g_0(\Lambda)$ near small g_0; rather, there is an essential singularity in g_0. (All derivatives of ϵ_B with respect to g_0 vanish at $g_0 = 0$.) You can't expand the dimensionful parameter in powers of the coupling. This means that you'll *never* see it in perturbation theory in g_0. Dimensional transmutation is an inherently non-perturbative phenomenon.

Still more:

$$g_0(\Lambda) = \frac{2\pi\hbar^2}{\log\left(1+\frac{\Lambda^2}{2\epsilon_B}\right)} \overset{\Lambda^2\gg\epsilon_B}{\to} \frac{2\pi\hbar^2}{\log\left(\frac{\Lambda^2}{2\epsilon_B}\right)} \overset{\Lambda^2\gg\epsilon_B}{\to} 0$$

– the bare coupling vanishes in this limit, since we are insisting that the parameter ϵ_B is fixed. This is called *asymptotic freedom* (AF): the bare coupling goes to zero (*i.e.* the theory becomes free) as the cutoff is removed. This also happens in QCD.

More: Define the *beta-function* as the logarithmic derivative of the bare coupling with respect to the cutoff:

$$\textbf{Def:} \qquad \beta(g_0) \equiv \Lambda\frac{\partial}{\partial\Lambda} g_0(\Lambda) \ .$$

For this theory

$$\beta(g_0) = \Lambda\frac{\partial}{\partial\Lambda}\left(\frac{2\pi\hbar^2}{\log\left(1+\frac{\Lambda^2}{2\epsilon_B}\right)}\right) \overset{\text{calculate}}{=} -\frac{g_0^2}{\pi\hbar^2}\left(\underbrace{1}_{\text{perturbative}} - \underbrace{e^{-2\pi\hbar^2/g_0}}_{\text{not perturbative}}\right) .$$

Notice that it's a function only of g_0, and not explicitly of Λ. Also, in this simple toy theory perturbation theory for the beta function happens to stop at order g_0^2.

Notice that β measures the failure of the cutoff to disappear from our discussion – it signals a quantum mechanical violation of scale invariance.

What's β for? Flow equations:

$$\dot{g}_0 = \beta(g_0).$$

[5] This is a tautology. The dot is

$$\dot{A} = \partial_s A, \quad s \equiv \log\Lambda/\Lambda_0 \implies \partial_s = \Lambda\partial_\Lambda.$$

[5] Warning: The sign in this definition carries a great deal of cultural baggage. With the definition given here, the flow (increasing s) is toward the UV, toward high energy. This is the high-energy particle physics perspective, where we learn more physics by going to higher energies. As we will see, there is a strong argument to be made for the other perspective, that the flow should be regarded as going from UV to IR, since we lose information as we move in that direction – in fact, the IR behavior does not determine the UV behavior in general.

(Λ_0 is some reference scale.) But forget for the moment that this is just a definition:

$$\dot{g}_0 = -\frac{g_0^2}{\pi\hbar^2}\left(1 - e^{-2\pi\hbar^2/g_0}\right) \ .$$

This equation tells you how g_0 changes as you change the cutoff. Think of it as a nonlinear dynamical system (fixed points, limit cycles...)

Def: A fixed point $g_0^\star$ of a flow is a point where the flow stops:

$$\boxed{0 = \dot{g}_0|_{g_0^\star} = \beta(g_0^\star)} \ ,$$

a zero of the beta function. (Note: if we have many couplings g_i, then we have such an equation for each g: $\dot{g}_i = \beta_i(g)$. So β_i is (locally) a vector field on the space of coupilngs.)

Where are the fixed points in our example?

$$\beta(g_0) = -\frac{g_0^2}{\pi\hbar^2}\left(1 - e^{-2\pi\hbar^2/g_0}\right) \ .$$

There's only one: $g_0^\star = 0$, near which $\beta(g_0) \sim -\frac{g_0^2}{\pi\hbar}$, the non-perturbative terms are small. What does the flow look like near this point? For $g_0 > 0$, $\dot{g}_0 = \beta(g_0) < 0$. With this (high-energy) definition of the direction of flow, $g_0 = 0$ is an attractive fixed point:

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$g_0^\star = 0$.

We already knew this. It just says $g_0(\Lambda) \sim \frac{1}{\log\Lambda^2} \to 0$ at large Λ. But the general lesson is that in the vicinity of such an AF fixed point, the non-perturbatuve stuff $e^{\frac{-2\pi\hbar^2}{g_0}}$ is small. So we can get good results near the fixed point from the perturbative part of β. That is: we can compute the behavior of the flow of couplings near an AF fixed point *perturbatively*, and be sure that it is an AF fixed point. This is the situation in QCD.

On the other hand, the d.t. phenomenon that we've shown here is something that we can't prove in QCD. The circumstantial evidence is very strong!

[End of Lecture 1]

Another example where this happens is quantum mechanics in any number of variables with a central potential $V = -\frac{g_0^2}{r^2}$. It is also classically scale invariant:

$$[r] = \frac{1}{2}, \quad [\frac{1}{r^2}] = -1 \quad \Longrightarrow \quad [g_0] = 0.$$

This model was studied in K.M. Case, *Phys Rev* **80** (1950) 797 and you will study it on pset 01. The resulting boundstates and d.t. phenomenon are called Efimov states; this model preserves a *discrete* scale invariance.

Here's a quote from Marty Halpern from his lecture on this subject:

I want you to study this set of examples very carefully, because it's the only time in your career when you will understand what is going on.

In my experience it's been basically true. For real QFTs, you get distracted by Feynman diagrams, gauge invariance, regularization and renormalization schemes, and the fact that you can only do perturbation theory.

1.2 Integrating out degrees of freedom

Here's a second parable from QM which gives some useful perspective on renormalization in QFT. It is also a valuable opportunity to understand the differences and connections between euclidean and real-time Green's functions.

[Banks p. 138] Consider a system of two coupled harmonic oscillators. We will assume one of the springs is much stiffer than the other: let's call their natural frequencies ω_0, Ω, with $\omega_0 \ll \Omega$. The euclidean-time action is

$$S[X,x] = \int dt \left[\frac{1}{2}\left(\dot{x}^2 + \omega_0^2 x^2\right) + \frac{1}{2}\left(\dot{X}^2 + \Omega^2 X^2\right) + gXx^2\right] \equiv S_{\omega_0}[x] + S_\Omega[X] + S_{\rm int}[X,x].$$

(The particular form of the x^2X coupling is chosen for convenience.) We can construct physical observables in this model by studying the path integral:

$$Z = \int [dX\,dx] e^{-S[X,x]}.$$

Since I put a minus sign rather than an i in the exponent (and the potential terms in the action have + signs), this is a euclidean path integral.

Let's consider what happens if we do the path integral over the heavy mode X, and postpone doing the path integral over x. This step, naturally, is called *integrating out* X, and we will see below why this is a good idea. The result just depends on x; we can think of it as an *effective action* for x:

$$\begin{aligned} e^{-S_{\rm eff}[x]} &:= \int [dX] e^{-S[x,X]} \\ &= e^{-S_{\omega_0}[x]} \langle e^{-S_{\rm int}[X,x]} \rangle_X \end{aligned}$$

Here $\langle ... \rangle_X$ indicates the expectation value of ... in the (free) theory of X, with the action $S_\Omega[X]$. It is a gaussian integral:

$$\langle e^{-S_{\rm int}[X,x]} \rangle_X = \int [dX] e^{-S_\Omega[X] - \int ds J(s) X(s)} = \mathcal{N} e^{\frac{1}{4}\int ds dt J(s) G(s,t) J(t)} \ .$$

You will show this last equality (just a property of gaussian integrals) on the homework. Here $J(s) \equiv g x(s)^2$. The normalization factor $\mathcal{N}$ is independent of J and hence of x. And $G(s,t)$ is the inverse of the linear operator appearing in S_Ω, the *green's function*:

$$S_\Omega[X] = \int ds dt X(s) G^{-1}(s,t) X(t).$$

More usefully, G satisfies

$$\left(-\partial_s^2 + \Omega^2\right) G(s,t) = \delta(s-t)$$

The fact that our system is time-translation invariant means $G(s,t) = G(s-t)$. We can solve this equation in fourier space: $G(s) = \int đ\omega e^{-i\omega s} G_\omega$ makes it algebraic:

$$G_\omega = \frac{1}{\omega^2 + \Omega^2}$$

and we have

$$G(s) = \int đ\omega e^{-\mathbf{i}\omega s} \frac{1}{\omega^2 + \Omega^2}. \tag{1.3}$$

So we have:

$$e^{-S_{\text{eff}}[x]} = e^{-S_{\omega_0}[x]} e^{-\int dtds \frac{g^2}{2} x(s)^2 G(s,t) x(t)^2}$$

or taking logs

$$S_{\text{eff}}[x] = S_{\omega_0}[x] + \int dtds \frac{g^2}{2} x(s)^2 G(s,t) x(t)^2 \ . \tag{1.4}$$

X mediates an interaction of four xs, an anharmonic term, a self-interaction of x. In Feynman diagrams, the leading term here comes from the diagram in Fig. 2. [6]

But it is *non-local*: we have two integrals over the time in the new quartic term. This is unfamiliar, and *bad*: *e.g.* classically we don't know how to pose an initial value problem.

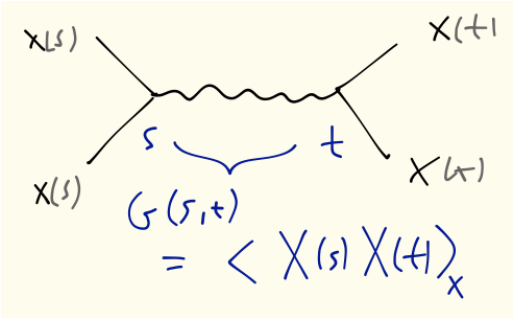


Figure 2: Interaction of x mediated by X.

But now suppose we are interested in times much longer than $1/\Omega$, say times comparable to the period of oscillation of the less-stiff spring $2\pi/\omega$. We can accomplish this by Taylor expanding under the integrand in (1.3):

$$G(s) \overset{s \gg 1/\Omega}{\simeq} \int đ\omega e^{-\mathbf{i}\omega s} \frac{1}{\Omega^2} \underbrace{\frac{1}{1 + \frac{\omega^2}{\Omega^2}}}_{= \sum_n (-1)^n \left(\frac{\omega^2}{\Omega^2}\right)^n} \simeq \frac{1}{\Omega^2}\delta(s) + \frac{1}{\Omega^4}\partial_s^2 \delta(s) + \ldots$$

Plug this back into (1.4):

$$S_{\text{eff}}[x] = S_{\omega_0}[x] + \int dt \frac{g^2}{2\Omega^2} x(t)^4 + \int dt \frac{g^2}{2\Omega^4} \dot{x}^2 x^2 + \ldots$$

The effects of the heavy mode X are now organized in a *derivative expansion*, with terms involving more derivatives suppressed by more powers of the high energy scale Ω.

[6] And the whole thing comes from exponentiating disconnected copies of this diagram. There are no other diagrams: once we make an X from two xs what can it do besides turn back into two xs? Nothing. And no internal x lines are allowed, they are just sources, for the purposes of the X integral.

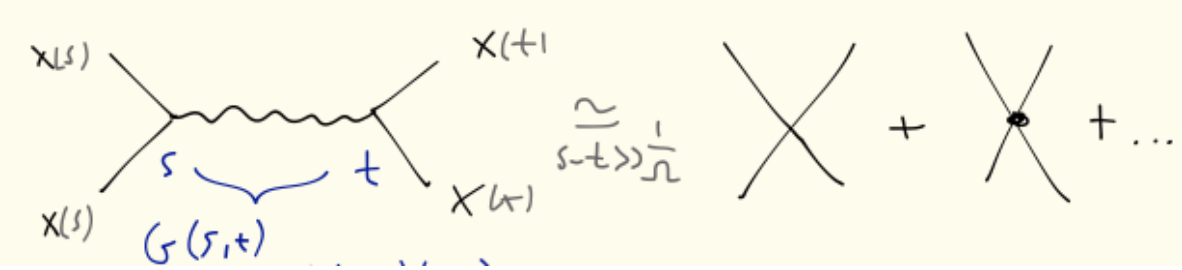


Figure 3: A useful mnemonic for integrating out the effects of X in terms of Feynman diagrams: to picture X as propagating for only a short time (compared to the external time $t - s$), we can contract its propagator to a point. The first term on the RHS shifts the x^4 term, the second shifts the kinetic term, the third involves four factors of $\dot{x}$...

On the RHS of this equation, we have various interactions involving four xs, which involve increasingly many derivatives. The first term is a quartic potential term for x: $\Delta V = \frac{g}{\Omega^2} x^4$; the leading effect of the fluctuations of X is to *shift* the quartic self-coupling of x by a finite amount (note that we could have included a bare $\lambda_0 x^4$ potential term).

Notice that if we keep going in this expansion, we get terms with *more than two derivatives* of x. This is OK. We've just derived the right way to think about such terms: they are part of a never-ending series of terms which become less and less important for low-energy questions. If we want to ask questions about x at energies of order ω, we can get answers that are correct up to effects of order $\left(\frac{\omega}{\Omega}\right)^{2n}$ by keeping the nth term in this expansion.

Conversely if we are doing an experiment with precision Δ at energy ω, we can measure the effects of up to the nth term, with

$$\left(\frac{\omega}{\Omega}\right)^{2n} \sim \Delta.$$

1.2.1 Attempt to consolidate understanding

We've just done some coarse graining: focusing on the dofs we care about (x), and actively ignoring the dofs we don't care about (X), except to the extent that they affect those we do (e.g. the self-interactions of x).

Above, we did a calculation in a QM model with two SHOs. This is a paradigm of QFT in many ways. For one thing, free quantum fields are bunches of harmonic oscillators with natural frequency depending on k. Here we keep just two of them for clarity. Perhaps more importantly, QM is just QFT in 0+1d.

The result of that calculation was that fluctuations of X mediate various x^4 interactions. It adds to the action for x the following: $\Delta S_{\text{eff}}[x] \sim \int dtdsx^2(t)G(t-s)x^2(s)$, as in Fig. 3.

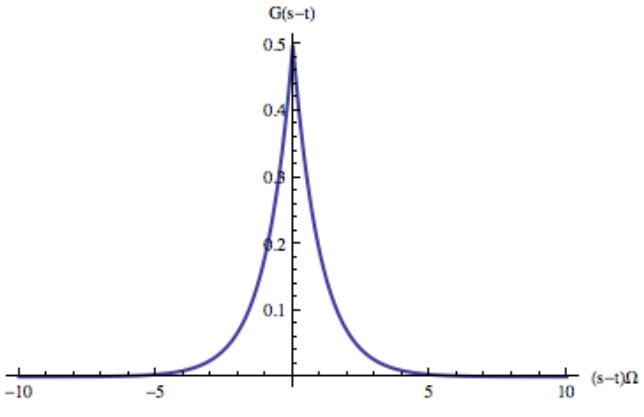
If we have the hubris to care about the exact answer, it's nonlocal in time. But if we want exact answers then we'll have to do the integral over x, too. On the other hand, the hierarchy of scales $\omega_0 \ll \Omega$ is useful if we ask questions about energies of order ω_0, e.g.

$$\langle x(t)x(0)\rangle \text{ with } t \sim \frac{1}{\omega_0} \gg \Omega$$

Then we can taylor expand the function $G(t-s)$, and we find a series of corrections in powers of $\frac{1}{t\Omega}$ (or more accurately, powers of $\frac{\partial_t}{\Omega}$).

(Notice that it's not so useful to integrate out light degrees of freedom to get an action for the heavy degrees of freedom; that would necessarily be nonlocal and stay nonlocal and we wouldn't be able to treat it using ordinary techniques.)

The crucial point is that the scary non-locality of the effective action that we saw only extends a distance of order $\frac{1}{\Omega}$; the kernel $G(s-t)$ looks like this:



One more attempt to drive home the central message of this discussion: the mechanism we've just discussed is an essential ingredient in getting *any* physics done at all. *Why* can we do physics despite the fact that we do not understand the theory of quantum gravity which governs Planckian distances? We happily do lots of physics without worrying about this! This is because the effect of those Planckian quantum gravity fluctuations – whatever they are, call them X – on the

degrees of freedom we do care about (*e.g.* the Standard Model, or an atom, or the sandwich you made this morning, call them collectively x) are encoded in terms in the effective action of x which are suppressed by powers of the high energy scale $M_{\rm Planck}$, whose role in the toy model is played by Ω. And the natural energy scale of your sandwich is much less than $M_{\rm Planck}$.

I picked the Planck scale as the scale to ignore here for rhetorical drama, and because we really are ignorant of what physics goes on there. But this idea is equally relevant for *e.g.* being able to describe water waves by hydrodynamics (a classical field theory) without worrying about atomic physics, or to understand the physics of atoms without needing to understand nuclear physics, or to understand the nuclear interactions without knowing about the Higgs boson, and so on deeper into the onion of physics.

This wonderful situation, which makes physics possible, has a price: since physics at low energies is so insensitive to high energy physics, it makes it hard to learn about high energy physics! People have been very clever and have learned a lot in spite of this vexing property of the RG. We can hope that will continue. (Cosmological inflation plays a similar role in hiding the physics of the early universe. It's like whoever designed this game is *trying* to hide this stuff from us.)

The explicit functional form of $G(s)$ (the inverse of the (euclidean) kinetic operator for X) is:

$$G(s) = \int đ\omega \frac{e^{-\mathbf{i}\omega s}}{\omega^2+\Omega^2} = e^{-\Omega|s|}\frac{1}{2\Omega}. \tag{1.5}$$

Do it by residues: the integrand has poles at $\omega = \pm\mathbf{i}\Omega$ (see the figure 4 below). The absolute value of $|s|$ is crucial, and comes from the fact that the contour at infinity converges in the upper (lower) half plane for $s<0$ ($s>0$).

Next, some comments about ingredients in the discussion of this subsection 1.2, which provides a useful opportunity to review/introduce some important QFT technology:

- Please don't be confused by the formal similarity of the above manipulations with the construction of the generating functional of correlation functions of X:

$$Z[J] \equiv \langle e^{\int dt X(t) J(t)} \rangle_X \ , \quad \langle X(t_1)X(t_2)... \rangle_X = \frac{\delta}{\delta J(t_1)}\frac{\delta}{\delta J(t_1)}... \log Z[J]$$

[7] It's true that what we did above amounts precisely to constructing $Z[J]$, and plugging

[7] Functional derivatives will be very useful to us. The definition is

$$\frac{\delta J(s)}{\delta J(t)} = \delta(s-t) \tag{1.6}$$

in $J = g_0 x^2$. But the motivation is different: in the above x is also a dynamical variable, so we don't get to pick x and differentiate with respect to it; we are merely postponing doing the path integral over x until later.

- Having said that, what is this quantity $G(s)$ above? It is the (euclidean) two-point function of X:

$$G(s,t) = \langle X(s)X(t)\rangle_X = \frac{\delta}{\delta J(t)}\frac{\delta}{\delta J(s)}\log Z[J].$$

  The middle expression makes it clearer that $G(s,t) = G(s-t)$ since nobody has chosen the origin of the time axis in this problem. This euclidean green's function – the inverse of $-\partial_\tau^2 + \Omega^2$ is unique, once we demand that it falls off at large separation. The same is not true of the real-time Green's function, which we discuss next in §1.2.2.

- **Adding more labels.** Quantum mechanics is quantum field theory in 0+1 dimensions. Except for our ability to do all the integrals, everything we are doing here generalizes to quantum field theory in more dimensions: quantum field theory *is* quantum mechanics (with infinitely many degrees of freedom). With more spatial dimensions, it becomes a good idea to call the fields something other than x, which we'll want to use for the spatial coordinates (which are just labels on the fields!). (I should have used q instead of x in anticipation of this step.)

  All the complications we'll encounter next (in §1.2.2) with choosing frequency contours are identical in QFT.

1.2.2 Wick rotation to real time.

For convenience, I have described this calculation in euclidean time (every t or s or τ that has appeared so far in this subsection has been a euclidean time). This is nice because the euclidean action is nice and positive, and all the wiggly and ugly configurations are manifestly highly suppressed in the path integral. Also, in real time[8] we have to make statements about states: *i.e.* in what state should we put the heavy mode?

---

plus the Liebniz properties (linearity, product rule). More prosaically, they are just partial derivatives, if we define a collection of values of the independent variable $\{s_i\}$ to regard as grid points, and let

$$J_i \equiv J(s_i)$$

so that (1.6) is just

$$\frac{\partial J_i}{\partial J_j} = \delta_{ij}.$$

If you are not yet comfortable with the machinery of functional derivatives, please work through pages 2-28 through 2-30 of this document now.

[8] aka Minkowski time aka Lorentzian time

[End of Lecture 2]

The answer is: in the groundstate – it costs more energy than we have to excite it. I claim that the real-time calculation which keeps the heavy mode in its groundstate is the analytic continuation of the one we did above, where we replace

$$\omega_{\text{Mink}} = e^{-\mathbf{i}(\pi/2-\epsilon)}\omega_{\text{above}} \tag{1.7}$$

where ϵ is (a familiar, I hope) infinitesimal. In the picture of the euclidean frequency plane in Fig. 4, this is a rotation by *nearly* 90 degrees. We don't want to go all the way to 90 degrees, because then we would hit the poles at $\pm\mathbf{i}\Omega$.

The replacement (1.7) just means that if we integrate over *real* ω_{Mink}, we rotate the contour in the integral over ω as follows:

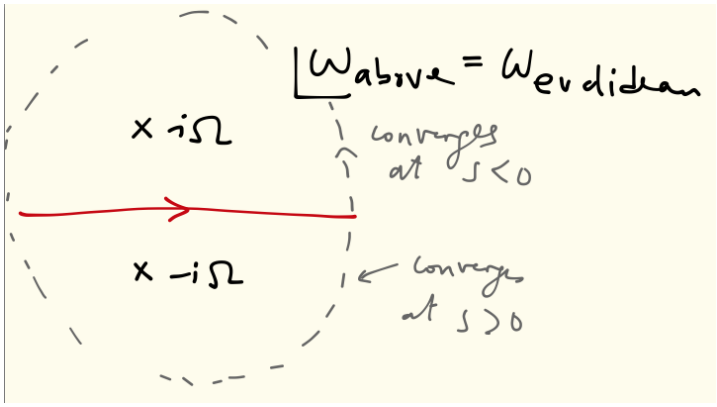
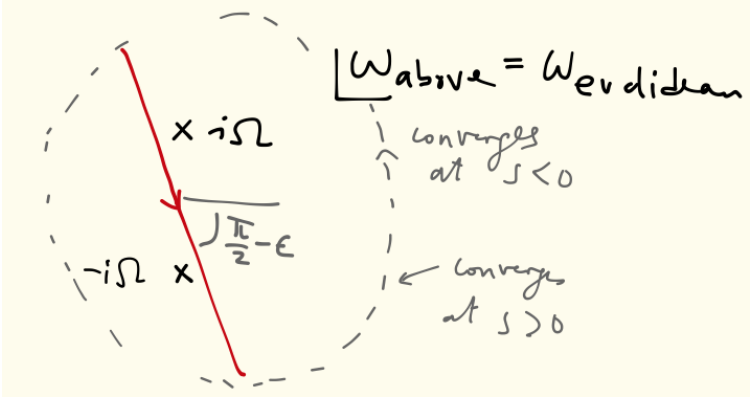


Figure 4: Poles of the integrand of the ω integral in (1.5).



as a result we pick up the same poles at $\omega_{\text{above}} = \pm\mathbf{i}\Omega$ as in the euclidean calculation. Notice that we had better also rotate the argument of the function, s, at the same time to maintain

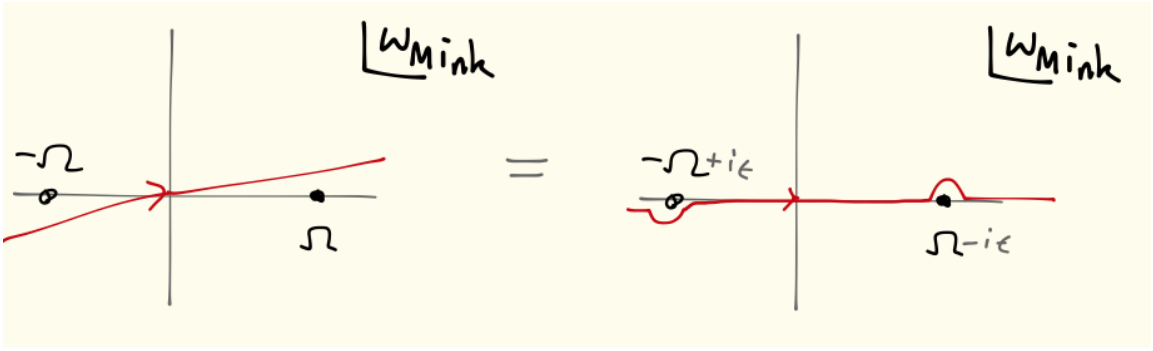


Figure 5: The Feynman contour in the $\omega_{\rm Mink}$ complex plane.

convergence, that is:

$$\omega_{\rm eucl} = -\mathbf{i}\omega_{\rm Mink}, \quad \omega_{\rm eucl}t_{\rm eucl} = \omega_{\rm Mink}t_{\rm Mink}, \quad t_{\rm eucl} = +\mathbf{i}t_{\rm Mink}. \tag{1.8}$$

So this is giving us a contour prescription for the real-frequency integral. The result is the *Feynman* propagator, with which you are familiar from previous quarters of QFT: depending on the sign of the (real) time separation of the two operators (recall that t is the difference), we close the contour around one pole or the other, giving the *time-ordered* propagator. (It is the same as shifting the heavy frequency by $\Omega \to \Omega - \mathbf{i}\epsilon$, as indicated in the right part of Fig. 5.)

Notice for future reference that the euclidean action and real-time action are related by

$$S_{\rm eucl}[X] = \int dt_{\rm eucl}\frac{1}{2}\left(\left(\frac{\partial X}{\partial t_{\rm eucl}}\right)^2 + \Omega^2 X^2\right) = -\mathbf{i}S_{\rm Mink}[X] = -\mathbf{i}\int dt_{\rm Mink}\frac{1}{2}\left(\left(\frac{\partial X}{\partial t_{\rm Mink}}\right)^2 - \Omega^2 X^2\right) .$$

because of (1.8). Notice that this means the path integrand is $e^{-S_{\rm eucl}} = e^{\mathbf{i}S_{\rm Mink}}$.

Why does the contour coming from the euclidean path integral put the excited mode into its groundstate? That's the the point in life of the euclidean path integral, to prepare the groundstate from an arbitrary state:

$$\int_{X_0} [dX] e^{-S[X]} = \langle X_0 | e^{-\mathbf{H}T} | ... \rangle = \psi_{\rm gs}(X_0) \tag{1.9}$$

– the euclidean-time propagator $e^{-\mathbf{H}T}$ beats down the amplitude of any excited state relative to the groundstate, for large enough T.

Let me back up one more step and explain (1.9) more. You know a path integral represen-

tation for the real-time propagator

$$\langle f|e^{-\mathbf{i}\mathbf{H}t}|i\rangle = \int [dx] e^{\mathbf{i}\int^t dt L}.$$

On the RHS here, we sum over all paths between i and f in time t, weighted by a phase $e^{\mathbf{i}\int dt L}$.

But that means you also know a representation for

$$\sum_f \langle f|e^{-\beta\mathbf{H}}|f\rangle \equiv \mathrm{tr} e^{-\beta\mathbf{H}}$$

– namely, you sum over all *periodic* paths in *imaginary time* $t = -\mathbf{i}\beta$. So:

$$Z(\beta) = \mathrm{tr} e^{-\beta\mathbf{H}} = \int [dx] e^{-\int_0^\beta d\tau L}$$

The LHS is the partition function in quantum statistical mechanics. The RHS is the euclidean functional integral we've been using. [For more on this, see Zee §V.2]

The period of imaginary time, $\beta \equiv 1/T$, is the inverse temperature. More accurately, we've been studying the limit as $\beta \to \infty$. Taking $\beta \to \infty$ means $T \to 0$, and you'll agree that at $T = 0$ we project onto the groundstate (if there's more than one groundstate we have to think more).

Time-ordering. To summarize the previous discussion: in real time, we must choose a state, and this means that there are many Green's functions, not just one: $\langle\psi|X(t)X(s)|\psi\rangle$ depends on $|\psi\rangle$, unsurprisingly.

But we found a special one which arises by analytic continuation from the euclidean Green's function, which is unique[9]. It is

$$G(s,t) = \langle \mathcal{T}\left(X(s)X(t)\right)\rangle_X \ ,$$

the time-ordered, or Feynman, Green's function, and I write the time-ordering symbol $\mathcal{T}$ to emphasize this. I emphasize that from our starting point above, the time ordering arose because we have to close the contour in the UHP (LHP) for $t < 0$ ($t > 0$).

Let's pursue this one more step. The same argument tells us that the generating functional for real-time correlation functions of X is

$$Z[J] = \langle \mathcal{T} e^{\mathbf{i}\int JX}\rangle = \langle 0|\mathcal{T} e^{\mathbf{i}\int JX}|0\rangle.$$

In the last step I just emphasized that the real time expectation value here is really a *vacuum* expectation value. This quantity has the picturesque interpretation as the *vacuum persistence amplitude*, in the presence of the source J.

Causality. In other treatments of this subject, you will see the Feynman contour motivated by ideas about causality. This was not the logic of our discussion but it is reassuring that we end up in the same place. Note that even in 0+1 dimensions there is a useful notion of causality: effects should come after their causes. I will have more to say about this later, when we have reason to discuss other real-time Green's functions.

[9] Another important perspective on the uniqueness of the euclidean Green's function and the non-uniqueness in real time: in euclidean time, we are inverting an operator of the form $-\partial_\tau^2 + \Omega^2$ which is *positive* ($\equiv$ all its eigenvalues are positive) – recall that $-\partial_\tau^2 = \hat{p}^2$ is the square of a hermitian operator. If all the eigenvalues are positive, the operator has no kernel, so it is completely and unambiguously invertible. This is why there are no poles on the axis of the (euclidean) ω integral in (1.5). In real time, in contrast, we are inverting something like $+\partial_t^2 + \Omega^2$ which annihilates modes with $\partial_t = \mathbf{i}\Omega$ (if we were doing QFT in $d > 0+1$ this equation would be the familiar $p^2 - m^2 = 0$) – on-shell states. So the operator we are trying to invert has a kernel and this is the source of the ambiguity. In frequency space, this is reflected in the presence of poles of the integrand *on* the contour of integration; the choice of how to negotiate them encodes the choice of Green's function.

1.3 Other ideas from systems with a finite number of degrees of freedom

If we had lots of time, I would continue this list of parables from quantum mechanics (by which I really mean systems with a finite number of degrees of freedom) with the following items:

- Semiclassical expansions
- Tunneling and instantons (for this, I have a good excuse: you can read the definitive treatment by Sidney Coleman here or in *Aspects of Symmetry*.)
- Large N expansions
- Supersymmetry
- Quantization of constrained systems and BRST formalism

We may have to have a section called 'Ideas from QM, part II'.

2 Renormalization in QFT

Next we will study the effect of adding those pesky extra position labels on our fields.

2.1 Naive scale invariance in field theory

[Halpern] Consider a field theory of a scalar field ϕ in D (euclidean) spacetime dimensions, with an action of the form

$$S[\phi] = \int d^D x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - g\phi^p \right)$$

for some constants p, g. Which value of p makes this scale invariant?

Naive dimensions:

$$[S] = [\hbar] = 0, \quad [x] \equiv 1, \quad [d^D x] = D, \quad [\partial] = -1$$

The kinetic term tells us the engineering dimensions of ϕ:

$$0 = [S_{\text{kinetic}}] = D - 2 + 2[\phi] \implies [\phi] = \frac{2-D}{2}.$$

Notice that the $D = 1$ case agrees with our quantum mechanics counting. Quantum field theory in $D = 1$ spacetime dimensions *is* quantum mechanics. (Quantum field theory in $D = 0$ spacetime dimensions is integrals. This sounds trivial but it actually has some useful lessons for us in the form of *random matrix theory*.)

Then the self-interaction term has dimensions

$$0 = [S_{\text{interaction}}] = D + [g] + p[\phi] \implies [g] = -(D + p[\phi]) = -\left(D + p\frac{2-D}{2} \right)$$

We expect scale invariance when $[g] = 0$ which happens when

$$\boxed{p = p_D \equiv \frac{2D}{D-2}},$$

i.e. the scale invariant scalar-field self-interaction in D spacetime dimensions is $\phi^{\frac{2D}{D-2}}$.

⋆ What is happening in $D = 2$? The field is dimensionless, and so *any* power of ϕ is naively scale invariant, as are more complicated interactions like $g(\phi)(\partial\phi)^2$. This allows for scale-invariant non-linear sigma models; we will explore this further later on.

D	1	2	3	4	5	6	...	∞
$[\phi]$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-3/2$	-2	...	$-D/2$
scale-inv't $p \equiv p_D$	-2	$\infty^\star$	6	4	$10/3$	3	...	2

In dimensions where we get fractional powers, this isn't so nice.

Notice that the mass term $\Delta S = \int d^D x \frac{m^2}{2}\phi^2$ gives

$$0 = D + 2[m] + 2[\phi] \quad \Longrightarrow \quad [m] = -1 \;\; \forall D < \infty.$$

What are the consequences of this engineering dimensions calculation in QFT? For $D > 2$, an interaction of the form $g\phi^p$ has

$$[g] = D \cdot \frac{p - p_D}{p_D} \begin{cases} > 0 \text{ when } p > p_D, & \text{non-renormalizable or irrelevant} \\ = 0 \text{ when } p = p_D, & \text{renormalizable or marginal} \\ < 0 \text{ when } p < p_D, & \text{super-renormalizable or relevant.} \end{cases}$$

Consider the 'non-renormalizable' case. Suppose we calculate in QFT some quantity f with $[f]$ as its naive dimension, in perturbation theory in g, *e.g.* by Feynman diagrams. We'll get:

$$f = \sum_{n=0}^{\infty} g^n c_n$$

with c_n independent of g. So

$$[f] = n[g] + [c_n] \quad \Longrightarrow \quad [c_n] = [f] - n[g]$$

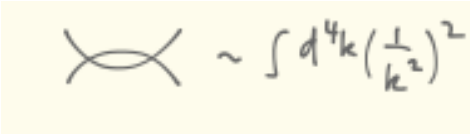
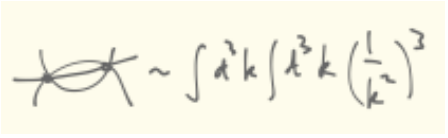
So if $[g] > 0$, c_n must have more and more powers of some mass (inverse length) as n increases. What dimensionful quantity makes up the difference?? Sometimes it is masses or external momenta. But generically, it gets made up by UV divergences (if everything is infinite, dimensional analysis can fail, nothing is real, I am the walrus). More usefully, in a meaningful theory with a UV cutoff, Λ_{UV}, the dimensions get made up by the UV cutoff, which has $[\Lambda_{UV}] = -1$. Generically: $c_n = \tilde{c}_n \left(\Lambda_{UV}\right)^{n[g]}$, where $\tilde{c}_n$ is dimensionless, and $n[g] > 0$ – it's higher and higher powers of the cutoff.

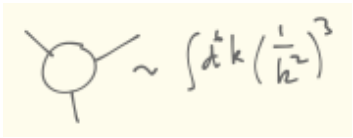
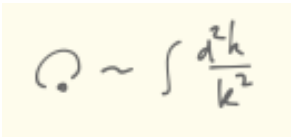
Consider the renormalizable (classically scale invariant) case: $[c_n] = [f]$, since $[g] = 0$. But in fact, what you'll get is something like

$$c_n = \tilde{c}_n \log^{\nu(n)}\left(\frac{\Lambda_{UV}}{\Lambda_{IR}}\right) \;,$$

where Λ_{IR} is an infrared cutoff, $[\Lambda_{IR}] = -1$.

Some classically scale invariant examples (so that $m = 0$ and the bare propagator is $1/k^2$) where you can see that we get logs from loop amplitudes:

ϕ^4 in $D = 4$:  ϕ^6 in $D = 3$: 

ϕ^3 in $D = 6$:  ϕ^1 in $D = 2$: 

Below I will convince you that these statements are true in general. But first we will need to think about about the structure of perturbation theory.

[End of Lecture 3]

2.2 Blob-ology: structure of diagrammatic perturbation theory

It will help streamline our discussion of perturbative renormalization if we organize our thinking about perturbation theory a bit.

Feynman diagrams reminder. [Zee I.7] But first: I should remind you what I mean by Feynman diagrams. As Zee correctly emphasizes, they are not magic; they are merely a useful tool for visualizing the perturbative expansion of the functional integral. This section is supposed to be about adding labels to our functional integration variables, but let's briefly retreat to QFT in $0 + 0$ dimensions. Suppose we want to do the integral

$$Z(J) = \int_{-\infty}^{\infty} dq\; e^{-\frac{1}{2}m^2q^2 - \frac{g}{4!}q^4 + Jq} \equiv \int dq\; e^{-S(q)} \; . \tag{2.1}$$

It is the path integral for ϕ^4 theory with fewer labels. For $g = 0$, this is a gaussian integral which we did on Problem Set 1. For $g \neq 0$ it's not an elementary function of its arguments. We can develop a (non-convergent!) series expansion in g by writing it as

$$Z(J) = \int_{-\infty}^{\infty} dq\; e^{-\frac{1}{2}m^2q^2 + Jq} \left(1 - \frac{g}{4!}q^4 + + \frac{1}{2}\left(-\frac{g}{4!}q^4\right)^2 + \cdots\right)$$

and integrating term by term. And the term with q^{4n} (that is, the coefficient of g^n) is

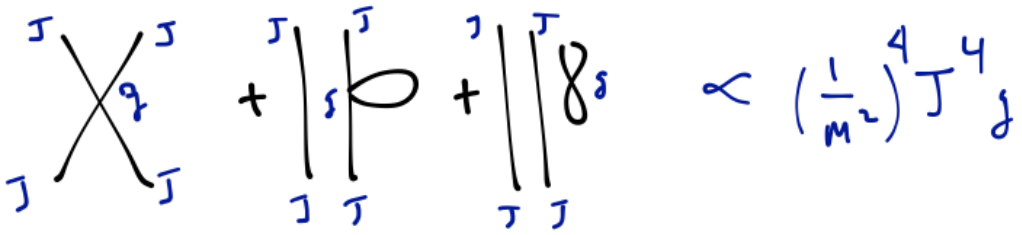
$$\int_{-\infty}^{\infty} dq\; e^{-\frac{1}{2}m^2q^2 + Jq} q^{4n} = \left(\frac{\partial}{\partial J}\right)^{4n} \int_{-\infty}^{\infty} dq\; e^{-\frac{1}{2}m^2q^2 + Jq} = \left(\frac{\partial}{\partial J}\right)^{4n} e^{\frac{1}{2}J\frac{1}{m^2}J} \sqrt{\frac{2\pi}{m^2}} \; .$$

So:

$$Z(J) = \sqrt{\frac{2\pi}{m^2}} e^{-\frac{g}{4!}\left(\frac{\partial}{\partial J}\right)^4} e^{\frac{1}{2}J\frac{1}{m^2}J} \; .$$

This is a double expansion in powers of J and powers of g. The process of computing the coefficient of $J^n g^m$ can be described usefully in terms of diagrams. There is a factor of $1/m^2$ for each line (the propagator), and a factor of $(-g)$ for each 4-point vertex (the coupling), and a factor of J for each external line (the source). For example, the coefficient of gJ^4 comes from:

$$\sim \left(\frac{1}{m^2}\right)^4 gJ^4.$$



There is a symmetry factor which comes from expanding the exponential: if the diagram has some symmetry preserving the external labels, the multiplicity of diagrams does not completely cancel the $1/n!$.

As another example, consider the analog of the two-point function:

$$G \equiv \langle q^2 \rangle|_{J=0} = \frac{\int dq\ q^2\ e^{-S(q)}}{\int dq\ e^{-S(q)}} = -2\frac{\partial}{\partial m^2} \log Z(J=0).$$

In perturbation theory this is:

$$\begin{aligned} G &\simeq \quad + \tfrac{1}{2} \quad + \tfrac{1}{4} \quad + \tfrac{1}{8} \quad + \tfrac{1}{6} \quad + \mathcal{O}(g^3) \\ &= m^{-2}\left(1 \quad - \frac{1}{2}gm^{-2} \quad + \quad \frac{2}{3}g^2 m^{-4} \quad + \mathcal{O}(g^3)\right) \end{aligned} \tag{2.2}$$

Brief comments about large orders of perturbation theory.

- How do I know the perturbation series about $g = 0$ doesn't converge? One way to see this is to notice that if I made g even infinitesimally negative, the integral itself would not converge (the potential would be unbounded below), and $Z_{g=-|\epsilon|}$ is not defined. Therefore Z_g as a function of g cannot be analytic in a neighborhood of $g = 0$. This argument is due to Dyson.
- The expansion of the exponential in the integrand is clearly convergent for each q. The place where we went wrong is exchanging the order of integration over q and summation over n.

- The integral actually does have a name – it's a Bessel function:

$$Z(J=0) = \frac{2}{\sqrt{m^2}}\sqrt{\rho}e^{\rho}K_{\frac{1}{4}}(\rho), \qquad \rho \equiv \frac{3m^4}{4g}$$

  (for $\mathrm{Re}\,\sqrt{\rho} > 0$), as `Mathematica` will tell you. Because we know about Bessel functions, in this case we can actually figure out what happens at strong coupling, when $g \gg m^4$.

- In this case, the perturbation expansion too can be given a closed form expression:

$$Z(0) \simeq \sqrt{\frac{2\pi}{m^2}} \sum_n \frac{(-1)^n}{n!} \frac{2^{2n+\frac{1}{2}}}{(4!)^n} \Gamma\left(2n+\frac{1}{2}\right) \left(\frac{g}{m^4}\right)^n. \tag{2.3}$$

- The expansion for G is of the form

$$G \simeq m^{-2} \sum_{n=0}^{\infty} c_n \left(\frac{g}{m^4}\right)^n.$$

  When n is large, the coefficients satisfy $c_{n+1} \overset{n\gg 1}{\simeq} -\frac{2}{3}nc_n$ (you can see this by looking at the coefficients in (2.3)) so that $|c_n| \sim n!$. This factorial growth of the number of diagrams is general in QFT and is another way to see that the series does not converge.

- The fact that the coefficients c_n grow means that there is a *best* number of orders to keep. The errors start getting bigger when $c_{n+1}\left(\frac{g}{m^4}\right) \sim c_n$, that is, at order $n \sim \frac{3m^4}{2g}$. So if you want to evaluate G at this value of the coupling, you should stop at that order of n.

- A technique called *Borel resummation* can sometimes produce a well-defined function of g from an asymptotic series whose coefficients diverge like $n!$. In fact it works in this case. I may say more about this.

- The function $G(g)$ can be analytically continued in g away from the real axis, and can in fact be defined on the whole complex g plane. It has a branch cut on the negative real axis, across which its discontinuity is related to its imaginary part. The imaginary part goes like $e^{-\frac{a}{|g|}}$ near the origin and can be computed by a tunneling calculation.

For a bit more about this, you might look at sections 3 and 4 of this recent paper from which I got some of the details here.

The idea of Feynman diagrams is the same in the case with more labels. Notice that each of the qs in our integral could come with a label, $q \to q_a$. Then each line in our diagram

would be associated with a matrix $(m^{-2})_{ab}$ which is the inverse of the quadratic term $q_a m^2_{ab} q_b$ in the action. If our diagrams have loops we get free sums over the label. If that label is conserved by the interactions, the vertices will have some delta functions.

In the case of translation-invariant field theories we can label lines by the conserved momentum k. Each comes with a factor of the free propagator $\frac{\mathbf{i}}{k^2+m^2+\mathbf{i}\epsilon}$, each vertex conserves momentum, so comes with $\mathbf{i}g\delta^D\left(\sum k\right)(2\pi)^D$, and we must integrate over momenta on internal lines $\int \text{đ}^D k$.

Now I will explain three general organizing facts about the diagrammatic expansion.

In thinking about the combinatorics below, we will represent collections of Feynman diagrams by blobs with legs sticking out, and think about how the blobs combine. Then we can just renormalize the appropriate blobs and be done.

The following discussion will look like I am talking about a field theory with a single scalar field. But really each of the ϕs is a collection of fields and all the indices are too small to see. This is yet another example of coarse-graining.

1. **Disconnected diagrams exponentiate.**

   [Zee, I.7, Banks, chapter 3]

   Recall that the Feynman rules come with a (often annoying, here crucial) statement about symmetry factors: we must divide the contribution of a given diagram by the order of the symmetry group of the diagram (preserving various external labels). For a diagram with k identical disconnected pieces, this symmetry group includes the permutation group S_k which permutes the identical pieces and has $k!$ elements. (Recall that the origin of the symmetry factors is that symmetric feynman diagrams fail to completely cancel the $1/n!$ in the Dyson formula. For a reminder about this, see *e.g.* Peskin p. 93.) Therefore:

$$Z = \sum (\text{all diagrams}) = e^{\sum(\text{connected diagrams})} = e^{\mathbf{i}W}.$$

   You can go a long way towards convincing yourself of this by studying the case where there are only two connected diagrams $A + B$ (draw whatever two squiggles you want) and writing out e^{A+B} in terms of disconnected diagrams with symmetry factors.

   Notice that this relationship is just like that of the partition function to the (Helmholtz) free energy $Z = e^{-\beta F}$ (modulo the factor of $\mathbf{i}$) in statistical mechanics (and is the same as that relationship when we study the euclidean path integral with periodic boundary

conditions in euclidean time). This statement is extremely general. It remains true if we include external sources:

$$Z[J] = \int [D\phi] e^{\mathbf{i}S[\phi]+\mathbf{i}\int \phi J} = e^{\mathbf{i}W[J]}.$$

Now the diagrams have sources J at which propagator lines can terminate; (the perturbation theory approximation to) $W[J]$ is the sum of all connected such diagrams.

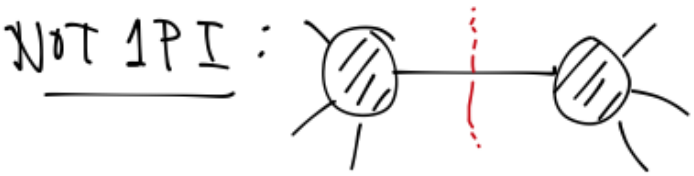
You probably knew this already, *e.g.* from stat mech. For example

$$\langle \phi(x) \rangle = \frac{1}{Z} \frac{\delta}{\mathbf{i}\delta J(x)} Z = \frac{\delta}{\mathbf{i}\delta J(x)} \log Z = \frac{\delta}{\delta J(x)} W$$

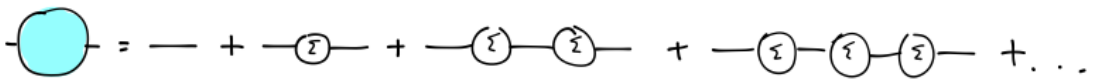
$$\langle \mathcal{T}\phi(x)\phi(y) \rangle = \frac{\delta}{\mathbf{i}\delta J(x)} \frac{\delta}{\mathbf{i}\delta J(y)} \log Z = \frac{\delta}{\mathbf{i}\delta J(x)} \frac{\delta}{\mathbf{i}\delta J(y)} \mathbf{i}W \ .$$

(Note that here $\langle \phi \rangle \equiv \langle \phi \rangle_J$ depends on J. You can set it to zero if you want, but the equation is true for any J.) If you forget to divide by the normalization Z, and instead look at just $\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} Z$, you get *disconnected* quantities like $\langle \phi \rangle \langle \phi \rangle$ (the terminology comes from the diagrammatic representation). [10] The point in life of W is that by differentiating it with respect to J we can construct all the connected Green's functions.

2. **Propagator corrections form a geometric series.** It is useful to define the notion of a *one-particle-irreducible* (1PI) diagram. This is a diagram which cannot be cut into two disconnected pieces by cutting a single propagator line.



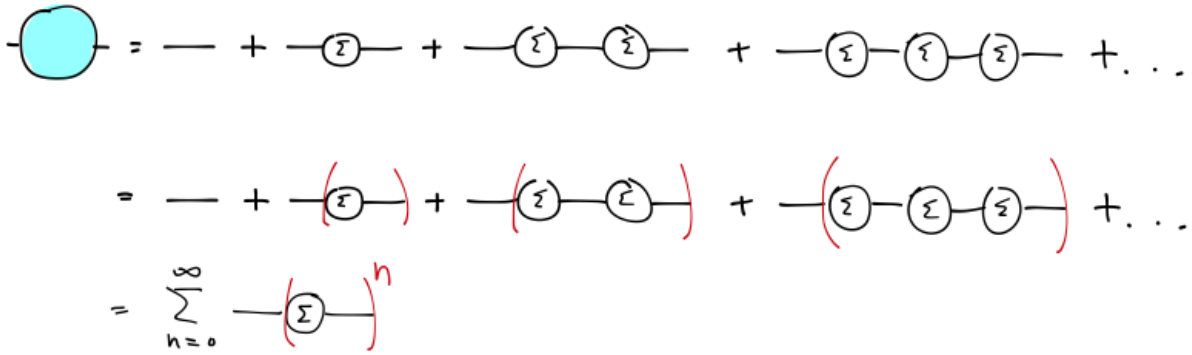
Consider the (connected) two-point function of the field G_2– the set of all (connected) diagrams with two external ϕ lines. Denote by a filled blob with little nubbins -O- the 1PI part of such diagrams (note that this omits the propagators for the external lines). The sum of these 1PI 2-point diagrams is called the *self-energy* Σ. Then the sum of all the diagrams is[11]



[10] More precisely: $\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} Z = \frac{\delta}{\delta J(x)} \left(\langle \phi(x) \rangle_J Z \right) = \langle \phi(x) \rangle_J \langle \phi(y) \rangle_J Z + \langle \phi(x)\phi(y) \rangle_J Z.$

[11] ascii feynman diagrams may be the way of the future, but this looks a little better.

where — denotes the free-field propagator G_2^0. You recognize this as a geometric series:



In the second line, the parentheses are to guide the eye. So the full propagator, in perturbation theory, is

$$G_2 = G_2^0 + G_2^0\Sigma G_2^0 + G_2^0\Sigma G_2^0\Sigma G_2^0 + ... = G_2^0\left(1 + \Sigma G_2^0 + \Sigma G_2^0\Sigma G_2^0 + ...\right) = G_2^0\frac{1}{1-\Sigma G_2^0}\,. \tag{2.4}$$

Recall that the name *propagator* is a good one: it propagates the state of the field in spacetime, and that means that really it is a matrix. The products in the previous expression, if we are working in position space, are actually convolutions: we have to sum over intermediate states. For example:

$$\left(G_2^0\Sigma G_2^0\right)(x,y) \equiv \int d^D z \int d^D w G_2^0(x,z)\Sigma(z,w)G_2^0(w,y).$$

(Aren't you glad I suppressed all those indices in (2.4)!) Notice that repeated labels are summed.

The convenience of momentum space (in translation-invariant examples, where it is available) is that these become simple products, because momentum is conserved, and so the momentum label is the same wherever we cut the diagram. This is true unless there is a loop, in which case the lines have to share the momentum. In that case the convolutions are just multiplication.

In momentum space (for a relativistic scalar field) these objects look like $G_2^0 = \frac{\mathbf{i}}{k^2-m^2-\mathbf{i}\epsilon}$. So

$$G_2 = \frac{\mathbf{i}}{k^2-m^2-\mathbf{i}\epsilon}\frac{1}{1-\Sigma\frac{\mathbf{i}}{k^2-m^2-\mathbf{i}\epsilon}} = \frac{\mathbf{i}}{k^2-m^2-\mathbf{i}\epsilon-\mathbf{i}\Sigma(k)}$$

– the effect of this sum is to shift the denominator of the propagator. (Notation warning: the thing I've called $\mathbf{i}\Sigma$ is what's usually called the self-energy Σ; I would have had to write lots more $\mathbf{i}$s above though.) [End of Lecture 4]

Consider Taylor expanding in k this quantity: $\Sigma(k) = \Sigma(0) + \frac{1}{2}k^2\Sigma''(0) + ...$ (I assumed Lorentz invariance). The term $\Sigma(0)$ shifts the mass term; the term $\Sigma''(0)$ rescales the kinetic term.

Notice that this shift in the denominator of the propagator would be effected by adding a quadratic term

$$\int dk\phi(k)\Sigma(k)\phi(-k) = \int dx\phi(x)\tilde{\Sigma}(x)\phi(x)$$

to the action. Here $\tilde{\Sigma}(x) = \int \text{đ}^D k e^{\mathbf{i}k_\mu x^\mu}\Sigma(k)$; this will be called Γ_2 below.

3. **The sum of all connected diagrams is the Legendre transform of the sum of the 1PI diagrams.**

   [Banks, 3.8; Zee IV.3; Srednicki §21] A simpler way to say our third fact is

   $$\sum(\text{connected diagrams}) = \sum(\text{connected } \textit{tree} \text{ diagrams with 1PI vertices})$$

   where a *tree* diagram is one with no loops. But the description in terms of Legendre transform will be extremely useful. Along the way we will show that the perturbation expansion is a semi-classical expansion. And we will construct a useful object called the 1PI effective action Γ. The basic idea is that we can construct the actual correct correlation functions by making *tree* diagrams ($\equiv$ diagrams with no loops) using the 1PI effective action as the action.

   Notice that this is a very good reason to care about the notion of 1PI: if we sum all the tree diagrams using the 1PI blobs, we clearly are including all the diagrams. Now we just have to see what machinery will pick out the 1PI blobs. The answer is: Legendre transform. There are many ways to go about showing this, and all involve a bit of complication. Bear with me for a bit; we will learn a lot along the way.

Def'n of ϕ_c, the 'classical field'. Consider the functional integral for a scalar field theory:

$$Z[J] = e^{\mathbf{i}W[J]} = \int [D\phi] e^{\mathbf{i}\left(S[\phi] + \int J\phi\right)} \ . \tag{2.5}$$

Define

$$\phi_c(x) \equiv \frac{\delta W[J]}{\delta J(x)} = \frac{1}{Z}\int [D\phi] e^{\mathbf{i}\left(S[\phi]+\int J\phi\right)}\phi(x) = \langle 0|\hat{\phi}(x)|0\rangle \ . \tag{2.6}$$

This is the vacuum expectation value of the field operator, in the presence of the source J. Note that $\phi_c(x)$ is a functional of J.

Warning: we are going to use the letter ϕ for many conceptually distinct objects here: the functional integration variable ϕ, the quantum field operator $\hat{\phi}$, the classical field ϕ_c. I will not always use the hats and subscripts.

Legendre Transform. Next we recall the notion of Legendre transform and extend it to the functional case: Given a function L of $\dot{q}$, we can make a new function H of p (the Legendre transform of L with respect to $\dot{q}$) defined by:

$$H(p,q) = p\dot{q} - L(\dot{q},q).$$

On the RHS here, $\dot{q}$ must be eliminated in favor of p using the relation $p = \frac{\partial L}{\partial \dot{q}}$. You've also seen this manipulation in thermodynamics using these letters:

$$F(T,V) = E(S,V) - TS, \quad T = \frac{\partial E}{\partial S}|_V \ .$$

The point of this operation is that it relates the free energies associated with different ensembles in which different variables are held fixed. More mathematically, it encodes a function (at least one with nonvanishing second derivative, *i.e.* one which is convex or concave) in terms of its envelope of tangents. For further discussion of this point of view, look here.

Now the functional version: Given a functional $W[J]$, we can make a new associated functional Γ of the conjugate variable ϕ_c:

$$\Gamma[\phi_c] \equiv W[J] - \int J\phi_c.$$

Again, the RHS of this equation defines a functional of ϕ_c implicitly by the fact that J can be determined from ϕ_c, using (2.6)[12].

Interpretation of ϕ_c. How to interpret ϕ_c? It's some function of spacetime, which depends on the source J. **Claim:** It solves

$$-J(x) = \frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} \tag{2.7}$$

So, in particular, when $J = 0$, it solves

$$0 = \frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)}|_{\phi_c=\langle\phi\rangle} \tag{2.8}$$

– the extremum of the effective action is $\langle\phi\rangle$. This gives a classical-like equation of motion for the field operator expectation value in QFT.

Proof of (2.7): $$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = \frac{\delta}{\delta\phi_c(x)}\left(W[J] - \int dy J(y)\phi_c(y)\right)$$

What do we do here? We use the functional product rule – there are three places where the derivative hits:

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = \frac{\delta W[J]}{\delta\phi_c(x)} - J(x) - \int dy \frac{\delta J(y)}{\delta\phi_c(x)}\phi_c(y)$$

[12] Come back later and worry about what happens if J is not determined uniquely.

In the first term we must use the functional chain rule:

$$\frac{\delta W[J]}{\delta\phi_c(x)} = \int dy \frac{\delta J(y)}{\delta\phi_c(x)} \frac{\delta W[J]}{\delta J(y)} = \int dy \frac{\delta J(y)}{\delta\phi_c(x)} \phi_c(y).$$

So we have:

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = \int dy \frac{\delta J(y)}{\delta\phi_c(x)} \phi_c(y) - J(x) - \int dy \frac{\delta J(y)}{\delta\phi_c(x)} \phi_c(y) = -J(x). \tag{2.9}$$

Now $\phi_c|_{J=0} = \langle\phi\rangle$. So if we set $J = 0$, we get the equation (2.8) above. So (2.8) replaces the action principle in QFT – to the extent that we can calculate $\Gamma[\phi_c]$. (Note that there can be more than one extremum of Γ. That requires further examination.)

Next we will build towards a demonstration of the diagrammatic interpretation of the Legendre transform; along the way we will uncover important features of the structure of perturbation theory.

Semiclassical expansion of path integral. Recall that the Legendre transform in thermodynamics is the leading term you get if you compute the partition function by saddle point – the classical approximation. In thermodynamics, this comes from the following manipulation: the thermal partition function is:

$$Z = e^{-\beta F} = \mathrm{tr} e^{-\beta \mathbf{H}} = \int dE \underbrace{\Omega(E)}_{\text{density of states with energy } E = e^{S(E)}} e^{-\beta E} \overset{\text{saddle}}{\approx} e^{S(E_\star) - \beta E_\star}|_{E_\star \text{ solves } \partial_E S = \beta} .$$

The log of this equation then says $F = E - TS$ with S eliminated in favor of T by $T = \frac{1}{\partial_E S}|_V = \partial_S E|_V$, *i.e.* the Legendre transform we discussed above. In simple thermodynamics the saddle point approx is justified by the thermodynamic limit: the quantity in the exponent is extensive, so the saddle point is well-peaked. This part of the analogy will not always hold, and we will need to think about fluctuations about the saddle point.

Let's go back to (2.5) and think about its semiclassical expansion. If we were going to do this path integral by stationary phase, we would solve

$$0 = \frac{\delta}{\delta\phi(x)} \left(S[\phi] + \int \phi J \right) = \frac{\delta S}{\delta\phi(x)} + J(x) \ . \tag{2.10}$$

This determines some function ϕ which depends on J; let's denote it here as $\phi^{[J]}(x)$. In the semiclassical approximation to $Z[J] = e^{\mathbf{i}W[J]}$, we would just plug this back into the exponent of the integrand:

$$W_c[J] = \frac{1}{g^2\hbar} \left(S[\phi^{[J]}] + \int J\phi^{[J]} \right).$$

So in this approximation, (2.10) is exactly the equation determining ϕ_c. This is just the Legendre transformation of the original bare action $S[\phi]$ (I hope this manipulation is also familiar from stat mech, and I promise we're not going in circles).

Let's think about expanding $S[\phi]$ about such a saddle point $\phi^{[J]}$ (or more correctly, a point of stationary phase). The stationary phase (or semi-classical) expansion familiar from QM is an expansion in powers of $\hbar$ (WKB):

$$Z = e^{\mathbf{i}W/\hbar} = \int dx\, e^{\frac{\mathbf{i}}{\hbar}S(x)} = \int dx e^{\frac{\mathbf{i}}{\hbar}\left(S(x_0)+(x-x_0)\underbrace{S'(x_0)}_{=0}+\frac{1}{2}(x-x_0)^2S''(x_0)+\ldots\right)} = e^{\mathbf{i}W_0/\hbar+\mathbf{i}W_1+\mathbf{i}\hbar W_2+\ldots}$$

with $W_0 = S(x_0)$, and W_n comes from (the exponentiation of) diagrams involving n contractions of $\delta x = x - x_0$, each of which comes with a power of $\hbar$: $\langle \delta x \delta x \rangle \sim \hbar$.

Expansion in $\hbar$ = expansion in coupling. Is this semiclassical expansion the same as the expansion in powers of the coupling? Yes, if there is indeed a notion of "*the* coupling", *i.e.* only one for each field. Then by a rescaling of the fields we can put all the dependence on the coupling in front:

$$S = \frac{1}{g^2}s[\phi]$$

so that the path integral is

$$\int [D\phi]\; e^{\mathbf{i}\frac{s[\phi]}{\hbar g^2}+\int \phi J}.$$

(It may be necessary to rescale our sources J, too.) For example, suppose we are talking about a QFT of a single field $\tilde{\phi}$ with action

$$S[\tilde{\phi}] = \int \left(\left(\partial\tilde{\phi}\right)^2 - \lambda\tilde{\phi}^p \right).$$

Then define $\phi \equiv \tilde{\phi}\lambda^\alpha$ and choose $\alpha = \frac{1}{p-2}$ to get

$$S[\phi] = \frac{1}{\lambda^{\frac{2}{p-2}}}\int \left((\partial\phi)^2 - \phi^p \right) = \frac{1}{g^2}s[\phi].$$

with $g \equiv \lambda^{\frac{1}{p-2}}$, and $s[\phi]$ independent of g. Then the path-integrand is $e^{\frac{\mathbf{i}}{\hbar g^2}s[\phi]}$ and so g and $\hbar$ will appear only in the combination $g^2\hbar$. (If we have more than one coupling term, this direct connection must break down; instead we can scale out some overall factor from all the couplings and that appears with $\hbar$.)

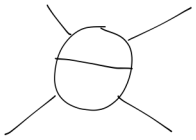
Loop expansion = expansion in coupling. Now I want to convince you that this is also the same as the loop expansion. The first correction in the semi-classical expansion comes from

$$S_2[\phi_0, \delta\phi] \equiv \frac{1}{g^2}\int dxdy\delta\phi(x)\delta\phi(y)\frac{\delta^2 s}{\delta\phi(x)\delta\phi(y)}|_{\phi=\phi_0}.$$

For the accounting of powers of g, it's useful to define $\Delta = g^{-1}\delta\phi$, so the action is

$$g^{-2}s[\phi] = g^{-2}s[\phi_0] + S_2[\Delta] + \sum_n g^{n-2}V_n[\Delta].$$

With this normalization, the power of the field Δ appearing in each term of the action is correlated with the power of g in that term. And the Δ propagator is independent of g.

So use the action $s[\phi]$, in an expansion about $\phi_\star$ to construct Feynman rules for correlators of Δ: the propagator is $\langle \mathcal{T}\Delta(x)\Delta(y)\rangle \propto g^0$, the 3-point vertex comes from V_3 and goes like $g^{3-2=1}$, and so on. Consider a diagram that contributes to an E-point function (of Δ) at order g^n, for example this contribution to the $(E = 4)$-point function at order $n = 6 \cdot (3-2) = 6$:  With our normalization of Δ, the powers of g come only from the vertices; a degree k vertex contributes $k-2$ powers of g; so the number of powers of g is

$$n = \sum_{\text{vertices, } i} (k_i - 2) = \sum_i k_i - 2V \tag{2.11}$$

where

V = # of vertices (This does not include external vertices.)
We also define:
n = # of powers of g
L = # of loops = #of independent internal momentum integrals
I = # of internal lines = # of internal propoagators
E = # of external lines

Facts about graphs:

- The total number of lines leaving all the vertices is equal to the total number of lines:

$$\sum_{\text{vertices, } i} k_i = E + 2I. \tag{2.12}$$

So the number of internal lines is

$$I = \frac{1}{2}\left(\sum_{\text{vertices, } i} k_i - E\right). \tag{2.13}$$

- For a connected graph, the number of loops is

$$L = I - V + 1 \tag{2.14}$$

since each loop is a sequence of internal lines interrupted by vertices. (This fact is probably best proved inductively. The generalization to graphs with multiple disconnected components is $L = I - V + C$.)

We conclude that[13]

$$L \overset{(2.14)}{=} I - V + 1 \overset{(2.13)}{=} \frac{1}{2}\left(\sum_i k_i - E\right) - V + 1 = \frac{n-E}{2} + 1 \overset{(2.11)}{=} \frac{n-E}{2} + 1.$$

This equation says:

$$L = \tfrac{n-E}{2} + 1: \quad \text{More powers of } g \text{ means (linearly) more loops.}$$

Diagrams with a fixed number of external lines and more loops are suppressed by more powers of g. (By rescaling the external field, it is possible to remove the dependence on E.)

We can summarize what we've learned by writing the sum of connected graphs as

$$W[J] = \sum_{L=0}^{\infty} \left(g^2\hbar\right)^{L-1} W_L$$

where W_L is the sum of connected graphs with L loops. In particular, the order-$\hbar^{-1}$ (classical) bit W_0 comes from *tree graphs*, graphs without loops. Solving the classical equations of motion sums up the tree diagrams.

Diagrammatic interpretation of Legendre transform. $\Gamma[\phi]$ is called the 1PI effective action[14]. And as its name suggests, Γ has a diagrammatic interpretation: it is the sum of just the 1PI connected diagrams. (Recall that $W[J]$ is the sum of *all* connected diagrams.) Consider the (functional) Taylor expansion Γ_n in ϕ

$$\Gamma[\phi] = \sum_n \frac{1}{n!} \int \Gamma_n(x_1 ... x_n)\phi(x_1)...\phi(x_n) d^D x_1 \cdots d^D x_n \ .$$

The coefficients Γ_n are called 1PI Green's functions (we will justify this name presently). To get the full connected Green's functions, we sum all tree diagrams with the 1PI Green's functions as vertices, using the full connected two-point function as the propagators.

[13] You should check that these relations are all true for some random example, like the one above, which has $I = 7, L = 2, \sum k_i = 18, V = 6, E = 4$. You will notice that Banks has several typos in his discussion of this in §3.4. His Es should be $E/2$s in the equations after (3.31).

[14] The 1PI effective action Γ must be distinguished from the S_{eff} that appeared in our second parable in §1.2 and the *Wilsonian effective action* which we will encounter later – the difference is that here we integrated over everybody, whereas the Wilsonian action integrates only high-energy modes. The different effective actions correspond to different choices about what we care about and what we don't, and hence different choices of what modes to integrate out.

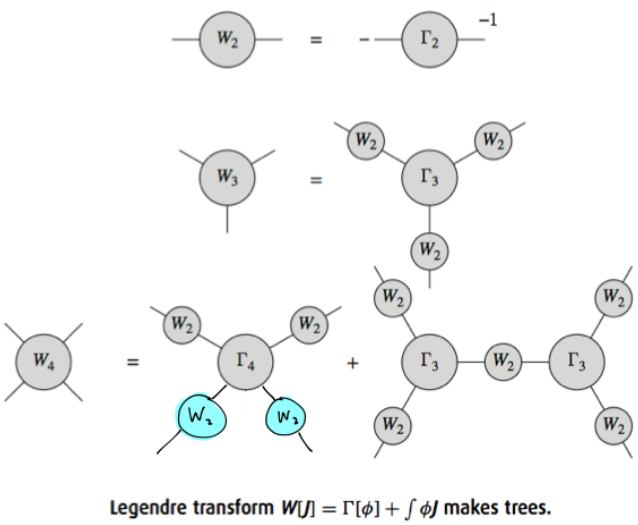


Figure 6: [From Banks, *Modern Quantum Field Theory*, slightly improved] W_n denotes the connected n-point function, $\left(\frac{\partial}{\partial J}\right)^n W[J] = \langle \phi^n \rangle$.

Perhaps the simplest way to arrive at this result is to consider what happens if we try to use Γ as the action in the path integral instead of S.

$$Z_{\Gamma,\hbar}[J] \equiv \int [D\phi] e^{\frac{\mathrm{i}}{\hbar}\left(\Gamma[\phi] + \int J\phi\right)}$$

By the preceding arguments, the expansion of $\log Z_\Gamma[J]$ in powers of $\hbar$, in the limit $\hbar \to 0$ is

$$\lim_{\hbar\to 0} \log Z_{\Gamma,\hbar}[J] = \sum_L \left(g^2\hbar\right)^{L-1} W_L^\Gamma \ .$$

The leading, tree level term in the $\hbar$ expansion, is obtained by solving

$$\frac{\delta\Gamma}{\delta\phi(x)} = -J(x)$$

and plugging the solution into Γ; the result is

$$\left(\Gamma[\phi] + \int \phi J\right)_{\frac{\partial\Gamma}{\partial\phi(x)} = -J(x)} \overset{\text{inverse Legendre transf}}{\equiv} W[J].$$

This expression is the definition of the inverse Legendre transform, and we see that it gives back $W[J]$: the generating functional of connected correlators! On the other hand, the counting of powers above indicates that the only terms that survive the $\hbar \to 0$ limit are tree diagrams where we use the terms in the Taylor expansion of $\Gamma[\phi]$ as the vertices. This is exactly the statement we were trying to demonstrate: the sum of all connected diagrams is the sum of tree diagrams made using 1PI vertices and the exact propagator (by definition of 1PI). Therefore Γ_n are the 1PI vertices.

[End of Lecture 5]

For a more arduous but more direct proof of this statement, see the problem set and/or Banks §3.5. There is an important typo on page 29 of Banks' book; it should say:

$$\frac{\delta^2 W}{\delta J(x)\delta J(y)} = \frac{\delta\phi(y)}{\delta J(x)} = \left(\frac{\delta J(x)}{\delta\phi(y)}\right)^{-1} \overset{(2.9)}{=} -\left(\frac{\delta^2\Gamma}{\delta\phi(x)\delta\phi(y)}\right)^{-1}. \tag{2.15}$$

(where $\phi \equiv \phi_c$ here). You can prove this from the definitions above. Inverse here means in the sense of integral operators: $\int d^D z K(x,z)K^{-1}(z,y) = \delta^D(x-y)$. So we can write the preceding result more compactly as:

$$W_2 = -\Gamma_2^{-1}.$$

Here's a way to think about why we get an inverse here: the 1PI blob is defined by removing the external propagators; but these external propagators are each W_2; removing two of them from one of them leaves -1 of them. You're on your own for the sign.

The idea to show the general case in Fig. 6 is to just compute W_n by taking the derivatives starting from (2.15): Differentiate again wrt J and use the matrix differentiation formula $dK^{-1} = -K^{-1}dKK^{-1}$ and the chain rule to get

$$W_3(x,y,z) = \int dw_1 \int dw_2 \int dw_3 W_2(x,w_1)W_2(y,w_2)W_2(z,w_3)\Gamma_3(w_1,w_2,w_3)\ .$$

To get the rest of the W_n requires an induction step.

This business is useful in at least two ways. First it lets us focus our attention on a much smaller collection of diagrams when we are doing our perturbative renormalization.

Secondly, this notion of effective action is extremely useful in thinking about the vacuum structure of field theories, and about spontaneous symmetry breaking. In particular, we can expand the functional in the form

$$\Gamma[\phi_c] = \int d^D x \left(-V_{\text{eff}}(\phi_c) + Z(\phi_c)\left(\partial\phi_c\right)^2 + ...\right)$$

(where the ... indicate terms with more derivatives of ϕ). In particular, in the case where ϕ_c is constant in spacetime we can minimize the *function* $V_{\text{eff}}(\phi_c)$ to find the vacuum. We will revisit this below (in §2.3).

(Finally this is the end of our discussion of the third organizing fact about diagrammatic expansions.)

LSZ

Here is a third useful formal conclusion we can draw from the above discussion. Suppose that we know that our quantum field ϕ can create a (stable) single-particle state from the vacuum with finite probability (this will not always be true). In equations, this says:

$$0 \neq \langle \vec{p} | \hat{\phi}(0) | \text{ground state} \rangle, \quad |\vec{p}\rangle \text{ is a 1-particle state with momentum } \vec{p} \text{ and energy } \omega_{\vec{p}}.$$

We will show below (in §2.4) that under this assumption, the exact propagator $W_2(p)$ has a pole at $p^2 = m^2$, where m is the mass of the particle (here I'm assuming Lorentz invariance). But then the expansion above shows that every W_n has such a pole on each external leg (as a function of the associated momentum through that leg)! The residue of this pole is (with some normalization) the S-matrix element for scattering those n particles. This statement is the LSZ formula. If provoked I will say more about it, but I would like to focus on observables other than the scattering matrix. The demonstration involves only bookkeeping (we would need to define the S-matrix).

2.3 Coleman-Weinberg(-Stone-Dasgupta-Ma-Halperin) potential

[Zee §IV.3, Xi Yin's notes §4.2]

Let us now take seriously the lack of indices on our field ϕ, and see about actually evaluating more of the semiclassical expansion of the path integral of a scalar field (eventually we will specify $D = 3 + 1$):

$$Z[J] = e^{\frac{\mathbf{i}}{\hbar}W[J]} = \int [D\phi] e^{\frac{\mathbf{i}}{\hbar}\left(S[\phi] + \int J\phi\right)} \ . \tag{2.16}$$

To add some drama to this discussion consider the following: if the potential V in $S = \int \left(\frac{1}{2}(\partial\phi)^2 - V(\phi)\right)$ has a *minimum* at the origin, then we expect that the vacuum has $\langle\phi\rangle = 0$. If on the other hand, the potential has a *maximum* at the origin, then the field will find a minimum somewhere else, $\langle\phi\rangle \neq 0$. If the potential has a discrete symmetry under $\phi \to -\phi$ (no odd powers of ϕ in V), then in the latter case ($V''(0) < 0$) this symmetry will be broken. If the potential is flat ($V''(0) = 0$)near the origin, what happens? Quantum effects matter.

The configuration of stationary phase is $\phi = \phi_\star$, which satisfies

$$0 = \frac{\delta\left(S + \int J\phi\right)}{\delta\phi(x)}|_{\phi=\phi_\star} = -\partial^2\phi_\star(x) - V'(\phi_\star(x)) + J(x) \ . \tag{2.17}$$

Change the integration variable in (2.16) to $\phi = \phi_\star + \varphi$, and expand in powers of the fluctuation φ:

$$\begin{aligned}
Z[J] &= e^{\frac{\mathbf{i}}{\hbar}\left(S[\phi_\star] + \int J\phi_\star\right)} \int [D\varphi] e^{\frac{\mathbf{i}}{\hbar}\int d^D x \frac{1}{2}\left((\partial\varphi)^2 - V''(\phi_\star)\varphi^2 + \mathcal{O}(\varphi^3)\right)} \\
&\overset{\text{IBP}}{=} e^{\frac{\mathbf{i}}{\hbar}\left(S[\phi_\star] + \int J\phi_\star\right)} \int [D\varphi] e^{-\frac{\mathbf{i}}{\hbar}\int d^D x \frac{1}{2}\left(\varphi\left(\partial^2 + V''(\phi_\star)\right)\varphi + \mathcal{O}(\varphi^3)\right)} \\
&\approx e^{\frac{\mathbf{i}}{\hbar}\left(S[\phi_\star] + \int J\phi_\star\right)} \frac{1}{\sqrt{\det\left(\partial^2 + V''(\phi_\star)\right)}} \\
&= e^{\frac{\mathbf{i}}{\hbar}\left(S[\phi_\star] + \int J\phi_\star\right)} e^{-\frac{1}{2}\mathrm{tr}\log\left(\partial^2 + V''(\phi_\star)\right)} .
\end{aligned}$$

In the second line, we integrated by parts to get the φ integral to look like a souped-up version of the gaussian integral from Problem Set 01 – just think of $\partial^2 + V''$ as a big matrix – and in the third line, we did that integral. In the last line we used the matrix identity $\mathrm{tr}\log = \log\det$. Note that all the $\phi_\star$s appearing in this expression are functionals of J, determined by (2.17).

So taking logs of the BHS of the previous equation we have the generating functional:

$$W[J] = S[\phi_\star] + \int J\phi_\star + \frac{\mathbf{i}\hbar}{2}\mathrm{tr}\log\left(\partial^2 + V''(\phi_\star)\right) + \mathcal{O}(\hbar^2) \ .$$

To find the effective potential, we need to Legendre transform to get a functional of ϕ_c:

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} \overset{\text{chain rule}}{=} \int d^D z \frac{\delta\left(S[\phi_\star] + \int J\phi_\star\right)}{\delta\phi_\star(z)} \frac{\delta\phi_\star(z)}{\delta J(x)} + \phi_\star(x) + \mathcal{O}(\hbar) \overset{(2.17)}{=} \phi_\star(x) + \mathcal{O}(\hbar) \quad .$$

The 1PI effective action is then:

$$\Gamma[\phi_c] \equiv W - \int J\phi_c = S[\phi_c] + \frac{\mathbf{i}\hbar}{2}\mathrm{tr}\log\left(\partial^2 + V''(\phi_c)\right) + \mathcal{O}(\hbar^2).$$

To leading order in $\hbar$, we just plug in the solution; to next order we need to compute the sum of the logs of the eigenvalues of a differential operator. This is challenging in general. In the special case that we are interested in ϕ_c which is constant in spacetime, it is doable. This case is also often physically relevant if our goal is to solve (2.8) to find the groundstate, which often preserves translation invariance (gradients cost energy). If $\phi_c(x) = \phi$ is spacetime-independent then we can write

$$\Gamma[\phi_c(x) = \phi] \equiv \int d^D x \; V_{\text{eff}}(\phi).$$

The computation of the trace-log is doable in this case because it is translation invariant, and hence we can use fourier space. We do this next.

2.3.1 The one-loop effective potential

The tr in the one-loop contribution is a trace over the space on which the differential operator (≡big matrix) acts; it acts on the space of scalar fields φ:

$$\left(\left(\partial^2 + V''(\phi)\right)\varphi\right)_x = \sum_y \left(\partial^2 + V''(\phi)\right)_{xy} \varphi_y \equiv \left(\partial_x^2 + V''(\phi)\right)\varphi(x)$$

with matrix element $\left(\partial^2 + V''\right)_{xy} = \delta^D(x-y)\left(\partial_x^2 + V''\right)$. (Note that in these expressions, we've assumed ϕ is a background field, not the same as the fluctuation φ – this operator is linear. Further we've assumed that that background field ϕ is a constant, which greatly simplifies the problem.) The trace can be represented as a position integral:

$$\mathrm{tr}\bullet = \int d^D x \langle x| \bullet |x\rangle$$

so

$$\begin{aligned}\mathrm{tr}\log\left(\partial^2 + V''(\phi)\right) &= \int d^D x \langle x| \log\left(\partial^2 + V''\right)|x\rangle \\ &= \int d^D x \int \mathrm{đ}^D k \int \mathrm{đ}^D k' \langle x|k'\rangle\langle k'|\log\left(\partial^2 + V''\right)|k\rangle\langle k|x\rangle \qquad \left(\mathbb{1} = \int \mathrm{đ}^D k |k\rangle\langle k|\right)\end{aligned}$$

$$= \int d^D x \int \text{đ}^D k \int \text{đ}^D k' \langle x|k'\rangle\langle k'| \log\left(-k^2 + V''\right)|k\rangle\langle k|x\rangle$$

$$(\langle k'| \log\left(-k^2 + V''\right)|k\rangle = \delta^D(k-k') \log\left(-k^2 + V''\right))$$

$$= \int d^D x \int \text{đ}^D k \log\left(-k^2 + V''\right), \qquad (\| \langle x|k\rangle \|^2 = 1)$$

The $\int d^D x$ goes along for the ride and we conclude that

$$V_{\text{eff}}(\phi) = V(\phi) - \frac{\mathbf{i}\hbar}{2} \int \text{đ}^D k \log\left(k^2 - V''(\phi)\right) + \mathcal{O}(\hbar^2).$$

What does it mean to take the log of a dimensionful thing? It means we haven't been careful about the additive constant (constant means independent of ϕ). And we don't need to be (unless we're worried about dynamical gravity); so let's choose the constant so that

$$V_{\text{eff}}(\phi) = V(\phi) - \frac{\mathbf{i}\hbar}{2} \int \text{đ}^D k \log\left(\frac{k^2 - V''(\phi)}{k^2}\right) + \mathcal{O}(\hbar^2). \tag{2.18}$$

$\boxed{V_{\mathbf{1\ loop}} = \sum_{\vec{k}} \frac{1}{2}\hbar\omega_{\vec{k}}}$. Here's the interpretation of the 1-loop potential: $V''(\phi)$ is the mass^2 of the field when it has the constant value ϕ; the one-loop term $V_{1\text{ loop}}$ is the vacuum energy $\int d^{D-1}\vec{k}\frac{1}{2}\hbar\omega_{\vec{k}}$ from the gaussian fluctuations of a field with that mass^2; it depends on the field because the mass depends on the field.

[Zee II.5.3] Why is $V_{1\text{ loop}}$ the vacuum energy? Recall that $k^2 \equiv \omega^2 - \vec{k}^2$ and $\text{đ}^D k = \text{đ}\omega \text{đ}^{D-1}\vec{k}$. Consider the integrand of the spatial momentum integrals: $V_{1\text{ loop}} = -\mathbf{i}\frac{\hbar}{2}\int \text{đ}^{D-1}\vec{k}\mathcal{I}$, with

$$\mathcal{I} \equiv \int \text{đ}\omega \log\left(\frac{k^2 - V''(\phi) + \mathbf{i}\epsilon}{k^2 + \mathbf{i}\epsilon}\right) = \int \text{đ}\omega \log\left(\frac{\omega^2 - \omega_k^2 + \mathbf{i}\epsilon}{\omega^2 - \omega_{k'}^2 + \mathbf{i}\epsilon}\right)$$

with $\omega_k = \sqrt{\vec{k}^2 + V''(\phi)}$, and $\omega_{k'} = |\vec{k}|$. The $\mathbf{i}\epsilon$ prescription is as usual inherited from the euclidean path integral. Notice that the integral is convergent – at large ω, the integrand goes like

$$\log\left(\frac{\omega^2 - A}{\omega^2 - B}\right) = \log\left(\frac{1 - \frac{A}{\omega^2}}{1 - \frac{B}{\omega^2}}\right) = \log\left(1 - \frac{A-B}{\omega^2} + \mathcal{O}\left(\frac{1}{\omega^4}\right)\right) \simeq \frac{A-B}{\omega^2}\ .$$

Integrate by parts:

$$\mathcal{I} = \int \text{đ}\omega \log\left(\frac{k^2 - V''(\phi) + \mathbf{i}\epsilon}{k^2 + \mathbf{i}\epsilon}\right) = -\int \text{đ}\omega\omega\partial_\omega \log\left(\frac{\omega^2 - \omega_k^2}{\omega - \omega_{k'}}\right)$$

$$= -2\int \text{đ}\omega\omega \left(\frac{\omega}{\omega^2 - \omega_k^2 + \mathbf{i}\epsilon} - (\omega_k \to \omega_{k'}) \right)$$
$$= -\mathbf{i}2\omega_k^2 \left(\frac{1}{-2\omega_k} \right) - (\omega_k \to \omega_{k'}) = \mathbf{i}\left(\omega_k - \omega_{k'}\right).$$

This is what we are summing (times $-\mathbf{i}\frac{1}{2}\hbar$) over all the modes $\int \text{đ}^{D-1}\vec{k}$.

2.3.2 Renormalization of the effective action

So we have a cute expression for the effective potential (2.18). Unfortunately it seems to be equal to infinity. The problem, as usual, is that we assumed that the parameters in the bare action $S[\phi]$ could be finite without introducing any cutoff. Let us parametrize (following Zee §IV.3) the action as $S = \int d^D x \mathcal{L}$ with

$$\mathcal{L} = \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4!}\lambda\phi^4 - A\left(\partial\phi\right)^2 - B\phi^2 - C\phi^4$$

and we will think of A, B, C as *counterterms*, in which to absorb the cutoff dependence.

So our effective potential is actually:

$$V_{\text{eff}}(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + B(\Lambda)\phi^2 + C(\Lambda)\phi^4 + \frac{\hbar}{2}\int^{\Lambda} \text{đ}^D k_E \log\left(\frac{k_E^2 + V''(\phi)}{k_E^2}\right) ,$$

(notice that A drops out in this special case with constant ϕ). We rotated the integration contour to euclidean space. This permits a nice regulator, which is just to limit the integration region to $\{k_E | k_E^2 \leq \Lambda^2\}$ for some big (Euclidean) wavenumber Λ.

Now let us specify to the case of $D = 4$, where the model with $\mu = 0$ is classically scale invariant. The integrals are elementary[15]

$$V_{\text{eff}}(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + B(\Lambda)\phi^2 + C(\Lambda)\phi^4 + \frac{\Lambda^2}{32\pi^2}V''(\phi) - \frac{\left(V''(\phi)\right)^2}{64\pi^2}\log\frac{\sqrt{e}\Lambda^2}{V''(\phi)}.$$

Notice that the leading cutoff dependence of the integral is Λ^2, and there is also a subleading logarithmically-cutoff-dependent term. ("log divergence" is certainly easier to say.)

Luckily we have two counterterms. Consider the case where V is a quartic polynomial; then V'' is quadratic, and $(V'')^2$ is quartic. In that case the two counterterms are in just the right form to absorb the Λ dependence. On the other hand, if V were sextic (recall that this is in the non-renormalizable category according to our dimensional analysis), we would

[15] This is not the same as 'easy'. The expressions here assume that $\Lambda \gg V''$.

have a fourth counterterm $D\phi^6$, but in this case $(V'')^2 \sim \phi^8$, and we're in trouble (adding a bare ϕ^8 term would produce $(V'')^2 \sim \phi^{12}$... and so on). We'll need a better way to think about such non-renormalizable theories. The better way (which we will return to in the next section) is simply to recognize that in non-renormalizable theories, *the cutoff is real* – it is part of the definition of the field theory. In renormalizable theories, we may pretend that it is not (though it usually is real there, too).

Renormalization conditions. Return to the renormalizable case, $V = \lambda\phi^4$ where we've found

$$V_{\rm eff} = \phi^2\left(\frac{1}{2}\mu^2 + B + \lambda\frac{\Lambda^2}{64\pi^2}\right) + \phi^4\left(\frac{1}{4!}\lambda + C + \frac{\Lambda^2}{16\pi^2}\log\frac{\phi^2}{\Lambda^2}\right) + \mathcal{O}(\lambda^3)\ .$$

(I've absorbed an additive $\log\sqrt{e}$ in C.)The counting of counterterms works out, but how do we determine them? We need to impose *renormalization conditions*; this is a fancy name for the should-be-obvious step of specifying some observable quantities to parametrize our model, in terms of which we can eliminate the silly letters in the lagrangian. We need two of these. Of course, what is observable depends on the physical system at hand. Let's suppose that we can measure some properties of the effective potential. For example, suppose we can measure the mass^2 when $\phi = 0$:

$$\mu^2 = \frac{\partial^2 V_{\rm eff}}{\partial\phi^2}|_{\phi=0} \qquad \Longrightarrow \text{ we should set } B = -\lambda\frac{\Lambda^2}{64\pi^2}.$$

For example, we could consider the case $\mu = 0$, when the potential is flat at the origin. With $\mu = 0$, have

$$V_{\rm eff}(\phi) = \left(\frac{1}{4!}\lambda + \frac{\lambda^2}{(16\pi)^2}\log\frac{\phi^2}{\Lambda^2} + C(\Lambda)\right)\phi^4 + \mathcal{O}(\lambda^3)\ .$$

And for the second renormalization condition, suppose we can measure the quartic term

$$\lambda_M = \frac{\partial^4 V_{\rm eff}}{\partial\phi^4}|_{\phi=M}\ . \tag{2.19}$$

Here M is some arbitrarily chosen quantity with dimensions of mass. We run into trouble if we try to set it to zero because of $\partial_\phi^4\left(\phi^4\log\phi\right) \sim \log\phi$. So the coupling depends very explicitly on the value of M at which we set the renormalization condition. Let's use (2.19) to eliminate C:

$$\lambda(M) \stackrel{!}{=} 4!\left(\frac{\lambda}{4!} + C + \left(\frac{\lambda}{16\pi}\right)^2\left(\log\frac{\phi^2}{\Lambda^2} + c_1\right)\right)|_{\phi=M} \tag{2.20}$$

(where c_1 is a numerical constant that you should determine) to get

$$V_{\rm eff}(\phi) = \frac{1}{4!}\lambda(M)\phi^4 + \left(\frac{\lambda(M)}{16\pi}\right)^2\left(\log\frac{\phi^2}{M^2} - c_1\right)\phi^4 + \mathcal{O}(\lambda(M)^3).$$

Here I used the fact that we are only accurate to $\mathcal{O}(\lambda^2)$ to replace $\lambda = \lambda(M) + \mathcal{O}(\lambda(M)^2)$ in various places. We can feel a sense of victory here: the dependence on the cutoff has disappeared. Further, the answer for V_{eff} does not depend on our renormalization point M:

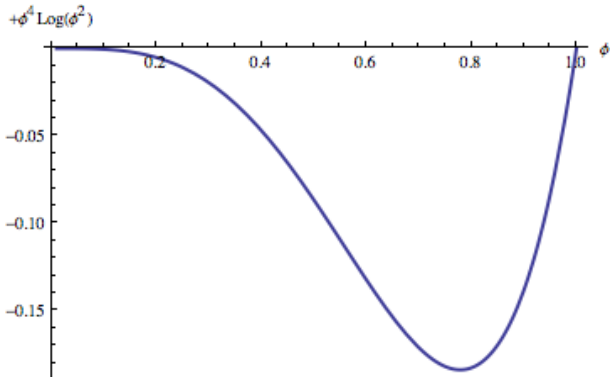
$$M\frac{d}{dM}V_{\text{eff}} = \frac{1}{4!}\phi^4\left(M\partial_M\lambda - \frac{2}{M}\frac{\lambda^2}{(16\pi^2)} + \mathcal{O}(\lambda^3)\right) = \mathcal{O}(\lambda^3) \tag{2.21}$$

which vanishes to this order from the definition of $\lambda(M)$ (2.20), which implies

$$M\partial_M\lambda(M) = \frac{3}{16\pi^2}\lambda(M)^2 + \mathcal{O}(\lambda^3).$$

The fact (2.21) is sometimes called the *Callan-Symanzik equation*, the condition that $\lambda(M)$ must satisfy in order that physics be independent of our choice of renormalization point M.

So: when $\mu = 0$ is the $\phi \to -\phi$ symmetry broken by the groundstate?



The effective potential looks like this for $\phi < M$:
Certainly it looks like this will push the field away from the origin. However, the minima lie in a region where our approximations aren't so great. In particular, the next correction looks like:

$$\lambda\phi^4\left(1 + \lambda\log\phi^2 + \left(\lambda\log\phi^2\right)^2 + \ldots\right)$$

– the expansion parameter is really $\lambda\log\phi$. (I haven't shown this yet, it is an application of the RG, below.) The apparent minimum lies in a regime where the higher powers of $\lambda\log\phi$ are just as important as the one we've kept.

Later I will comment on some physical realizations of this business.

We can get around this issue by studying a system where the fluctuations producing the extra terms in the potential for ϕ come from some *other* field whose mass depends on ϕ.

For example, consider a fermion field whose mass depends on ϕ:

$$S[\psi,\phi] = \int d^Dx\bar{\psi}\left(\mathbf{i}\partial\!\!\!/ - m - g\phi\right)\psi$$

– then $m_\psi = m + g\phi$. The $\sum \frac{1}{2}\hbar\omega$s from the fermion will now depend on ϕ, and we get a reliable answer for $\langle\phi\rangle \neq 0$ from this phenomenon of *radiative symmetry breaking*.

[End of Lecture 6]

2.3.3 Useful properties of the effective action

[For a version of this discussion which is better in just about every way, see Coleman, *Aspects of Symmetry* §5.3.7. I also highly recommend all the preceding sections! And the ones that come after. This book is available electronically from the UCSD library.]

V_{eff} as minimum energy with fixed ϕ. Recall that $\langle\phi\rangle$ is the configuration of ϕ_c which extremizes the effective action $\Gamma[\phi_c]$. Even away from its minimum, the effective potential has a useful physical interpretation. It is the natural extension of the interpretation of the potential in classical field theory, which is: $V(\phi)$ = the value of the energy density if you fix the field equal to ϕ everywhere. Consider the space of states of the QFT where the field has a given expectation value:

$$|\Omega\rangle \quad \text{such that} \quad \langle\Omega|\phi(x)|\Omega\rangle = \phi_0(x) \quad ; \tag{2.22}$$

one of them has the smallest energy. I claim that its energy is $V_{\text{eff}}(\phi_0)$. This fact, which we'll show next, has some useful consequences.

Let $|\Omega_{\phi_0}\rangle$ be the (normalized) state of the QFT which minimizes the energy subject to the constraint (2.22). The familiar way to do this (familiar from QM, associated with Rayleigh and Ritz)[16] is to introduce Lagrange multipliers to impose (2.22) and the normalization condition and extremize without constraints the functional

$$\langle\Omega|\mathbf{H}|\Omega\rangle - \alpha\left(\langle\Omega|\Omega\rangle - 1\right) - \int d^{D-1}\vec{x}\beta(\vec{x})\left(\langle\Omega|\phi(\vec{x},t)|\Omega\rangle - \phi_0(\vec{x})\right)$$

with respect to $|\Omega\rangle$ and the functions on space α, β. [17]

[16] The more familiar thing is to find the state which extremizes $\langle a|\mathbf{H}|a\rangle$ subject to the normalization condition $\langle a|a\rangle = 1$. To do this, we vary $\langle a|\mathbf{H}|a\rangle - E\left(\langle a|a\rangle - 1\right)$ with respect to both $|a\rangle$ and the Lagrange multiplier E. The equation from varying $|a\rangle$ says that the extremum occurs when $(\mathbf{H} - E)|a\rangle = 0$, *i.e.* $|a\rangle$ is an energy eigenstate with energy E. Notice that we could just as well have varied the simpler thing

$$\langle a|\left(\mathbf{H} - E\right)|a\rangle$$

and found the same answer.

[17] Here is the QM version (*i.e.* the same thing without all the labels): we want to find the extremum of $\langle a|\mathbf{H}|a\rangle$ with $|a\rangle$ normalized *and* $\langle a|\mathbf{A}|a\rangle = A_c$ some fixed number. Then we introduce two Lagrange

Clearly the extremum with respect to α, β imposes the desired constraints. Extremizing with respect to $|\Omega\rangle$ gives:

$$\mathbf{H}|\Omega\rangle = \alpha|\Omega\rangle + \int d^{D-1}\vec{x}\beta(\vec{x})\phi(\vec{x},t)|\Omega\rangle \tag{2.23}$$

or

$$\left(\mathbf{H} - \int d^{D-1}\vec{x}\beta(\vec{x})\phi(\vec{x},t)\right)|\Omega\rangle = \alpha|\Omega\rangle \tag{2.24}$$

Note that α, β are functionals of ϕ_0. We can interpret the operator $\mathbf{H}_\beta \equiv \mathbf{H} - \int d^{D-1}\vec{x}\beta(\vec{x})\phi(\vec{x},t)$ on the LHS of (2.24) as the hamiltonian with a source β; and α is the groundstate energy in the presence of that source. (Note that that source is *chosen* so that $\langle\phi\rangle = \phi_0$ – it is a functional of ϕ_0.)

This groundstate energy is related to the generating functional $W[J=\beta]$ as we've seen several times – $e^{\mathbf{i}W[\beta]}$ is the vacuum persistence amplitude in the presence of the source

$$e^{\mathbf{i}W[\beta]} = \langle 0|\mathcal{T}e^{\mathbf{i}\int\beta\phi}|0\rangle = \langle 0_\beta|e^{-\mathbf{i}T\mathbf{H}_\beta}|0_\beta\rangle = e^{-\mathbf{i}\alpha T} \tag{2.25}$$

where T is the time duration. (If you want, you could imagine that we are adiabatically turning on the interactions for a time duration T.)

The actual extremal energy (of the unperturbed hamiltonian, with constrained expectation value of ϕ) is obtained by taking the overlap of (2.23) with $\langle\Omega|$ (really all the Ωs below are Ω_{ϕ_0}s):

$$\begin{aligned}\langle\Omega|\mathbf{H}|\Omega\rangle &= \alpha\langle\Omega|\Omega\rangle + \int d^{D-1}\vec{x}\beta(\vec{x})\langle\Omega|\phi(\vec{x},t)|\Omega\rangle \\ &= \alpha + \int d^{D-1}\vec{x}\beta(\vec{x})\phi_0(\vec{x})\end{aligned}$$

---

multipliers E, J and vary without constraint the quantity

$$\langle a|\left(\mathbf{H} - E - J\mathbf{A}\right)|a\rangle$$

(plus irrelevant constants). The solution satisfies

$$\left(\mathbf{H} - E - J\mathbf{A}\right)|a\rangle = 0$$

so $|a\rangle$ is an eigenstate of the perturbed hamiltonian $\mathbf{H} - J\mathbf{A}$, with energy E. J is an auxiliary thing, which really depends on our choice A_c, via

$$A_c = \langle a|\mathbf{A}|a\rangle = -\frac{dE}{dJ}.$$

(If you like, we used the Feynman-Hellmann theorem, $\frac{dE}{dJ} = \langle\frac{d\mathbf{H}}{dJ}\rangle$.) The quantity we extremized is

$$\langle a|\mathbf{H}|a\rangle = E + JA_c = E - J\frac{dE}{dJ}\ .$$

This Legendre transform is exactly (the QM analog of) the effective potential.

$$\overset{(2.25)}{=} \frac{1}{T}\left(-W[\beta] + \int d^D x \beta(\vec{x})\phi_0(\vec{x})\right)$$
$$\overset{\text{Legendre}}{=} -\frac{1}{T}\Gamma[\phi_0] \overset{\phi=\phi_0,\text{ const}}{=} \int d^{D-1}\vec{x} V_{\text{eff}}(\phi_0).$$

Cluster decomposition. The relationship (2.25) between the generating functional $W[J]$ (for time-independent J) and the energy in the presence of the source is very useful. (You've already used it on problem set 2 to compute the potential between static sources.) Notice that it gives an independent proof that W only gets contributions from connected amplitudes. Amplitudes with n connected components, $\underbrace{\langle....\rangle\langle...\rangle\langle...\rangle}_{n \text{ of these}}$, go like T^n (where T is the time duration) at large T. Since $W = -E_J T$ goes like T^1, we conclude that it has one connected component (terms that went like $T^{n>1}$ would dominate at large T and therefore must be absent). This *extensivity* of W in T is of the same nature as the extensivity in volume of the free energy in thermodynamics.

[Brown, 6.4.2] Another important reason why W must be connected is called the *cluster decomposition property*. Consider a source which has the form $J(x) = J_1(x) + J_2(x)$ where the two parts have support in widely-separated (spacelike separated) spacetime regions. If all the fields are massive, 'widely-separated' means precisely that the distance between the regions is $R \gg 1/m$, much larger than the range of the interactions mediated by ϕ. In this case, measurements made in region 1 cannot have any effect on those in region 2, and they should be uncorrelated. If so, the probability amplitude factorizes

$$Z[J_1 + J_2] = Z[J_1]Z[J_2]$$

which by the magic of logs is the same as

$$W[J_1 + J_2] = W[J_1] + W[J_2].$$

If W were not connected, it would not have this additive property.

There are actually some exceptions to cluster decomposition arising from situations where we prepare an initial state (it could be the groundstate for some hamiltonian) in which there are correlations between the excitations in the widely separated regions. Such a thing happens in situations with spontaneous symmetry breaking, where the value of the field is the same everywhere in space, and therefore correlates distant regions.

Convexity of the effective potential. Another important property of the effective potential is $V''_{\text{eff}}(\phi) > 0$ – the effective potential is *convex* (sometimes called 'concave up'). We can see this directly from our previous work. Most simply, recall that the functional Taylor coefficients of $\Gamma[\phi]$ are the 1PI Green's functions; V_{eff} is just Γ evaluated for constant ϕ, *i.e.* zero momentum; therefore the Taylor coefficients of V_{eff} are the 1PI Green's functions

at zero momentum. In particular, $V''_{\text{eff}}(\phi) = \langle \phi_{k=0}\phi_{k=0}\rangle$: the ground state expectation value of the square of a hermitian operator, which is positive.[18] [19]

On the other hand, it seems that if $V(\phi)$ has a maximum, or even any region of field space where $V''(\phi) < 0$, we get a *complex* one-loop effective potential (from the log of a negative V''). What gives? One resolution is that in this case the minimum energy state with fixed $\langle\phi\rangle$ is not a ϕ eigenstate.

For example, consider a quartic potential $\frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4$ with $m^2 < 0$, with minima at $\phi_\pm \equiv \pm\sqrt{\frac{6|m|^2}{g}}$. Then for $\langle\phi\rangle \in (\phi_-, \phi_+)$, rather we can lower the energy below $V(\phi)$ by considering a state

$$|\Omega\rangle = c_+|\Omega_+\rangle + c_-|\Omega_-\rangle, \quad \langle\Omega|\phi|\Omega\rangle = |c_+|^2\phi_+ + |c_-|^2\phi_-.$$

The one-loop effective potential at ϕ only knows about some infinitesimal neighborhood of the field space near ϕ, and fails to see this non-perturbative stuff. In fact, the correct effective potential is exactly flat in between the two minima. More generally, if the two minima have unequal energies, we have

$$V_{\text{eff}} = \langle\Omega|\mathbf{H}|\Omega\rangle = |c_+|^2V(\phi_+) + |c_-|^2V(\phi_-)$$

– the potential interpolates *linearly* between the energies of the two surrounding minima.

The imaginary part of $V_{\text{1 loop}}$ is a decay rate. If we find that the (perturbative approximation to) effective potential $E \equiv V_{1\text{ loop}}$ is complex, it means that the amplitude for our state to persist is not just a phase:

$$\mathcal{A} \equiv \langle 0|e^{-\mathbf{i}T\mathbf{H}}|0\rangle = e^{-\mathbf{i}E\mathcal{V}T}$$

[18] More explicitly: Begin from $V_{\text{eff}} = -\frac{\Gamma}{\mathcal{V}}$.

$$\frac{\partial}{\partial\phi_0}V_{\text{eff}}(\phi_0) = -\int\frac{d^Dx}{\mathcal{V}}\frac{\delta}{\delta\phi(x)}\frac{\Gamma[\phi]}{\mathcal{V}}|_{\phi(x)=\phi_0} \overset{(2.7)}{=} -\frac{1}{\mathcal{V}}\int\frac{d^Dx}{\mathcal{V}}\left(-J(x)\right)|_{\phi(x)=\phi_0}.$$

In the first expression here, we are averaging over space the functional derivative of Γ. The second derivative is then

$$\left(\frac{\partial}{\partial\phi_0}\right)^2 V_{\text{eff}}(\phi_0) = \frac{1}{\mathcal{V}}\int\frac{d^Dy}{\mathcal{V}}\frac{\delta}{\delta\phi(y)}\int\frac{d^Dx}{\mathcal{V}}\left(J(x)\right)|_{\phi(x)=\phi_0} = +\frac{1}{\mathcal{V}^3}\int_y\int_x\frac{\delta J(x)}{\delta\phi(y)}|_{\phi(x)=\phi_0}$$

Using (2.15), this is

$$V''_{\text{eff}} = +\frac{1}{\mathcal{V}^3}\int_y\int_x\left(W_2^{-1}\right)_{xy}$$

– the inverse is in a matrix sense, with x, y as matrix indices. But W_2 is a positive operator – it is the groundstate expectation value of the square of a hermitian operator.

[19] In fact, the whole effective action $\Gamma[\phi]$ is a convex functional: $\frac{\delta^2\Gamma}{\delta\phi(x)\delta\phi(y)}$ is a positive integral operator. For more on this, I recommend Brown, *Quantum Field Theory*, Chapter 6.

has a modulus different from one ($\mathcal{V}$ is the volume of space). Notice that the $|0\rangle$ here is our perturbative approximation to the groundstate of the system, which is *wrong* in the region of field space where $V'' < 0$. The modulus of this object is

$$P_{\text{no decay}} = \| \mathcal{A} \|^2 = e^{-\mathcal{V}T 2\text{Im}\, E}$$

– we can interpret $2\text{Im}\, E$ as the (connected!) decay probability of the state in question per unit time per unit volume. (Notice that this relation means that the imaginary part of $V_{\text{1-loop}}$ had *better* be positive, so that the probability stays less than one! In the one-loop approximation, this is guaranteed by the correct $\mathbf{i}\epsilon$ prescription.)

For more on what happens when the perturbative answer becomes complex and non-convex, and how to interpret the imaginary part, see: this paper by E. Weinberg and Wu.

2.4 The spectral density and consequences of unitarity

Next I would like to talk about the notion of *density of states* in QFT, and in particular the notion of the *density of states contributing to a correlation function* G, also known as the *spectral density* of G. In high-energy physics this idea is associated with the names Källen-Lehmann and is part of a program of trying to use complex analysis to make progress in QFT. In cond-mat there are no names because it is everywhere.

[Zee III.8, Appendix 2] In the following we will consider a (time-ordered) two-point function of an operator $\mathcal{O}$. We will make hardly any assumptions about this operator. We will assume it is a scalar under rotations, and will assume translation invariance in time and space. But we need not assume that $\mathcal{O}$ is 'elementary'. This is an extremely loaded term, a useful definition for which is: a field governed by a nearly-quadratic action. Also: try to keep an eye out for where (if anywhere) we assume Lorentz invariance.

So, let

$$\mathbf{i}\mathcal{D}(x) \equiv \langle 0|\mathcal{T}\mathcal{O}(x)\mathcal{O}(0)^\dagger|0\rangle.$$

Notice that we do not assume that $\mathcal{O}$ is hermitian. Use translation invariance to move the left operator to the origin: $\mathcal{O}(x) = e^{\mathbf{i}\mathbf{P}x}\mathcal{O}(0)e^{-\mathbf{i}\mathbf{P}x}$. This follows from the statement that $\mathbf{P}$ generates translations [20]

$$\partial_\mu \mathcal{O}(x) = \mathbf{i}[\mathbf{P}_\mu, \mathcal{O}(x)] \ .$$

And let's unpack the time-ordering symbol:

$$\mathbf{i}\mathcal{D}(x) = \theta(t)\langle 0|e^{\mathbf{i}\mathbf{P}x}\mathcal{O}(0)e^{-\mathbf{i}\mathbf{P}x}\mathcal{O}(0)|0\rangle + \theta(-t)\langle 0|\mathcal{O}(0)e^{\mathbf{i}\mathbf{P}x}\mathcal{O}(0)e^{-\mathbf{i}\mathbf{P}x}|0\rangle. \tag{2.26}$$

Now we need a resolution of the identity operator on the *entire* QFT $\mathcal{H}$:

$$\mathbb{1} = \sum_n |n\rangle\langle n|.$$

This innocent-looking n summation variable is hiding an enormous sum! Let's also assume that the groundstate $|0\rangle$ is translation invariant:

$$\mathbf{P}|0\rangle = 0.$$

We can label each state $|n\rangle$ by its total momentum:

$$\mathbf{P}|n\rangle = p_n|n\rangle.$$

[20] Note that $\mathbf{P}$ here is a D-component vector of operators

$$\mathbf{P}_\mu = (\mathbf{H}, \vec{\mathbf{P}})_\mu$$

which includes the Hamiltonian – we are using relativistic notation – but we haven't actually required any assumption about the action of boosts.

Let's examine the first term in (2.26); sticking the $\mathbb{1}$ in a suitable place:

$$\langle 0|e^{\mathbf{i}\mathbf{P}x}\mathcal{O}(0)\mathbb{1}e^{-\mathbf{i}\mathbf{P}x}\mathcal{O}(0)|0\rangle = \sum_n \langle 0|\mathcal{O}(0)|n\rangle\langle n|e^{-\mathbf{i}\mathbf{P}x}\mathcal{O}(0)|0\rangle = \sum_n e^{-\mathbf{i}p_n x}\| \mathcal{O}_{0n} \|^2 ,$$

with $\mathcal{O}_{0n} \equiv \langle 0|\mathcal{O}(0)|n\rangle$ the matrix element of our operator between the vacuum and the state $|n\rangle$. Notice the absolute value: unitarity of our QFT requires this to be positive and this will have valuable consequences.

Next we work on the time-ordering symbol. I claim that :

$$\theta(x^0) = \theta(t) = -\mathbf{i}\int \text{đ}\omega \frac{e^{+\mathbf{i}\omega t}}{\omega - \mathbf{i}\epsilon} \quad ; \qquad \theta(-t) = +\mathbf{i}\int \text{đ}\omega \frac{e^{+\mathbf{i}\omega t}}{\omega + \mathbf{i}\epsilon} .$$

Just like in our discussion of the Feynman contour, the point of the $\mathbf{i}\epsilon$ is to push the pole inside or outside the integration contour. The half-plane in which we must close the contour depends on the sign of t. There is an important sign related to the orientation with which we circumnavigate the pole. Here is a check that we got the signs and factors right:

$$\frac{d\theta(t)}{dt} = -\mathbf{i}\partial_t \int \text{đ}\omega \frac{e^{\mathbf{i}\omega t}}{\omega - \mathbf{i}\epsilon} = \int \text{đ}\omega e^{\mathbf{i}\omega t} = \delta(t).$$

Consider now the fourier transform of $\mathcal{D}(x)$:

$$\mathbf{i}\mathcal{D}(q) = \int d^D x e^{\mathbf{i}qx}\mathbf{i}\mathcal{D}(x) = \mathbf{i}(2\pi)^{D-1}\sum_n \| \mathcal{O}_{0n} \|^2 \left(\frac{\delta^{(D-1)}(\vec{q} - \vec{p}_n)}{q^0 - p_n^0 + \mathbf{i}\epsilon} - \frac{\delta^{(D-1)}(\vec{q} + \vec{p}_n)}{q^0 - p_n^0 + \mathbf{i}\epsilon} \right) .$$

With this expression in hand, you could imagine measuring the $\mathcal{O}_{0n}$s and using that to determine $\mathcal{D}$.

Suppose that our operator $\mathcal{O}$ is capable of creating a single particle (for example, suppose, if you must, that $\mathcal{O} = \phi$, a perturbative quantum field). Such a state is labelled only by its spatial momentum: $|\vec{k}\rangle$. The statement that $\mathcal{O}$ can create this state from the vacuum means

$$\langle \vec{k}|\mathcal{O}(0)^\dagger|0\rangle = \frac{Z^{\frac{1}{2}}}{\sqrt{(2\pi)^{D-1}\, 2\omega_{\vec{k}}}} \tag{2.27}$$

where $\omega_{\vec{k}}$ is the energy of the particle as a function of $\vec{k}$. For a Lorentz invariant theory, we can parametrize this as

$$\omega_{\vec{k}} \overset{\text{Lorentz!}}{\equiv} \sqrt{\vec{k}^2 + m^2}$$

in terms of m, the *mass* of the particle. [21] What is Z? It's the probability that $\mathcal{O}$ creates this 1-particle state. In the free field theory it's 1. $1 - Z$ measures the extent to which $\mathcal{O}$ does anything *besides* create this 1-particle state.

[End of Lecture 7]

The identity of the one-particle Hilbert space (relatively tiny!) $\mathcal{H}_1$ is

$$\mathbb{1}_1 = \int \mathrm{đ}^{D-1}\vec{k}|\vec{k}\rangle\langle\vec{k}|, \quad \langle\vec{k}|\vec{k}'\rangle = \delta^{(D-1)}(\vec{k} - \vec{k}').$$

This is a summand in the whole horrible resolution:

$$\mathbb{1} = \mathbb{1}_1 + \cdots .$$

I mention this because it lets us define the part of the horrible $\sum_n$ which comes from 1-particle states:

$$\begin{aligned} \implies \quad \mathbf{i}\mathcal{D}(q) &= \ldots + \mathbf{i}(2\pi)^{D-1}\int \mathrm{đ}^{D-1}\vec{k}\frac{Z}{2\omega_k}\left(\frac{\delta^{D-1}(\vec{q}-\vec{k})}{q^0 - \omega_{\vec{k}} + \mathbf{i}\epsilon} - (\omega_k \to -\omega_k)\right) \\ &= \ldots + \mathbf{i}\frac{Z}{2\omega_q}\left(\frac{1}{q^0 - \omega_q + \mathbf{i}\epsilon} - \frac{1}{q^0 + \omega_q - \mathbf{i}\epsilon}\right) \\ &= \ldots + \mathbf{i}\frac{Z}{q^2 - m^2 + \mathbf{i}\epsilon} \end{aligned}$$

(Here again ... is contributions from states involving something else, *e.g.* more than one particle.) The big conclusion here is that even in the interacting theory, even if $\mathcal{O}$ is composite and complicated, if $\mathcal{O}$ can create a 1-particle state with mass m with probability Z, then its 2-point function has a pole at the right mass, and the residue of that pole is Z. (This result was promised earlier when we mentioned LSZ.)[22]

[21] To get comfortable with the appearance of $\omega^{-\frac{1}{2}}$ in (2.27), recall the expansion of a free scalar field in creation an annihilation operators:

$$\phi(x) = \int \frac{\mathrm{đ}^{D-1}\vec{p}}{\sqrt{2\omega_{\vec{p}}}}\left(\mathbf{a}_{\vec{p}}e^{-\mathbf{i}px} + \mathbf{a}^\dagger_{\vec{p}}e^{\mathbf{i}px}\right) \quad .$$

For a free field $|\vec{k}\rangle = \mathbf{a}^\dagger_{\vec{k}}|0\rangle$, and $\langle\vec{k}|\phi(0)|0\rangle = \frac{1}{\sqrt{(2\pi)^{D-1}2\omega_{\vec{k}}}}$. The factor of $\omega^{-\frac{1}{2}}$ is required by the ETCRs:

$$[\phi(\vec{x}), \pi(\vec{x}')] = \mathbf{i}\delta^{D-1}(\vec{x} - \vec{x}'), \quad [\mathbf{a}_{\vec{k}}, \mathbf{a}^\dagger_{\vec{k}'}] = \delta^{D-1}(\vec{k} - \vec{k}') \ ,$$

where $\pi = \partial_t\phi$ is the canonical field momentum. It is just like in the simple harmonic oscillator, where

$$\mathbf{q} = \sqrt{\frac{\hbar}{2m\omega}}\left(\mathbf{a} + \mathbf{a}^\dagger\right), \quad \mathbf{p} = \mathbf{i}\sqrt{\frac{\hbar\omega}{2}}\left(\mathbf{a} - \mathbf{a}^\dagger\right).$$

[22] If we hadn't assumed Lorentz invariance, this would be replaced by the statement: if the operator $\mathcal{O}$ can create a state with energy ω from the vacuum with probability Z, then its Green's function has a pole at that frequency, with residue Z.

The imaginary part of $\mathcal{D}$ is called the *spectral density* $\boldsymbol{\rho}$ (beware that different physicists have different conventions for the factor of $\mathbf{i}$ in front of the Green's function; the spectral density is not always the imaginary part, but it's always positive (in unitary theories)!

Using

$$\mathrm{Im}\,\frac{1}{Q-\mathbf{i}\epsilon} = \pi\delta(Q), \qquad (\text{for } Q \text{ real}). \tag{2.28}$$

we have

$$\mathrm{Im}\,\mathcal{D}(q) = \pi\,(2\pi)^{D-1}\sum_n \|\,\mathcal{O}_{0n}\,\|^2\left(\delta^D(q-p_n)+\delta^D(q+p_n)\right).$$

More explicitly:

$$\mathrm{Im}\,\mathbf{i}\int d^Dx\, e^{\mathbf{i}qx}\langle 0|\mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle = \pi\,(2\pi)^{D-1}\sum_n \|\,\mathcal{O}_{0n}\,\|^2\left(\delta^D(q-p_n) - \underbrace{\delta^D(q+p_n)}_{=0 \text{ for } q^0>0 \text{ since } p_n^0>0}\right).$$

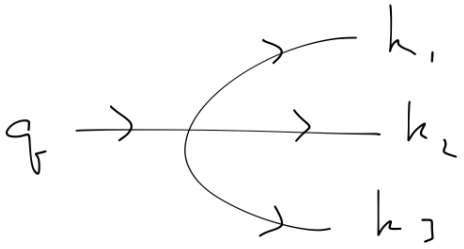
The second term on the RHS vanishes when $q^0 > 0$, since states in $\mathcal{H}$ have energy bigger than the energy of the groundstate.

Using (2.28), the contribution of a 1-particle state to the spectral density is:

$$\mathrm{Im}\,\mathcal{D}(q) = \ldots + \pi Z\delta(q^2-m^2).$$

This quantity $\mathrm{Im}\,\mathcal{D}(q)$ is called the *spectral density* of $\mathcal{O}$, and is positive because it is the *number* of states (with D-momentum in an infinitesimal neighborhood of q), weighted by the modulus of their overlap with the state engendered by the operator on the groundstate.

Now what about multiparticle states? The associated sum over such states involves mutliple (spatial) momentum integrals, not fixed by the total momentum *e.g.* in ϕ^4 theory:



The three particles must share the momentum q. In this case the sum over all 3-particle states is

$$\sum_{n,\text{ 3-particle states with momentum } q} \propto \int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \delta^D(k_1+k_2+k_3-q)$$

Now instead of an isolated pole, we have a whole collection of poles right next to each other. This is a branch cut. In this example, the branch cut begins at $q^2 = (3m)^2$. $3m$ is the lowest energy q^0 at which we can produce three particles of mass m (they have to be at rest).

Note that in ϕ^3 theory, we would instead find that the particle can decay into two particles, and the sum over two particle states would look like

$$\sum_{n,\text{ 2-particle states with momentum } q} \propto \int d\vec{k}_1 d\vec{k}_2 \delta^D(k_1 + k_2 - q)$$

Recall some complex analysis, in the form of the *Kramers-Kronig (or dispersion) relations*:

$$\mathrm{Re}\, G(z) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega \frac{\mathrm{Im}\, G(\omega)}{\omega - z}$$

(valid if $\mathrm{Im}\, G(\omega)$ is analytic in the UHP of ω and falls off faster than $1/\omega$). These equations, which you are supposed to learn in E&M but no one seems to, and which relate the real and imaginary parts of an analytic function by an integral equation, can be interpreted as the statement that the imaginary part of a complex integral comes from the singularities of the integrand, and conversely that those singularities completely determine the function.

An even more dramatic version of these relations (whose imaginary part is the previous eqn) is

$$f(z) = \frac{1}{\pi}\int dw \frac{\rho(w)}{w - z}\ , \qquad \rho(w) \equiv \mathrm{Im}\, f(w + \mathbf{i}\epsilon).$$

The imaginary part determines the whole function.

Comments:

- The spectral density $\mathrm{Im}\,\mathcal{D}(q)$ *determines* $\mathcal{D}(q)$. When people get excited about this it is called the "S-matrix program".
- The result we've shown protects physics from our caprices in choosing field variables. If someone else uses a different field variable $\eta \equiv Z^{\frac{1}{2}}\phi + \alpha\phi^3$, the result above with $\mathcal{O} = \eta$ shows that

  $$\int d^D x e^{\mathbf{i}qx} \langle \mathcal{T}\eta(x)\eta(0)\rangle$$

  still has a pole at $q^2 = m^2$ and a cut starting at the three-particle threshold, $q^2 = (3m)^2$.

- A sometimes useful fact which we've basically already shown:

$$-\mathrm{Im}\,\mathcal{D}(q) = (2\pi)^D \sum_n \| \mathcal{O}_{0n} \|^2 \left(\delta^D(q - p_n) + \delta^D(q + p_n)\right) = \frac{1}{2}\int d^D x e^{\mathbf{i}qx} \langle 0|[\mathcal{O}(x), \mathcal{O}(0)]|0\rangle \ .$$

We can summarize what we've learned in the Lorentz-invariant case as follows:

In a Lorentz invariant theory, the spectral density for a scalar operator is a scalar function of p^μ with

$$\sum_s \delta^D(p - p_s) \| \langle 0|\phi(0)|s\rangle \|^2 = \frac{\theta(p^0)}{(2\pi)^{D-1}} \rho(p^2) \ .$$

The function $\rho(s)$ is called the spectral density for this Green's function. Claims:

- $\rho(s) = \mathcal{N}\mathrm{Im}\,\mathcal{D}$ for some number $\mathcal{N}$, when $s > 0$.
- $\rho(s) = 0$ for $s < 0$. There are no states for spacelike momenta.
- $\rho(s) \geq 0$ for $s > 0$. The density of states for timelike momenta is positive or zero.
- With our assumption about one-particle states, $\rho(s)$ has a delta-function singularity at $s = m^2$, with weight Z. More generally we have shown that

$$\mathcal{D}(k^2) = \int ds \ \pi\rho(s) \frac{1}{k^2 - s + \mathbf{i}\epsilon}.$$

This is called the Källen-Lehmann spectral representation of the propagator; it represents it as a sum of *free* propagators with different masses, determined by the spectral density.

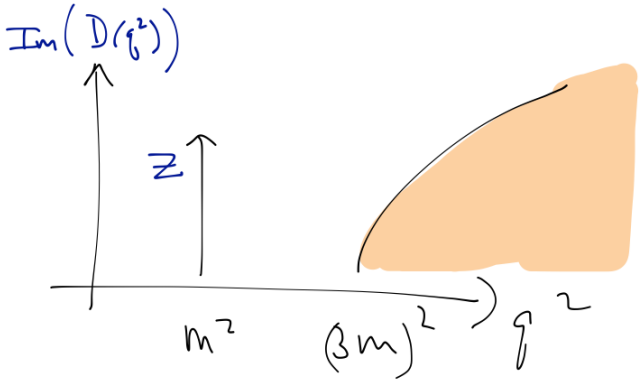


Figure 7: The spectral density of ϕ in massive ϕ^4 theory.

Taking into account our assumption about single-particle states, this is

$$\mathcal{D}(k^2) = \frac{Z}{k^2 - m^2 + \mathbf{i}\epsilon} + \int_{(3m)^2}^{\infty} ds\ \rho_c(s) \frac{1}{k^2 - s + \mathbf{i}\epsilon}$$

where ρ_c is just the continuum part. The pole at the particle-mass^2 survives interactions, with our assumption. (The value of the mass need not be the same as the bare mass!)

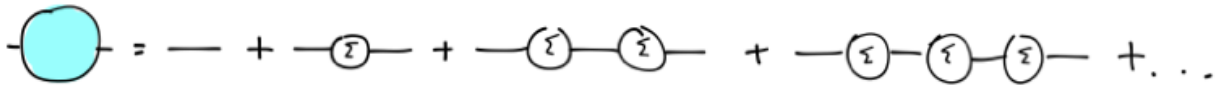
The idea of spectral representation and spectral density is more general than the Lorentz-invariant case. In particular, the spectral density of a Green's function is an important concept in cond-mat. For example, the spectral density for the electron 2-point function is the thing that actually gets measured in angle-resolved photoemission experiments (ARPES).

2.4.1 Cutting rules

[Zee §III.8] Consider the two point function of a relativistic scalar field ϕ which has a perturbative cubic interaction:

$$S = \int d^D x \left(\frac{1}{2} \left((\partial\phi)^2 + m^2\phi^2 \right) + \frac{g}{3!}\phi^3 \right).$$

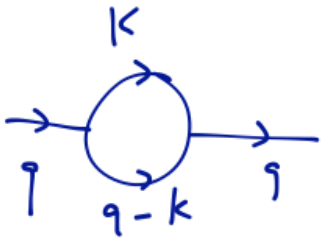
Sum the geometric series of 1PI insertions to get



$$\mathbf{i}\mathcal{D}_\phi(q) = \frac{\mathbf{i}}{q^2 - m^2 + \Sigma(q) + \mathbf{i}\epsilon}$$

where $\Sigma(q)$ is the 1PI two point vertex.

The leading contribution to Σ comes from the one loop diagram at right and is



$$\mathbf{i}\Sigma_{1\text{ loop}}(q^2) = (\mathbf{i}g)^2 \int \mathrm{đ}^D k \frac{\mathbf{i}}{k^2 - m^2 + \mathbf{i}\epsilon} \frac{\mathbf{i}}{(q-k)^2 - m^2 + \mathbf{i}\epsilon}.$$

Consider this function for real q, for which there are actual states of the scalar field – timelike q^μ, with $q^0 > m$. The real part of Σ shifts the mass. What does it mean if this function has an imaginary part?

Claim: ImΣ is a decay rate.

It moves the energy of the particle off of the real axis from m to

$$\sqrt{m^2 - \mathbf{i}\mathrm{Im}\,\Sigma(m^2)} \overset{\text{small } \mathrm{Im}\Sigma \sim g^2}{\simeq} m - \mathbf{i}\frac{\mathrm{Im}\,\Sigma(m^2)}{2m}.$$

The fourier transform to real time is an amplitude for a state with complex energy $\mathcal{E}$: its wavefunction evolves like $\psi(t) \sim e^{-\mathbf{i}\mathcal{E}t}$ and has norm

$$\| \psi(t) \|^2 \sim \| e^{-\mathbf{i}\left(E - \mathbf{i}\frac{1}{2}\Gamma\right)t} \|^2 = e^{-\Gamma t}.$$

In our case, we have $\Gamma = \mathrm{Im}\,\Sigma(m^2)/m$, and we interpret that as the rate of decay of the norm of the single-particle state. There is a nonzero probability that the state turns into something else as a result of time evolution in the QFT: the single particle must decay into some other state – multiple particles. (We will see next how to figure out into what it decays.)

The absolute value of the Fourier transform of this quantity $\psi(t)$ is the kind of thing you would measure in a scattering experiment. This is

$$F(\omega) = \int dt\; e^{-\mathbf{i}\omega t}\psi(t) = \int_0^\infty dt\; e^{-\mathbf{i}\omega t} e^{\mathbf{i}\left(M - \frac{1}{2}\mathbf{i}\Gamma\right)t} = \frac{1}{\mathbf{i}\,(\omega - M) - \frac{1}{2}\Gamma}$$

$$\| F(\omega) \|^2 = \frac{1}{(\omega - M)^2 + \frac{1}{4}\Gamma^2}$$

is a Lorentzian in ω with width Γ. so Γ is sometimes called a *width*.

So: what is $\mathrm{Im}\,\Sigma_{1\text{ loop}}$ in this example?

We will use

$$\frac{1}{k^2 - m^2 + \mathbf{i}\epsilon} = \mathcal{P}\frac{1}{k^2 - m^2} - \mathbf{i}\pi\delta(k^2 - m^2) \equiv \mathcal{P} - \mathbf{i}\Delta$$

where $\mathcal{P}$ denotes 'principal part'. Then

$$\mathrm{Im}\,\Sigma_{1\text{ loop}}(q) = -g^2 \int d\Phi\; (\mathcal{P}_1\mathcal{P}_2 - \Delta_1\Delta_2)$$

with $d\Phi = đk_1 đk_2 (2\pi)^D \delta^D(k_1 + k_2 - q)$.

This next trick, to get rid of the principal part bit, is from Zee's book (the second edition on p.214; he also does the calculation by brute force in the appendix to that section). We can find a representation for the 1-loop self-energy in terms of real-space propagators: it's the fourier transform of the amplitude to create two ϕ excitations at the origin at time zero

with a single ϕ field (this is ig), to propagate them both from 0 to x (this is $\mathbf{i}\mathcal{D}(x)^2$) and then destroy them both with a single ϕ field (this is ig again). Altogether:

$$\begin{aligned}\mathbf{i}\Sigma(q) &= \int d^dx\; e^{\mathbf{i}qx}\,(\mathbf{i}g)^2\,\mathbf{i}\mathcal{D}(x)\mathbf{i}\mathcal{D}(x) \\ &= g^2\int d\Phi \frac{1}{k_1^2-m_1^2-\mathbf{i}\epsilon}\frac{1}{k_2^2-m_2^2-\mathbf{i}\epsilon}\end{aligned} \tag{2.29}$$

In the bottom expression, the $\mathbf{i}\epsilon$s are designed to produce the *time-ordered* $\mathcal{D}(x)$s. Consider instead the strange combination

$$\begin{aligned}0 &= \int d^dx\; e^{\mathbf{i}qx}\,(\mathbf{i}g)^2\,\mathbf{i}\mathcal{D}_{\text{adv}}(x)\mathbf{i}\mathcal{D}_{\text{ret}}(x) \\ &= g^2\int d\Phi \frac{1}{k_1^2-m_1^2-\sigma_1\mathbf{i}\epsilon}\frac{1}{k_2^2+\sigma_2 m_2^2-\mathbf{i}\epsilon}\end{aligned} \tag{2.30}$$

where $\sigma_{1,2}\equiv \text{sign}(k^0_{1,2})$. This expression vanishes because the integrand is identically zero: there is no value of t for which both the advanced and retarded propagators are nonzero. Therefore, we can add the imaginary part of zero

$$-\text{Im}\,(0) = -g^2\int d\Phi\,(\mathcal{P}_1\mathcal{P}_2+\sigma_1\sigma_2\Delta_1\Delta_2)$$

to our expression for $\text{Im}\,\Sigma_{\text{1-loop}}$ to cancel the annoying principal part bits:

$$\text{Im}\,\Sigma_{\text{1-loop}} = g^2\int d\Phi\,((1+\sigma_1\sigma_2)\,\Delta_1\Delta_2)\,.$$

The quantity $(1+\sigma_1\sigma_2)$ is only nonzero when k_1^0 and k_2^0 have the same sign; but in $d\Phi$ is a delta function which sets $q^0 = k_1^0+k_2^0$. WLOG we can take $q^0>0$ since we only care about the propagation of positive-energy states. Therefore both k_1^0 and k_2^0 must be positive.

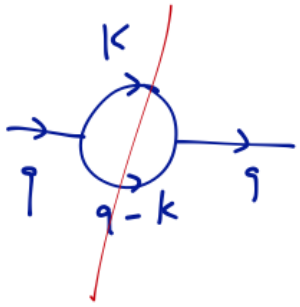
The result is that the only values of k on the RHS that contribute are ones with *positive* energy, which satisfy all the momentum conservation constraints:

$$\begin{aligned}\text{Im}\,\Sigma &= g^2\int d\Phi\theta(k_1^0)\theta(k_2^0)\Delta_1\Delta_2 \\ &= \frac{g^2}{2}\int \frac{\text{đ}^{D-1}\vec{k}_1}{2\omega_{\vec{k}_1}}\frac{\text{đ}^{D-1}\vec{k}_2}{2\omega_{\vec{k}_2}}(2\pi)^D\delta^D(k_1+k_2-q)\;.\end{aligned}$$

In summary:

$$\text{Im}\,\Sigma = \sum_{\substack{\text{actual states } n \text{ of 2 particles} \\ \text{into which } \phi \text{ can decay}}} \|\,\mathcal{A}_{\phi\to n}\,\|^2 \tag{2.31}$$

In this example the decay amplitude $\mathcal{A}$ is just $\mathbf{i}g$.



This result is generalized by the *Cutkosky cutting rules* for finding the imaginary part of a feynman diagram describing a physical process. The rough rules are the following. Assume the diagram is amputated – leave out the external propagators. Then any line drawn through the diagram which separates initial and final states (as at right) will 'cut' through some number of internal propagators; replace each of the cut propagators by $\theta(p^0)\pi\delta(p^2-m^2) = \theta(p^0)\frac{\pi\delta(p_0-\epsilon_p)}{2\epsilon_p}$. As Tony Zee says: the amplitude becomes imaginary when the intermediate particles become real (as opposed to virtual), aka 'go on-shell'.

The general form of (2.31) is a general consequence of unitarity. Recall that the S-matrix is

$$\mathcal{S}_{fi} = \langle f|e^{-\mathbf{i}\mathbf{H}T}|i\rangle \equiv (\mathbb{1} + \mathbf{i}\mathcal{T})_{fi}\,.$$

$$\mathbf{H} = \mathbf{H}^\dagger \implies \mathbb{1} = \mathcal{S}\mathcal{S}^\dagger \implies 2\mathrm{Im}\,\mathcal{T} \equiv \mathbf{i}\left(\mathcal{T}^\dagger - \mathcal{T}\right) \overset{\mathbb{1}=\mathcal{S}\mathcal{S}^\dagger}{=} \mathcal{T}^\dagger\mathcal{T}.$$

This is called the optical theorem and it is the same as the one taught in some QM classes. In terms of matrix elements:

$$2\mathrm{Im}\,\mathcal{T}_{fi} = \sum_n \mathcal{T}^\dagger_{fn}\mathcal{T}_{ni}$$

Here we've inserted a resolution of the identity (again on the QFT Hilbert space, the same scary sum) in between the two $\mathcal{T}$ operators. In the one-loop approximation, in the ϕ^3 theory here, the intermediate states which can contribute to $\sum_n$ are two-particle states, so that $\sum_n \to \int \text{đ}\vec{k}_1\,\text{đ}\vec{k}_2$, the two-particle density of states.

Recall that for real x the imaginary part of a function of one variable with a branch cut, (like $\mathrm{Im}\,(x+\mathbf{i}\epsilon)^\nu = \frac{1}{2}\left((x+\mathbf{i}\epsilon)^\nu - (x-\mathbf{i}\epsilon)^\nu\right)$) is equal to (half) the *discontinuity* of the function $((x)^\nu)$ across the branch cut. Problem Set 4 mentions a second example which is more complicated than the one above in that there is more than one way to cut the diagram. Different ways of cutting the diagram correspond to discontinuities in different kinematical variables. To get the whole imaginary part, we have to add these up.

One important comment (which is elaborated further in Zee's discussion) is: there had better not be any cutoff dependence in the imaginary part. If there is, we'll have trouble cancelling it by adding counterterms – an imaginary part of the action will destroy unitarity.

[End of Lecture 8]

3 The Wilsonian perspective on renormalization

[Fradkin, 2d edition, chapter 4; Cardy; Zee §VI; Álvarez-Gaumé and Vázquez-Mozo, *An Invitation to QFT*, chapter 8.4-5 ($\simeq$ §7.3-4 of hep-th/0510040)]

The following discussion describes a perspective which can be applied to any system of (many) extensive degrees of freedom. This includes many statistical-mechanics systems, condensed-matter systems and also QFTs in high energy physics. The great insight of Kadanoff and Wilson about such systems is that we should organize our thinking about them by length scale. We should think about a *family* of descriptions, labelled by the resolution of our microscope.

Before explaining this perspective in detail, let's spend some time addressing the following basic and instructive question:

3.1 Where do field theories come from?

3.1.1 A model with finitely many degrees of freedom per unit volume

Consider the following system of extensive degrees of freedom – it is an example of a very well-regulated (euclidean) QFT. At each site i of a square lattice we place a two-valued (classical) degree of freedom $s_i = \pm 1$, so that the path 'integral' measure is

$$\int [ds] ... \equiv \sum_{\{s_i\}} ... = \prod_{\text{sites, } i} \sum_{s_i = \pm 1} ... \ .$$

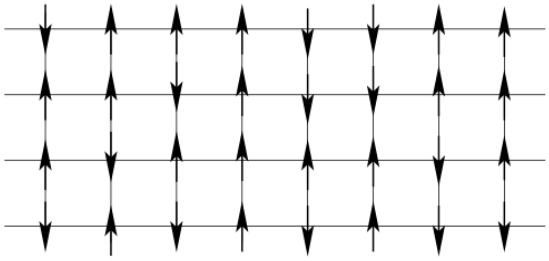


Figure 8: A configuration of classical Ising spins on the 2d square lattice. [from Álvarez-Gaumé and Vázquez-Mozo, hep-th/0510040]

Let's choose the euclidean action to be

$$S[s] = -\beta J \sum_{\langle i,j \rangle} s_i s_j \ .$$

Here βJ is some coupling; the notation $\langle i, j \rangle$ means 'sites i and j which are nearest neighbors'. The partition function is

$$Z = \int [ds] e^{-S[s]} = \sum_{\{s_i\}} e^{+\beta J \sum_{\langle i,j \rangle} s_i s_j}. \tag{3.1}$$

(I can't hide the fact that this is the thermal partition function $Z = \mathrm{tr} e^{-\beta H}$ for the *classical Ising model on the square lattice*, with $H = -J \sum_{\langle i,j \rangle} s_i s_j$, and $\beta \equiv 1/T$ is the *coolness*[23], *i.e.* the inverse temperature.)

In the thermodynamic limit (the number of sites goes to infinity), this model has a special value of $\beta J > 0$ below which there is spontaneous breaking of the $\mathbb{Z}_2$ symmetry $s_i \to -s_i$ by a nonzero *magnetization*, $\langle s_i \rangle \neq 0$.

Kramers-Wannier duality. To see that there is a special value of βJ, we can make the following observation, due to Kramers and Wannier, and generalized by Wegner, which is now a subject of obsession for many theoretical physicists. It is called *duality*. Consider a configuration of the spins. The action $S[s]$ is determined by the number of links across which the spins disagree (positive βJ favors contributions from spins which agree). It is possible to rewrite the partition sum in terms of these disagreements. (For more on this, see the lecture notes here.) The answer is identical to the original model, except with βJ replaced by $a(\beta J)^{-1}$ for some number a! At high temperature the model is obviously disordered, at low temperature the dual model is obviously disordered, but that means that the original model is ordered. In between something happens. If only *one* something happens, it must happen at the special value $\beta J = a(\beta J)^{-1}$.

For a more complete discussion of this subject of *duality* I recommend this review by Kogut, §4. I hope we will have the opportunity to come back to this later in the quarter.

Onsager solution. Lars Onsager solved the model above exactly (published in 1944) and showed for sure that it has a critical point $(\beta J)_\star = \frac{1}{2} \tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)$. For our present purposes this landmark result is a distraction.

Comment on analyticity in βJ *versus* the critical point. [Zee §V.3] The Ising model defined by (3.1) is a model of a magnet (more specifically, when $\beta J > 0$ which makes neighboring spins want to align, a *ferro*magnet). Some basic phenomenology: just below the Curie temperature T_c, the magnetization (average magnetic moment per unit volume) behaves like

$$|M| \sim (T_c - T)^\beta$$

where β is a pure number (it depends on the number of spatial dimensions)[24]. In terms of

[23] This nomenclature, due to the condensed matter physicist Miles Stoudenmire, does a great job of reminding us that at lower temperatures, quantum mechanics has more dramatic consequences.

[24] The name is conventional; don't confuse it with the inverse temperature.

the Ising model, the magnetization is[25]

$$\langle M \rangle = \frac{1}{Z} \sum_{\{s_i\}} e^{-H(s)/T} \frac{\sum_i s_i}{\mathcal{V}} \quad . \tag{3.2}$$

($\mathcal{V}$ is the number of sites of the lattice, the volume of space.) How can you get such a *non-analytic* (at $T = T_c \neq 0$) function of T by adding a bunch of terms of the form $e^{-E/T}$? It is clearly impossible if there is only a finite number of terms in the sum, each of which is analytic near $T_c \neq 0$. It *is* actually possible if the number of terms is infinite – phase transitions only happen in the thermodynamic limit.

3.1.2 Landau and Ginzburg guess the answer.

Starting from Z, even with clever tricks like Kramers-Wannier duality, and even for Onsager, it is pretty hard to figure out what the answer is for the magnetization. But the answer is actually largely determined on general grounds, as follows.

We want to ask what is the free energy G *at fixed magnetization*. This $G[M]$ is just the same idea as the euclidean effective action $\Gamma[\phi_c]$ (divided by β) – it is a Legendre transform of the usual F in $Z = e^{-\beta F}$. [26] So as we've been discussing, G is the thing we should minimize to find the groundstate.

LG Effective Potential. We can even consider a model where the magnetization is a vector. If $\vec{M}$ is independent of position $\vec{x}$ [27] then rotation invariance (or even just $M \to -M$ symmetry) demands that

$$G = V \left(a\vec{M}^2 + b\left(\vec{M}^2\right)^2 + ... \right)$$

where a, b[28] are some functions of T that we don't know, and the dots are terms with more

[25] In many real magnets, the magnetization can point in any direction in three-space – it's a vector $\vec{M}$. We are simplifying our lives.

[26] To be more explicit, we can add a source for the magnetization and compute

$$e^{-\beta F[J]} = \mathrm{tr} e^{-\beta(H + \sum MJ)}.$$

Now pick some magnetization M_c, and *choose* $J^{[M_c]}$ so that

$$\langle M \rangle = -\frac{\partial F}{\partial J} = M_c.$$

Then $G[M_c] \equiv F[J^{[M_c]}] - \sum M_c J^{[M_c]}$. Make sure you agree that this is identical to our construction of $\Gamma[\phi_c]$. In this context, the source J is (minus) an external magnetic (Zeeman) field.

[27] In (3.2), I've averaged over all space; instead we could have averaged over just a big enough patch to make it look smooth. We'll ask 'how big is big enough?' next – the answer is 'the correlation length'.

[28] Don't confuse a with the lattice spacing; sorry, ran out of letters.

Ms. These functions $a(T)$ and $b(T)$ have no reason not to be smooth functions of T. Now suppose there is a value of T for which $a(T)$ vanishes:

$$a(T) = a_1(T - T_c) + ...$$

with $a_1 > 0$ a pure constant. For $T > T_c$, the minimum of G is at $\vec{M} = 0$; for $T < T_c$, the unmagnetized state becomes unstable and new minima emerge at $|\vec{M}| = \sqrt{-\frac{a}{2b}} \sim (T_c - T)^{\frac{1}{2}}$. This is the *mean field theory* description of a second-order phase transition. It's not the right value of β (it's about 1/3) for the 3d Curie point, but it shows very simply how to get an answer that is not analytic at T_c.

LG Effective Action. Landau and Ginzburg can do even better. $G(M)$ with constant M is like the effective *potential*; if we let $M(\vec{x})$ vary in space, we can ask and answer what is the effective *action*, $G[M(\vec{x})]$. The Landau-Ginzburg effective action is

$$G[M] = \int d^d\vec{x} \left(a\vec{M}^2 + b\left(\vec{M}^2\right)^2 + c\partial_i\vec{M} \cdot \partial_i\vec{M} + ... \right) \tag{3.3}$$

– now we are allowed to have gradients. c is a new unknown function of T; let's set it to 1 by rescaling M. This just a scalar field theory (with several scalars) in euclidean space. Each field has a mass $\sqrt{a}$ (they are all the same as a consequence of the spin rotation symmetry). So $\frac{1}{\sqrt{a}}$ is a length scale, to which we turn next.

Definition of correlation length. Suppose we perturb the system by turning on an external (we pick it) magnetic field (source for $\vec{M}$) $\vec{H}$, which adds to the hamiltonian by $-\vec{H} \cdot \vec{M}$. Pick the field to be small, so its effect is small and we can study the linearized equations (let's do it for $T > T_c$, so we're expanding around $M = 0$):

$$\left(-\partial^2 + a\right)\vec{M} = \vec{H} \ .$$

Recall here the result of problem set 2 problem 1 on the Green's function G_2 of a massive scalar field. There you solved this equation in the case where H is a delta function. Since the equation is linear, that solution determines the solution for general H (this was why Green introduced Green's functions):

$$\begin{aligned} M(x) &= \int d^3y G_2(x,y)H(y) = \int d^3y \left(\int \mathrm{đ}^3k \frac{e^{\mathbf{i}\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2 + a} \right) H(y) \\ &= \int d^3y \frac{1}{4\pi|\vec{x}-\vec{y}|} e^{-\sqrt{a}|\vec{x}-\vec{y}|} H(y). \end{aligned} \tag{3.4}$$

The Green's function

$$G_2^{IJ}(x) = \langle \vec{M}^I(x)\vec{M}^J(0)\rangle = \delta^{IJ}\frac{1}{4\pi|\vec{x}|}e^{-\sqrt{a}|\vec{x}|}$$

is diagonal in the vector index I, J so I've suppressed it in (3.4). G_2 is the answer to the question: if I perturb the magnetization at the origin, how does it respond at x? The answer is that it dies off like

$$\langle \vec{M}(x)\vec{M}(0)\rangle \sim e^{-|x|/\xi}$$

– this relation defines the *correlation length* ξ, which will depend on the parameters. In the LG mean field theory, we find $\xi = \frac{1}{\sqrt{a}}$. The LG theory predicts the behavior of ξ as we approach the phase transition to be $\xi \sim \frac{1}{(T-T_c)^\nu}$ with $\nu = \frac{1}{2}$. Again the exponent is wrong in detail (we'll see why below), but it's a great start.

Now let's return to the microscopic model (3.1). Away from the special value of βJ, the correlation functions behave as

$$\langle s_i s_j\rangle_{\text{connected}} \sim e^{-\frac{r_{ij}}{\xi}}$$

where $r_{ij} \equiv$ distance between sites i and j. Notice that the subscript *connected* means that we need not specify whether we are above or below T_c, since it subtracts out the disconnected bit $\langle s_i\rangle\langle s_j\rangle$ by which their form differs. From the more microscopic viewpoint, ξ is the length scale over which the values of the spins are highly correlated. This allows us to answer the question of how much coarse-graining we need to do to reach a continuum approximation: The continuum description in terms of

$$M(x) \equiv \frac{\sum_{i\in R_x}\langle s_i\rangle}{\text{Vol}(R_x)}$$

is valid if we average over regions R (centered around the point x) with linear size bigger than ξ.

3.1.3 Coarse-graining by block spins.

We want to understand the connection between the microscopic spin model and the macroscopic description of the magnetization better, for example to systematically improve upon the quantitative failures of the LG mean field theory for the critical exponents. Kadanoff's idea is to consider a sequence of *blocking transformations*, whereby we group more and more spins together, to interpolate between the spin at a single site s_i, and the magnetization averaged over the whole system.

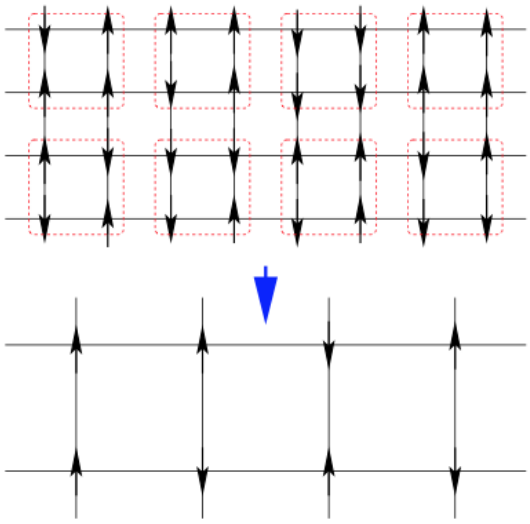


Figure 9: A blocking transformation. [from Álvarez-Gaumé and Vázquez-Mozo, hep-th/0510040]

The blocking (or 'decimation') transformation can be implemented in more detail for ising spins on the 2d square lattice as follows (Fig. 9). Group the spins into blocks of four as shown; we will construct a new coarser Ising system, where the sites of the new lattice correspond to the blocks of the original one, and the spin at the new site is an average of the four. One way to do this is majority rule:

$$s_{\text{block, } b} \equiv \text{sign}\left(\sum_{i \in \text{block},b} s_i\right)$$

where we break a tie by defining $\text{sign}(0) = +1$.

We want to write our original partition function in terms of the averaged spins on a lattice with twice the lattice spacing. We'll use the identity

$$1 = \sum_{s_{\text{block}}} \delta\left(s_{\text{block}} - \text{sign}(\sum_{i \in \text{block}} s_i)\right) \ .$$

This is true for each block; we can insert one of these for each block. Split the original sum into nested sums, the outer one over the blocks, and the inner one over the spins within the block:

$$Z = \sum_{\{s\}} e^{-\beta H[s_i]} = \sum_{\{s_{\text{block, } b}\}} \sum_{s \in \text{block},b} \prod_{\text{blocks}} \delta\left(s_{\text{block},b} - \text{sign}\left(\sum_{i \in \text{block},b} s_i\right)\right) e^{-\beta H^{(a)}[s]} \ .$$

The superscript (a) on the Hamiltonian is intended to indicate that the lattice spacing is a. Now we interpret the inner sum as another example of integrating out stuff we don't care about to generate an effective interaction between the stuff we do care about:

$$\sum_{s \in \text{block},b} \prod_{\text{blocks}} \delta\left(s^{(2a)} - \text{sign}\left(\sum_{i \in \text{block},b} s_i\right)\right) e^{-\beta H^a[s]} \ \equiv e^{-\beta H^{(2a)}[s^{(2a)}]}$$

These sums are hard to actually do, except in 1d. But we don't need to do them to understand the form of the result.

As in our QM example from the first lecture, the new Hamiltonian will be less local than the original one – it won't just be nearest neighbors in general:

$$H^{(2a)}[s^{(2a)}] = -J^{(2a)} \sum_{\langle i,j \rangle} s_i^{(2a)} s_j^{(2a)} + -K^{(2a)} \sum_{\langle\langle i,j \rangle\rangle} s_i^{(2a)} s_j^{(2a)} + ...$$

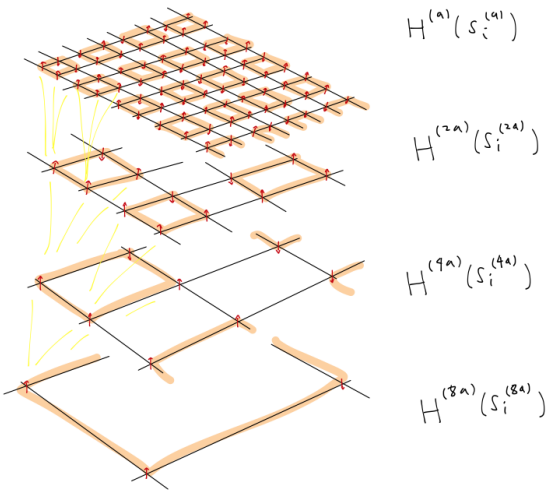
where $\langle\langle i,j \rangle\rangle$ means next-neighbors. Notice that I've used the same labels i, j for the coarser lattice. We have rewritten the partition function as the same *kind* of model, on a coarser lattice, with different values of the couplings:

$$Z = \sum_{\{s^{(2a)}\}} e^{-\beta H^{(2a)}[s^{(2a)}]} \ .$$

Now we can do it again. The decimation operation defines *a map on the space of (in this case Ising) Hamiltonians*:

$$H^{(a)} \mapsto H^{(2a)} \mapsto H^{(4a)} \mapsto H^{(8a)} \mapsto ...$$

The couplings $J, K...$ are coordinates on the space of Hamiltonians. Each time we do it, we double the lattice spacing; the correlation length in units of the lattice spacing gets halved, $\xi \mapsto \xi/2$. This operation is called a 'renormalization group transformation' but notice that it is very much not invertible; we lose information about the short-distance stuff by integrating it out.



RG fixed points. Where can it end? One thing that can happen is that the form of the Hamiltonian can stop changing:

$$H^{(a)} \mapsto H^{(2a)} \mapsto H^{(4a)} \mapsto H^{(8a)} \mapsto ... \mapsto H_\star \mapsto H_\star \mapsto H_\star \, ...$$

The fixed point hamiltionian $H_\star$, which is not changed by the rescaling operation, is scale invariant. What can its correlation length be if it is invariant under $\xi \to \xi/2$? Either $\xi = 0$ (the mass of the fields go to infinity and there is nothing left to integrate) or $\xi = \infty$ (the mass goes to zero and we have more to discuss, we can call this a nontrivial fixed point).

Near a nontrivial fixed point, once $\xi \gg a$, the original lattice spacing, we are quite justified in using a continuum description, to which we return in subsection 3.2.

Perturbations of a fixed point. Before doing any more work, though, we can examine the possible behaviors of the RG flow near a fixed point. Consider a fixed point Hamiltonian $H_\star$, and move away from it slightly by changing one of the couplings a little bit:

$$H = H_\star + \delta g \mathcal{O}.$$

What does the RG do to this to leading order in δg? The possibilities are:

- If the flow takes it back to the original fixed point, $\mathcal{O}$ (and its associated coupling δg) is called *irrelevant*.
- If the flow takes it away from the original fixed point, $\mathcal{O}$ is called a *relevant* perturbation of $H_\star$.
- The new H might also be a fixed point, at least to this order in δg. Such a coupling (and the associated operator $\mathcal{O}$) is called *marginal*. If the new H really is a new fixed point, not just to leading order in δg, then $\mathcal{O}$ is called *exactly marginal*. Usually it goes one way or the other and is called *marginally relevant* or *marginally irrelevant*.

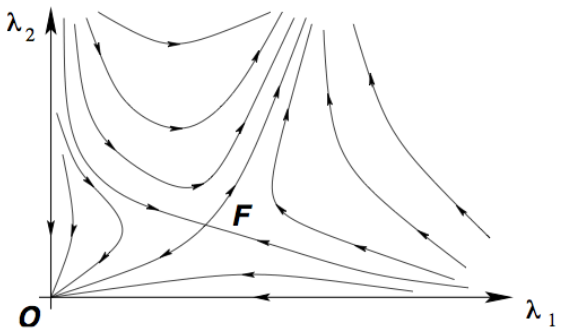


Figure 10: A possible set of RG flows for a system with two couplings $\lambda_{1,2}$. [from Álvarez-Gaumé and Vázquez-Mozo, hep-th/0510040]

Note the infrared-centric terminology.

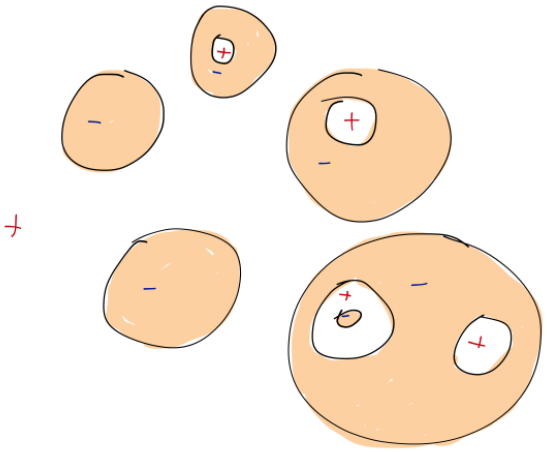
Comment on *Universality*: The Ising model is a model of many microscopically-different-looking systems. It can be a model of spins like we imagined above. Or it could be a model of a lattice gas – we say spin up at site i indicates the presence of a gas molecule there, and spin down represents its absence. These different models will naturally have different microscopic interactions. But there will only be so many fixed points of the flow in the space of Hamiltonians on this system of 2-valued variables. This idea of the paucity of fixed points underlies Kadanoff and Wilson's explanation of the experimental phenomenon of *universality*: the same critical exponents arise from very different-seeming systems (*e.g.* the Curie point of a magnet and the liquid-gas critical point).

3.2 The continuum version of blocking

[Zee, §VI.8 (page 362 of 2d Ed.)]

Here is a very different starting point from which to approach the same critical point as in the previous subsection:

Consider the ϕ^4 theory in Euclidean space, with negative m^2 (and no ϕ^k terms with odd k). This potential has two minima and a $\mathbb{Z}_2$ symmetry that interchanges them, $\phi \to -\phi$. If we squint at a configuration of ϕ, we can label regions of space by the sign of ϕ (as in the figure at right). The kinetic term for ϕ will make nearby regions want to agree, just like the $J\sum_{\langle ij\rangle}\sigma_i\sigma_j$ term in the Ising model. So the critical point described by taking m^2 near zero is plausibly the same as the one obtained from the lattice Ising model described above[29].



So we want to understand the integral

$$Z_\Lambda \equiv \int_\Lambda [D\phi] e^{-\int d^D x \mathcal{L}(\phi)}. \tag{3.5}$$

Here the specification $\int_\Lambda$ says that we integrate over field configurations $\phi(x) = \int đ^D k e^{\mathbf{i}kx}\phi_k$ such that $\phi_k = 0$ for $|k| \equiv \sqrt{\sum_{i=1}^D k_i^2} > \Lambda$. Think of $2\pi/\Lambda$ as the lattice spacing[30] – there just aren't modes of shorter wavelength.

So we are using (again) a cutoff on the euclidean momenta $k_E^2 \leq \Lambda^2$.

We want to understand (3.5) by some coarse-graining procedure. Let us imitate the block spin procedure. Field variations *within* blocks of space of linear size na have wavenumbers greater than $\frac{2\pi}{na}$. (These modes average to zero on larger blocks; modes with larger wavenumber encode the variation *between* these blocks.) So the analog of the partition function after

[29] For a more sophisticated argument for this equivalence, see page 7-9 of Polyakov, *Gauge Fields and Strings*.

[30] This cutoff is not precisely the same as have a lattice; with a lattice, the momentum space is periodic: $e^{\mathbf{i}kx_n} = e^{\mathbf{i}k(na)} = e^{\mathbf{i}\left(k+\frac{2\pi}{a}\right)(na)}$ for $n \in \mathbb{Z}$. Morally it is the same.

a single blocking step is the following: Break up the configurations into pieces:

$$\phi(x) = \int \text{đ}k e^{\mathbf{i}kx}\phi_k \equiv \phi^< + \phi^> \ .$$

Here $\phi^<$ has nonzero fourier components only for $|k| \leq \Lambda - \delta\Lambda$ and $\phi^>$ has nonzero fourier components only for $\Lambda - \delta\Lambda \leq |k| \leq \Lambda$. Zee calls the two parts 'smooth' and 'wiggly'. They could also be called 'slow' and 'fast' or 'light' and 'heavy'. We want to do the integral over the heavy/wiggly/fast modes to develop an effective action for the light/smooth/slow modes:

$$Z_\Lambda = \int_{\Lambda-\delta\Lambda} [D\phi^<] e^{-\int d^D x \mathcal{L}(\phi^<)} \int [D\phi^>] e^{-\int d^D x \mathcal{L}_1(\phi^<,\phi^>)}$$

where $\mathcal{L}_1$ contains all the dependence on $\phi^>$ (and no other terms).

[End of Lecture 9]

Just as with the spin sums, these integrals are hard to actually do, except in a gaussian theory. But again we don't need to do them to understand the form of the result. First give it a name:

$$e^{-\int d^D x \delta L(\phi^<)} \equiv \int [D\phi^>] e^{-\int d^D x \mathcal{L}_1(\phi^<,\phi^>)} \tag{3.6}$$

so once we've done the integral we'll find

$$Z_\Lambda = \int_{\Lambda-\delta\Lambda} [D\phi^<] e^{-\int d^D x (\mathcal{L}(\phi^<)+\delta\mathcal{L}(\phi^<))} \ . \tag{3.7}$$

To get a feeling for the form of $\delta\mathcal{L}$ (and because there is little reason not to) consider the more general Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \sum_n g_n \phi^n + ... \tag{3.8}$$

where we include *all* possible terms consistent with the symmetries (rotation invariance, maybe $\phi \to -\phi$...). Then we can find an explicit expression for $\mathcal{L}_1$:

$$\int d^D x \mathcal{L}_1(\phi^<,\phi^>) = \int d^D x \left(\frac{1}{2}(\partial\phi^>)^2 + \frac{1}{2}m^2 (\phi^>)^2 + ... \right)$$

(I write the integral so that I can ignore terms that integrate to zero such as $\partial\phi^<\partial\phi^>$.) This is the action for a scalar field $\phi^>$ interacting with itself and with a (slowly-varying) background field $\phi^<$. But what can the result $\delta\mathcal{L}$ be but something of the form (3.8) again, with different coefficients? The result is to shift the couplings $g_n \to g_n + \delta g_n$. (This includes the coefficient of the kinetic term and also of the higher-derivative terms which are hidden in the ... in (3.8). You will see in a moment the logic behind which terms I hid.)

Finally, so that we can compare steps of the procedure to each other, we rescale our rulers. We'd like to change units so that $\int_{\Lambda-\delta\Lambda}$ is a $\int_\Lambda$ with different couplings; we accomplish this by defining

$$\Lambda - \delta\Lambda \equiv b\Lambda, \qquad b < 1.$$

In $\int_{\Lambda-\delta\Lambda}$, we integrate over fields with $|k| < b\Lambda$. Change variables: $k = bk'$ so now $|k'| < \Lambda$. So $x = x'/b, \partial' \equiv \partial/\partial x' = \frac{1}{b}\partial_x$ and wavefunctions are preserved $e^{\mathbf{i}kx} = e^{\mathbf{i}k'x'}$. Plug this into the action

$$\int d^D x \mathcal{L}(\phi^<) = \int d^D x' b^{-D} \left(\frac{1}{2} b^2 \left(\partial' \phi^<\right)^2 + \sum_n \left(g_n + \delta g_n\right) \left(\phi^<\right)^n + ... \right)$$

We can make this look like $\mathcal{L}$ again by rescaling the field variable: $b^{2-D} \left(\partial' \phi^<\right)^2 \equiv \left(\partial' \phi'\right)^2$ (*i.e.* $\phi' \equiv b^{\frac{1}{2}(2-D)} \phi^<$):

$$\int d^D x' \mathcal{L}(\phi^<) = \int d^D x' \left(\frac{1}{2} \left(\partial' \phi'\right)^2 + \sum_n \left(g_n + \delta g_n\right) b^{-D+\frac{n(D-2)}{2}} \left(\phi'\right)^n + ... \right)$$

So the end result is that integrating out a momentum shell of thickness $\delta\Lambda \equiv (1-b)\Lambda$ results in a change of the couplings to

$$g'_n = b^{\frac{n(D-2)}{2} - D} \left(g_n + \delta g_n\right) \ .$$

This procedure produces a *flow* on the space of actions.

Ignore the interaction corrections, δg_n, for a moment. Then, since $b < 1$, the couplings with $\frac{n(D-2)}{2} - D > 0$ get smaller and smaller as we integrate out more shells. If we are interested in only the longest-wavelength modes, we can ignore these terms. They are *irrelevant*. Couplings ('operators') with $\frac{n(D-2)}{2} - D < 0$ get bigger and are *relevant*.

The mass term has $n = 2$ and $(m')^2 = b^{-2} m^2$ is always relevant for any $D < \infty$. So far, the counting is the same as our naive dimensional analysis. That's because we left out the δL term! This term can make an important difference, even in perturbation theory, for the fate of marginal operators (such as ϕ^4 in $D = 4$), where the would-be-big tree-level term is agnostic about whether they grow or shrink in the IR.

Notice that starting from (3.5) we are assuming that the system has a rotation invariance in euclidean momentum. If one of those euclidean directions is time, this follows from Lorentz invariance. This simplifies the discussion. But for non-relativistic systems, it is often necessary to scale time differently from space. The relative scaling z in $\vec{x}' = b\vec{x}, t' = b^z t$ is called the *dynamical critical exponent*.

The definition of the beta function and of a fixed point theory is just as it was in the first lecture.

At this point we need to pick an example in which to include the interaction term.

3.3 An extended example: a complex scalar field

[R. Shankar, *Rev. Mod. Phys.* **66** (1994) 129]

Consider complex bosons in D dimensions. I am a little tired of a real scalar field, so instead we will study *two* real scalar fields $\phi = \phi_1 + \mathbf{i}\phi_2$. We can define this model, for example, on a euclidean lattice, by an action of the form

$$S[\phi, \phi^\star] = \frac{1}{2}\sum_{n,i} |\phi(n) - \phi(n+i)|^2 + \sum_n u_0 |\phi(n)|^4 \quad . \tag{3.9}$$

Here n labels sites of some (*e.g.* hypercubic) lattice and i labels the (8 in the 4d hypercubic case) links connecting neighboring sites. We'll call the lattice spacing $2\pi/\Lambda_1$. In terms of Fourier modes, this is

$$S[\phi, \phi^\star] = -\int_{|k|<\Lambda_0} \mathrm{đ}^D k \phi^\star(k) J(k) \phi(k) + S_{\rm int} \ .$$

For the hyper-cubic lattice, we get

$$J(k) = 2\left(\sum_{\mu=1}^{D} (\cos k_\mu - 1)\right) \stackrel{k\to 0}{\simeq} k^2 + ...$$

[31] The path integral is defined by

$$\begin{aligned} Z \equiv \int \underbrace{[\mathrm{d}\phi^\star \mathrm{d}\phi]_{|k|<\Lambda_0}} \quad e^{-S[\phi,\phi^\star]} \\ \equiv \prod_{|k|<\Lambda_0} \frac{\mathrm{dRe}\,\phi(k)\mathrm{dIm}\,\phi(k)}{\pi} \\ = \prod_{|k|<\Lambda_0} \frac{\mathrm{d}\phi^\star(k)\mathrm{d}\phi(k)}{2\pi\mathbf{i}} \end{aligned} \tag{3.10}$$

There is a $U(1)$ global symmetry which acts by

$$\phi(k) \to e^{\mathbf{i}\theta}\phi(k), \phi^\star(k) \to e^{-\mathbf{i}\theta}\phi^\star(k) \ . \tag{3.11}$$

With $u_0 = 0$, this is a bunch of gaussian integrals, and everything can be computed by Wick from the two-point function:

$$\langle \phi^\star(k_1)\phi(k_2)\rangle = (2\pi)^D \delta^D(k_1 - k_2)\frac{1}{k_1^2} = (2\pi)^D \delta^D(k_1 - k_2) G(k_1).$$

Although this gaussian model is trivial, we can still do the RG to it. (We will turn on the interactions in a moment.) An RG step has three ingredients, of which I've emphasized only two so far:

[31] Confession: the restriction on the momenta in the exact lattice model should be to a fundamental domain for the identification $k^\mu \equiv k^\mu + \Lambda_1^\mu$; I am going to replace this right away with a rotation-invariant cutoff on the magnitude $k^2 \equiv k^\mu k_\mu \leq \Lambda_0$ of the euclidean momentum. This is an unimportant lie for our purposes.

1. Integrate out the fast modes, *i.e.* $\phi^>$, with $|k| \in (\Lambda - \delta\Lambda, \Lambda)$. I will call $\Lambda - \delta\Lambda \equiv \Lambda/s$, and[32] $s > 1$, we will regard s as close to 1: $s - 1 \ll 1$.

$$Z = \int \prod_{0\le|k|\le\Lambda/s} \mathrm{d}\phi_<(k) \left(\int \prod_{\Lambda/s\le|k|\le\Lambda} \mathrm{d}\phi_>(k) e^{-\left(\underbrace{S_0[\phi^<] + S_0[\phi^>]}_{\text{quadratic}} + \underbrace{S_{\text{int}}[\phi^<, \phi^>]}_{\text{mixes fast and slow}} \right)} \right)$$
$$= \int [\mathrm{d}\phi^<] e^{-S_0[\phi^<]} \underbrace{\langle e^{-S_{\text{int}}[\phi^<, \phi^\cdot]} \rangle_{0,>}}_{\text{average over } \phi^>, \text{ with gaussian measure}} Z_{0,>} \tag{3.12}$$

   The factor of $Z_{0,>}$ is independent of $\phi^<$ and can be ignored.

2. Rescale momenta so that we may compare successive steps: $\tilde{k} \equiv sk$ lies in the same interval $|\tilde{k}| \in (0, \Lambda)$.

3. Are the actions $s(\phi) = r\phi^2 + u\phi^4$ and $\tilde{s}(\psi) = 4r\psi^2 + 16u\psi^4$ different? No: let $2\psi \equiv \phi$. We can resacle the field variable at each step:

$$\tilde{\phi}(\tilde{k}) \equiv \zeta^{-1} \phi_<(\tilde{k}/s).$$

   We will choose the 'wavefunction renormalization' factor ζ so that the kinetic terms are fixed.

RG for free field

If $S_{\text{int}} = 0$, then (3.12) gives

$$\tilde{S}[\phi_<] = \int_{|k|<\Lambda/s} \text{đ}^D k \phi^\star_<(k) k^2 \phi_<(k) \overset{\text{steps 2 and 3}}{=} s^{-D-2} \zeta^2 \int_{|\tilde{k}|<\Lambda} \tilde{\phi}^\star(\tilde{k}) \tilde{k}^2 \tilde{\phi}(\tilde{k}) \text{đ}^D \tilde{k} \ .$$

With $\zeta \equiv s^{\frac{D+2}{2}}$, the Gaussian action is a fixed point of the RG step:

$$\tilde{S}[\tilde{\phi}] = S[\phi] = S^\star.$$

Warning: the field $\phi(k)$ is not the same as the field $\phi(x)$ that we considered above! They are different by an integral over space or momenta: $\phi(x) = \int \text{đ}^D k \phi(k) e^{\mathbf{i}kx}$. So they scale differently. The result that $\zeta = s^{\frac{D+2}{2}}$ is perfectly consistent with our earlier result that $\phi(x)$ scales like $s^{\frac{2-D}{2}}$.

---

[32] I note that $s = 1/b$ from the previous subsection; sorry.

Now we consider perturbations. We'll only study those that preserve the symmetry (3.11). We can order them by their degree in ϕ. The first nontrivial case preserving the symmetry is

$$\delta S_2[\phi] = \int_{|k|<\Lambda} \mathrm{đ}^D k \phi^\star(k)\phi(k) r(k) \ .$$

Here $r(k)$ is a coupling *function*. If its position-space representation is local, it has a nice Taylor expansion about $k = 0$:

$$r(k) = \underbrace{r_0}_{\equiv m_0^2} + k^2 r_2 + ...$$

(I also assumed rotation invariance.) The same manipulation as above gives

$$\widetilde{\delta S_2}[\tilde\phi(\tilde k)] = s^{-D+\frac{D+2}{2}2=2} \int_{|\tilde k|<\Lambda} \tilde\phi^\star(\tilde k)\tilde r(\tilde k)\tilde\phi(\tilde k) \mathrm{đ}^D\tilde k$$

with $\tilde r(\tilde k) = s^2 r(\tilde k/s)$, so that

$$\underbrace{\tilde r_0 = s^2 r_0}_{\text{relevant}} \ , \qquad \underbrace{\tilde r_2 = s^0 r_2}_{\text{marginal by design}} \ , \quad \underbrace{\tilde r_4 = s^{-2} r_4}_{\text{irrelevant}} ...$$

Quartic perturbation

$$\delta S_4 = S_{\text{int}} = \int_\Lambda \phi^\star(4)\phi^\star(3)\phi(2)\phi(1) u(4321)$$

This is some shorthand notation for

$$\delta S_4 = S_{\text{int}} = \frac{1}{(2!)^2} \int \prod_{i=1}^4 \mathrm{đ}^D k_i (2\pi)^D \delta^D(k_4 + k_3 - k_2 - k_1)\phi^\star(k_4)\phi^\star(k_3)\phi(k_2)\phi(k_1) u(k_4 k_3 k_2 k_1) \ .$$

The delta function maintains translation invariance in real space. Here $u(4321)$ is some general function, but only the bit with $u(4321) = u(3421) = u(4312)$ matters. This interaction couples the fast and slow modes. We need to evaluate

$$e^{-\tilde S[\phi_<]} = e^{-S_0[\phi_<]} \langle e^{-\delta S[\phi_<,\phi_>]} \rangle_{0,>} \ .$$

A tool at our disposal is the *cumulant expansion*:

$$\langle e^{-\Omega} \rangle = e^{-\langle\Omega\rangle + \frac{1}{2}\left(\langle\Omega^2\rangle - \langle\Omega\rangle^2\right) + ...}$$

So

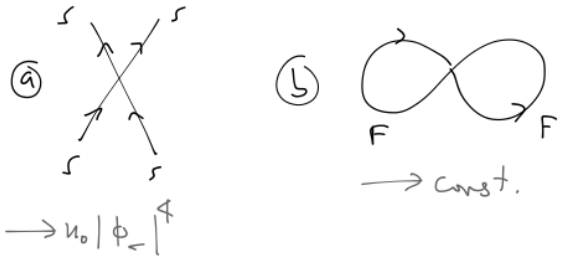
$$\widetilde{\delta S} = \underbrace{\langle \delta S\rangle_{>,0}}_{\sim u_0} - \frac{1}{2} \underbrace{\left(\langle \delta S^2\rangle_{>,0} - \langle\delta S\rangle^2_{>,0}\right)}_{\sim u_0^2} + ...$$

So this expansion is a perturbative expansion in u_0.

First the first term ($\sim u_0$):

$$\langle \delta S\rangle_{>,0} = \frac{1}{(2!)^2} \int_{|k|<\Lambda} \langle (\phi_< + \phi_>)^\star_4 (\phi_< + \phi_>)^\star_3 (\phi_< + \phi_>)_2 (\phi_< + \phi_>)_1 u(4321)\rangle_{>,0}$$

This is made of 16 terms which can be decomposed as follows, and illustrated by the Feynman diagrams at right. These Feynman diagrams are just like the usual ones with the important difference that the loop momenta only run over the shell from $|k| = \Lambda/s$ to $|k| = \Lambda$. They all have a single 4-point vertex; the only allowed external lines are the slow modes.

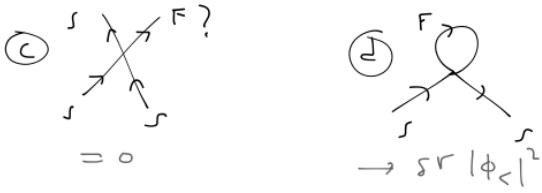


(a) 1 diagram with all external lines being slow modes. This gives the tree level interaction term for the slow modes.

(b) 1 diagram with only fast modes involved in the vertex. This contributes to the irrelevant constant $Z_{0,>}$.

(c) 8 diagrams with an odd number of fast modes; these all vanish by the usual Wick business.

(d) 6 diagrams with 2 slow 2 fast. The fast modes must be contracted and this makes a loop. The arrows (representing the flow of the U(1) charge) must work out to allow nonzero contractions (recall that $\langle\phi\phi\rangle = 0$ by charge conservation).



So the only interesting ones are diagrams of type (d), which give

$$\begin{aligned}\widetilde{\delta S_2}(\phi_<) &= \frac{u_0}{(2!)^2}\int_{|k|<\Lambda} \langle(\phi^\star_>(4)\phi^\star_<(3) + \phi^\star_>(3)\phi^\star_<(4))(\phi_>(2)\phi_<(1) + \phi_>(1)\phi_<(2))\rangle_{0,>} \\ &= u_0\int_{|k|<\Lambda/s} \text{đ}^D k\phi^\star_<(k)\phi_<(k) \quad \cdot \underbrace{\int_{\Lambda/s}^{\Lambda} \text{đ}^D p \frac{1}{p^2}}_{= \frac{\Omega_{D-1}}{(2\pi)^D}\int_{\Lambda/s}^{\Lambda} k^{D-3}dk} \\ &\overset{D=4}{=} \frac{2\pi^2}{(2\pi)^4}\frac{\Lambda^2}{2}\left(1 - s^{-2}\right).\end{aligned} \tag{3.13}$$

$$\widetilde{\delta S}_2[\tilde\phi_<(\tilde k)] = u_0 s^2 \int_{|\tilde k|<\Lambda} \text{đ}^4 k \tilde\phi^\star(\tilde k)\tilde\phi(\tilde k)\frac{\Lambda^2}{16\pi^2}(1 - s^{-2}).$$

$$\boxed{\delta r_0 = \frac{u_0 \Lambda^2}{16\pi^2}(s^2-1)}.$$

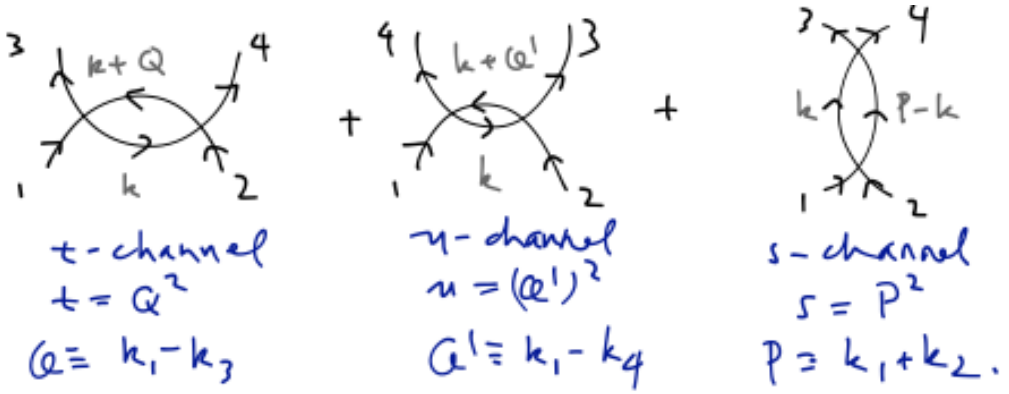
The correction to the mass is of order the cutoff.

In D dimensions, we get instead

$$\delta r_0 = \frac{\Omega_{D-1}}{(2\pi)^D} u_0 \Lambda^{D-2}(s^2 - s^{4-D}).$$

The next term in the cumulant expansion

Now for the $\mathcal{O}(u_0^2)$ term in $\widetilde{\delta S}$. The diagrammatic representation of $\frac{1}{2}\left(\langle \delta S^2\rangle - \langle \delta S\rangle^2\right)$ is: all connected diagrams containing *two* 4-point vertices, with only external slow lines. The second term cancels all disconnected diagrams. Diagrammatically, these are:



These correct the quartic coupling $u = u_0 + u_1 k^2 + \ldots$. We care about the sign of δu_0, because in $D = 4$ it is marginal. Even small corrections will make a big difference.

$$\tilde{u}(\tilde{k}_4, \ldots \tilde{k}_1) = u_0 - u_0^2 \underbrace{\int_{\Lambda/s}^{\Lambda} \text{đ}^D k}_{\equiv \int_{d\Lambda}} \left(\frac{1}{k^2 |k - (\tilde{k}_3 - \tilde{k}_1)/s|^2} + \frac{1}{k^2 |k - (\tilde{k}_4 - \tilde{k}_1)/s|^2} + \frac{1}{2} \frac{1}{k^2 | - k - (\tilde{k}_1 + \tilde{k}_2)/s|^2} \right)$$

Note the symmetry factor in the s-channel diagram, which you can see directly from the cumulant expression.

The most interesting part of this expression is the correction to u_0, which is when we set the external momenta to zero:

$$\tilde{u}(k=0) = \tilde{u}_0 = u_0 - u_0^2 \frac{5}{2} \underbrace{\int_{d\Lambda} \frac{k^3 dk}{k^4}}_{= \log s} \cdot \underbrace{\frac{\Omega_3}{(2\pi)^4}}_{= \frac{1}{16\pi^2}}.$$

Let $\Lambda(s) \equiv \Lambda_0/s \equiv \Lambda_0 e^{-\ell}$ so $s = e^{\ell}, \ell = \log \Lambda_0/\Lambda$ and $\Lambda \frac{d}{d\Lambda} = s\partial_s = \partial_\ell$. Large ℓ is the IR.

$$\begin{cases} \frac{du_0}{d\ell} = -\frac{5}{16\pi^2} u_0^2 \equiv -b u_0^2 \\ \frac{d\hat{r}_0}{d\ell} = 2\hat{r}_0 + \frac{u_0}{16\pi^2} = 2r_0 + a u_0 \end{cases} . \tag{3.14}$$

Here $a, b > 0$ are constants, and $\hat{r}_0 \equiv r_0 \Lambda^2$ is the mass^2 in units of the cutoff. (Note that the usual high-energy definition of the beta function has the opposite sign, $\frac{dg}{d\ell} = -\beta_g$.)

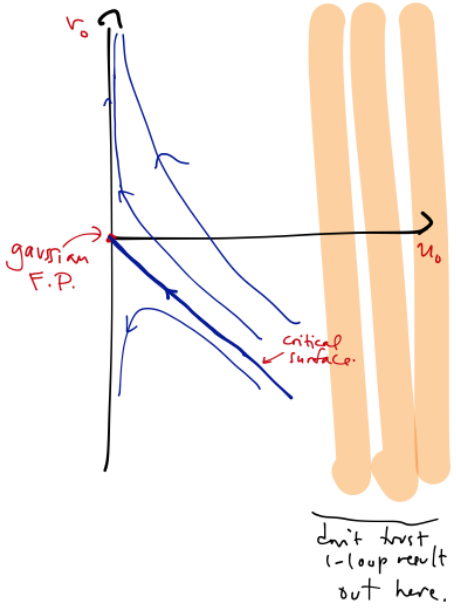
[End of Lecture 10]

These equations can be solved in terms of two initial conditions:

$$u_0(\ell) = \frac{u_0(0)}{1 + b u_0(0) \ell} \overset{\ell \to \infty, u_0(0) > 0}{\sim} \frac{1}{\ell} = \frac{1}{\log \Lambda_0/\Lambda} \to 0.$$

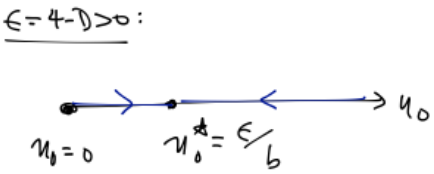
u_0 is a marginally irrelevant perturbation of the gaussian fixed point. This theory is not asymptotically free[33] The phase diagram is at right. There's just the one fixed Gaussian point. Notice that it's not true that an arbitrary small u_0 added to the gaussian FP runs back to the gaussian FP. r_0 runs too:

$$r_0(\ell) = e^{2\ell} \left[r_0(0) + \int_0^\ell e^{-2\ell'} \frac{a u_0(0)}{1 + b u_0(0) \ell'} \mathrm{d}\ell' \right] .$$



There is a curve of choices of initial data in $(u_0(0), r_0(0))$ which ends up at the origin – it's when the thing in brackets vanishes; for small u_0, this is the line $r_0(0) = -\frac{a}{2} u_0(0)$.

Following Wilson and Fisher, it is an extremely good idea to consider dimensions other than 4, $D \equiv 4 - \epsilon$. Now the quartic interaction is no longer marginal at tree level, but scales like s^ϵ. The RG equation is modified to



$$\frac{du_0}{dt} = \epsilon u_0 - b u_0^2 \quad . \tag{3.15}$$

For $\epsilon > 0$ ($D < 4$) there is another fixed point at $u_0^\star = \epsilon/b > 0$. And in fact the Gaussian FP is unstable, and this *Wilson-Fisher fixed point* is the stable one in the IR (see fig at right, which is drawn along the critical surface leading to $r_0(\infty) = 0$.). This situation allows one to calculate (universal) critical exponents at the fixed point in an expansion in ϵ.

As $\epsilon \to 0$, the two fixed points coalesce.

[33] This statement was for $u_0(0) > 0$. For $u_0(0) < 0$, it is AF (this was an observation of Symanzik, before the study of Yang-Mills), but seems likely to be unstable. For an interesting claim to the contrary, see here if you are feeling brave. It would be nice to know for sure.

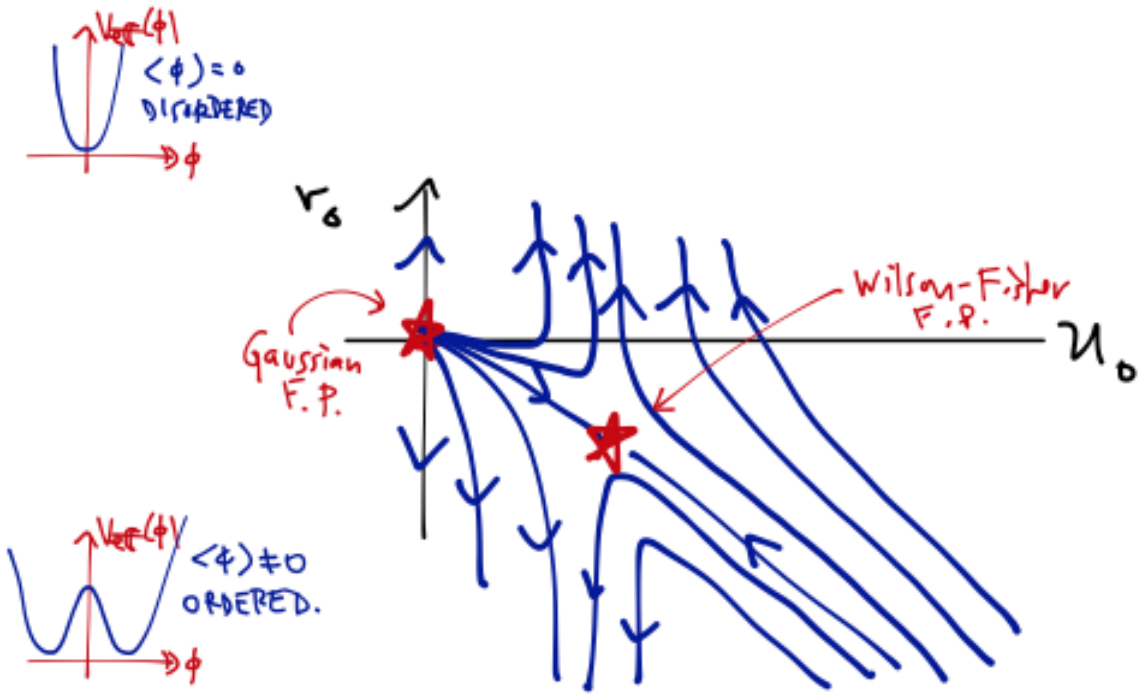


Figure 11: The ϕ^4 phase diagram. If $r_0(\ell = \infty) > 0$, the effective potential for the uniform 'magnetization' has a minimum at the origin; this is the disordered phase, where there is no magnetization. If $r_0(\ell = \infty) = V''_{\text{eff}} < 0$, the effective potential has minima away from the origin, and the groundstate breaks the symmetry (here $\phi \to e^{i\theta}\phi$); this is the ordered phase.

The W-F fixed point describes a continuous phase transition between ordered and disordered phases. An external variable (roughly r_0) must be tuned to reach the phase transition. A physical realization of this is the following: think of our euclidean path integral as a thermal partition function at temperature $1/\beta$:

$$Z = \int [D\phi] e^{-\beta H[\phi]} \ ;$$

here we are integrating over thermal fluctuations of classical fields. WLOG, we can choose normalize our fields so that the coefficient β determines r_0. The critical value of r_0 then realizes the critical temperature at which this system goes from a high-temperature disordered phase to a low-temperature ordered phase. For this kind of application, $D \leq 3$ is most interesting physically. We will see that the ϵ expansion about $D = 4$ is nevertheless quite useful.

You could ask me what it means for the number of dimensions D to be not an integer. One correct answer is that we have constructed various well-defined functions of continuous

D simply by keeping D arbitrary; basically all we need to know is the volume of a D-sphere for continuous D. You have likely seen this defined in Peskin, via Euler Gamma functions:

$$\left(\frac{\pi}{a}\right)^{D/2} = \int d^D x e^{-a\vec{x}^2} = \Omega_{D-1} \int_0^\infty x^{D-1} dx e^{-ax^2} = \frac{1}{2} a^{-\frac{D}{2}} \Gamma\left(\frac{D}{2}\right) \Omega_{D-1} \tag{3.16}$$

defines Ω_{D-1} for general D.

An also-correct answer that some people (*e.g.* me) find more satisfying is is the following. Suppose we can define our QFT by a discrete model, defined on a discretized space (like in (3.9)). Then we can also put the model on a graph whose *fractal dimension* is not an integer. Evidence that this is a physical realization of QFT in non-integer dimensions is given in [Gefen-Meir-Mandelbrot-Aharony] and [Gefen-Mandelbrot-Aharony].

3.3.1 Important lessons

- Elimination of modes does not introduce new singularities into the couplings. At each step of the RG, we integrate out a finite-width shell in momentum space – we are doing integrals which are convergent in the infrared and ultraviolet.

- The RG plays nicely with symmetries. In particular any symmetry *of the regulated model* is a symmetry of the long-wavelength effective action. The extra qualifier about the regulated model[34] is important because some symmetries of continuum classical field theories cannot be realized as symmetries of well-defined quantum field theories. We will discuss this phenomenon, called *anomalies*, in the near future.

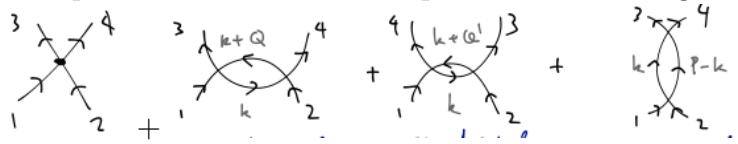
- Some people conclude from the field theory calculation of the ϕ^4 beta function that ϕ^4 theory "does not exist" or "is trivial", in the sense that *if we demand that this description is valid up to arbitrarily short distances*, we would need to pick $u(\Lambda = \infty) = \infty$ in order to get a finite interaction strength at long wavelengths. You can now see that this is a ridiculous conclusion. Obviously the theory exists in a useful sense. It can easily be defined at short distances (for example) in terms of the lattice model we wrote at the beginning of this subsection. Similar statements apply to QED.

- The corrections to the mass of the scalar field are of order of the cutoff. This makes it hard to understand how you could arrive in the IR and find that an interacting scalar field has a mass which is much smaller than the cutoff. Yet, there seems to be a Higgs boson with $m \simeq 125$ GeV, and no cutoff on the Standard Model in sight. This is a mystery.

- As Tony Zee says, a better name than 'renormalization group' would be 'the trick of doing the path integral a little at a time'.

[34] Thanks to Brian Shotwell for emphasizing this important point.

3.3.2 Comparison with renormalization by counterterms

Is this procedure the same as 'renormalization' in the high-energy physics sense of sweeping divergences under the rug of bare couplings? Suppose we impose the renormalization condition that $\Gamma_4(k_4...k_1) \equiv \Gamma(4321)$, the 1PI 4-point vertex, is cutoff independent. Its leading contributions come from the diagrams: 

(where now they denote amputated amplitudes, and the integrals run over all momenta up to the cutoff). Clearly there is already a big similarity. In more detail, this is

$$\Gamma(4321) = u_0 - u_0^2 \int_0^\Lambda \text{đ}^D k \left(\frac{1}{(k^2+r_0)(|k+k_3-k_1|^2+r_0)} + \frac{1}{(k^2+r_0)(|k+k_4-k_1|^2+r_0)} + \frac{1}{2}\frac{1}{(k^2+r_0)(|-k+k_1+k_2|^2+r_0)} \right)$$

And in particular, the bit that matters is

$$\Gamma(0000) = u_0 - u_0^2 \frac{5}{32\pi^2} \log \frac{\Lambda^2}{r_0}.$$

Demanding that this be independent of the cutoff $\Lambda = e^{-\ell}\Lambda_0$,

$$0 = \partial_\ell \left(\Gamma(0000)\right) = -\Lambda \frac{d}{d\Lambda}\Gamma(0000)$$

gives

$$0 = \frac{du_0}{d\ell} + \frac{5}{16\pi^2}u_0^2 + \mathcal{O}(u_0^3)$$

$$\implies \boxed{\beta_{u_0} = -\frac{5}{16\pi^2}u_0^2} + \mathcal{O}(u_0^3)$$

as before. (The bit that would come from $\partial_\ell u_0^2$ in the second term is of order u_0^3 and so of the order of things we are already neglecting.)

I leave it to you to show that the flow for r_0 that results from demanding that $\langle \phi(k)\phi^\star(k)\rangle$ have a pole at $k^2 = -m^2$ (with m independent of the cutoff) gives the same flow we found above.

It is worth noting that although the continuum field theory perspective with counterterms is less philosophically satisfying, it is often easier for actual calculations than integrating momentum shells.

3.3.3 Comment on critical exponents

[Zinn-Justin, chapter 25, Peskin, chapter 12.5, Stone, chapter 16, and the original Kogut-Wilson]

Recall that the Landau-Ginzburg mean field theory made a (wrong) prediction for the critical exponents at the Ising transition:

$$\langle M \rangle \sim (T_c - T)^\beta \qquad \text{for } T < T_c, \quad \xi \sim (T_c - T)^{-\nu}$$

with $\beta_{MFT} = \frac{1}{2}$, $\nu_{MFT} = \frac{1}{2}$. This answer was wrong (*e.g.* for the Ising transition in (euclidean) $D = 3$, which describes uniaxial magnets (spin is ± 1) or the liquid-gas critical point) because it simply ignored the effects of fluctuations of the modes of nonzero wavelength, *i.e.* the δL bit in (3.7). I emphasize that these numbers are worth getting right because they are universal – they are properties of a fixed point, which are completely independent of any microscopic details.

Now that we have learned to include the effects of fluctuations at all length scales on the long-wavelength physics, we can do better. We've done a calculation which includes fluctuations at the transition for an XY magnet (the spin has two components, and a U(1) symmetry that rotates them into each other), and is also relevant to certain systems of bosons with conserved particle number. The mean field theory prediction for the exponents is the same as for the Ising case (recall that we did the calculation for a magnetization field with an arbitrary number N of components, and in fact the mean field theory prediction is independent of $N \geq 1$; we will study the case of general N next).

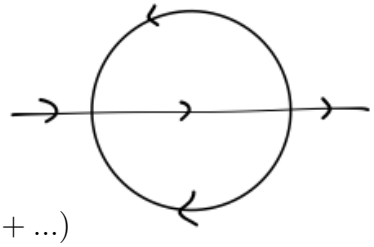
In general there are many *scaling relations* between various critical exponents, which can be understood beginning from the effective action. So not all of them are independent. For illustration, we will briefly discuss two independent exponents.

Order parameter exponent, η. The simplest critical exponent to understand from what we've done so far is η, the exponent associated with the anomalous dimension of the field ϕ itself. (It is not the easiest to actually calculate, however.) This is defined in terms of the (momentum-space) 1PI two-point function of ϕ as

$$\Gamma_2(p) = -W_2(p)^{-1} \overset{\xi^{-1} \ll p \ll \Lambda}{\simeq} \left(\frac{p}{\Lambda}\right)^{2-\eta}$$

where ξ is the correlation length and Λ is the UV cutoff. This looks a bit crazy – at nonzero η, the full propagator has a weird power-law singularity instead of a $\frac{1}{p^2 - m^2}$, and in position space it is a power law $G_2(x) \sim \frac{1}{|x|^{D-2+\eta}}$, instead of an exponential decay. You have seen an example of this already in the form of the operator $e^{\mathbf{i}\alpha X}$ the massless scalar field X in 1+1 dimensions.

But how can this happen in perturbation theory? Consider physics near the gaussian fixed point, where η must be small, in which case we can expand:



$$\Gamma_2(p) \overset{\xi^{-1} \ll p \ll \Lambda, \eta \ll 1}{\simeq} \left(\frac{p}{\Lambda}\right)^2 \left(e^{-\eta \log(p/\Lambda)}\right) = \left(\frac{p}{\Lambda}\right)^2 (1 - \eta \log(p/\Lambda) + ...)$$

In the ϕ^4 theory, $\eta = 0$ at one loop. The leading correction to η comes from the 'sunrise' (or 'eyeball') diagram at right, at two loops. So in this model, $\eta \sim g_\star^2 \sim \epsilon^2$. Recall that $\Gamma_2(p)$ is the 1PI momentum space 2-point vertex, *i.e.* the kinetic operator. We can interpret a nonzero η as saying that the dimension of ϕ, which in the free theory was $\Delta_0 = \frac{2-D}{2}$, has been modified by the interactions to $\Delta = \frac{2-D}{2} - \eta/2$. $\eta/2$ is the *anomalous dimension* of ϕ. Quantum mechanics violates (naive) dimensional analysis; it must, since it violates classical scale invariance. Of course (slightly more sophisticated) dimensional analysis is still true – the extra length scale is the UV cutoff, or some other scale involved in the renormalization procedure.

[End of Lecture 11]

Correlation length exponent, ν. Returning to the correlation length exponent ν, we can proceed as follows. First we relate the scaling of the correlation length to the scaling behavior of the relevant perturbation that takes us away from from the fixed point. The latter we will evaluate subsequently in our example. (There is actually an easier way to do this, which we discuss in §3.3.4, but this will be instructive.)

The correlation length is the length scale above which the relevant perturbation gets big and cuts off the critical fluctuations of the fixed point. As the actual fixed point is approached, this never happens and ξ diverges at a rate determined by the exponent ν. Suppose we begin our RG procedure with a perturbation of a fixed point Hamiltonian by a relevant operator $\mathcal{O}$:

$$H(\xi_1) = H_\star + \delta_1 \mathcal{O} \quad .$$

Under a step of the RG, $\xi_1 \to s^{-1}\xi_1$, $\delta_1 \to s^\Delta \delta_1$, where I have defined Δ to be the scaling dimension of the operator $\mathcal{O}$. Then after N steps, $\delta = s^{N\Delta}\delta_1, \xi = s^{-N}\xi_1$. Eliminating s^N from these equations we get the relation

$$\xi = \xi_1 \left(\frac{\delta}{\delta_1}\right)^{-\frac{1}{\Delta}} \tag{3.17}$$

which is the definition of the correlation length exponent ν, and we conclude that $\nu = \frac{1}{\Delta}$.

Here is a better way to think about this. At the critical point, the two-point function of the order parameter $G(x) \equiv \langle \phi(x)\phi(0)\rangle$ is a power law in x, specified by η. Away from the critical point, there is another scale, namely the size of the perturbation – the deviation

of the microscopic knob δ_0 from its critical value, such as $T - T_c$. Therefore, dimensional analysis says that $G(x)$ takes the form

$$G(x) = \frac{1}{|x|^{D-2}} \left(\frac{1}{|x|/a}\right)^{\eta} \Phi\left(|x|\delta_0^{1/\Delta}\right)$$

where the argument of the *scaling function* Φ is dimensionless. (I emphasized that the lattice spacing makes up the extra engineering dimensions to allow for an anomalous dimension of the field.) When $x \ll$ all other length scales, $G(x)$ should decay exponentially, and the decay length must then be $\xi \sim \delta_0^{-\frac{1}{\Delta}}$ which says $\nu = \frac{1}{\Delta}$.

In the case of ϕ^4 theory, r_0 is the parameter that an experimentalist must carefully tune to access the critical point (what I just called δ_0) – it is the coefficient of the relevant operator $\mathcal{O} = |\phi|^2$ which takes us away from the critical point; it plays the role of $T - T_c$.

At the free fixed point the dimension of $|\phi|^2$ is just twice that of ϕ, and we get $\nu^{-1} = \Delta^{(0)}_{|\phi|^2} = 2\frac{D-2}{2} = D-2$. At the nontrivial fixed point, however, notice that $|\phi|^2$ is a composite operator in an interacting field theory. In particular, its scaling dimension is not just twice that of ϕ! This requires a bit of a digression.

Renormalization of composite operators.

[Peskin §12.4] Perturbing the Wilson-Fisher fixed point by this seemingly-innocuous quadratic operator, is then no longer quite so innocent. In particular, we must define what we mean by the operator $|\phi|^2$! One way to define it (from the counterterms point of view, now, following Peskin and Zinn-Justin) is by adding an extra renormalization condition[35]. We can define the normalization of the composite operator $\mathcal{O}(k) \equiv |\phi|^2(k)$ by the condition that its (amputated) 3-point function gives

$$\langle \mathcal{O}_\Lambda(k)\phi(p)\phi^\star(q)\rangle = 1 \quad \text{at } p^2 = q^2 = k^2 = -\Lambda^2 \ .$$

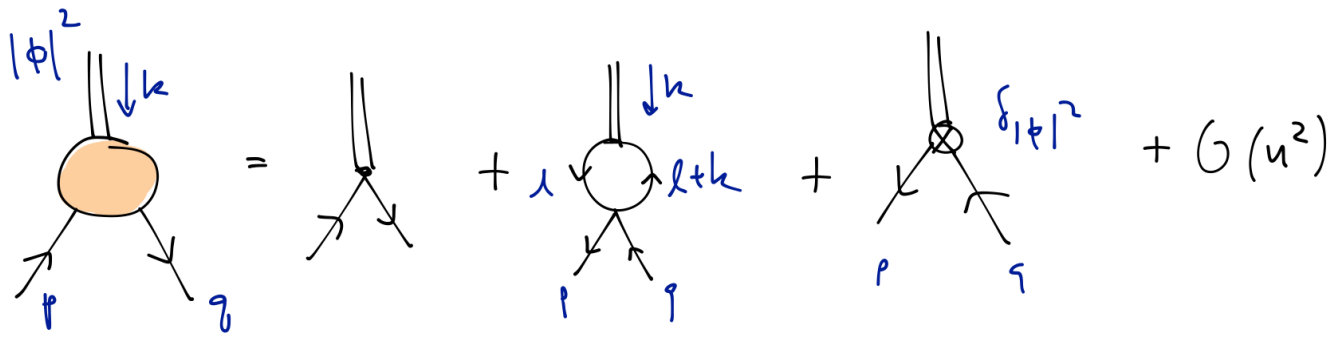
The subscript on $\mathcal{O}_\Lambda(k)$ is to emphasize that its (multiplicative) normalization is defined by a renormalization condition at scale (spacelike momentum) Λ. Just like for the 'elementary fields', we can define a wavefunction renormalization factor:

$$\mathcal{O}_\Lambda \equiv Z_{\mathcal{O}}^{-1}(\Lambda)\mathcal{O}_\infty$$

where $\mathcal{O}_\infty \equiv \phi^\star\phi$ is the bare product of fields.

We can represent the implementation of this prescription diagramatically. In the diagram above, the double line is a new kind of thing – it represents the insertion of $\mathcal{O}_\Lambda$. The vertex

[35] Note that various factors differ from Peskin's discussion in §12.4 because I am discussing a complex field $\phi \neq \phi^\star$; this changes the symmetry factors.



where it meets the two ϕ lines is *not* the 4-point vertex associated with the interaction – two ϕs can turn into two ϕs even in the free theory. The one-loop, 1PI correction to this correlator is (the second diagram on the RHS of the figure)[36]

$$(-u_0)\int_0^\infty \mathrm{d}^D\ell \frac{1}{\ell^2}\frac{1}{(k+\ell)^2} = -u_0\frac{c}{k^{4-D}}$$

where c is a number (I think it is $c = \frac{\Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^2}$) and we know the k dependence of the integral by scaling. If you like, I am using dimensional regularization here, thinking of the answer as an analytic function of D.

Imposing the renormalization condition requires us to add a counterterm diagram (part of the definition of $|\phi|^2$, indicated by the $\otimes$ in the diagrams above) which adds

$$Z_{\mathcal{O}}^{-1}(\Lambda) - 1 \equiv \delta_{|\phi|^2} = \frac{u_0 c}{\Lambda^{4-D}} \ .$$

We can infer the dimension of (the well-defined) $|\phi|^2_\Lambda$ by writing a renormalization group equation for our 3-point function

$$G^{(2;1)} \equiv \langle |\phi|^2_\Lambda(k)\phi(p)\phi^\star(q)\rangle.$$

$$0 = \left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(u)\frac{\partial}{\partial u} + n\gamma_\phi + \gamma_{\mathcal{O}}\right) G^{(n;1)} \ .$$

This (Callan-Symanzik equation) is the demand that physics is independent of the cutoff. $\gamma_{\mathcal{O}} \equiv \Lambda\frac{\partial}{\partial\Lambda}\log Z_{\mathcal{O}}(\Lambda)$ is the *anomalous dimension* of the operator $\mathcal{O}$, roughly the addition to its engineering dimension coming from the interactions (similarly $\gamma_\phi \equiv \Lambda\frac{\partial}{\partial\Lambda}\log Z_\phi(\Lambda)$). To leading order in u_0, we learn that

$$\gamma_{\mathcal{O}} = \Lambda\frac{\partial}{\partial\Lambda}\left(-\delta_{\mathcal{O}} + \frac{n}{2}\delta_Z\right)$$

[36] At higher order in u_0, the wavefunction renormalization of ϕ will also contribute to the renormalization of $|\phi|^2$.

which for our example with $n = 2$ gives the anomalous dimension of $|\phi|^2$ to be (just the first term to this order since δ_Z is the wavefunction renormalization of ϕ, which as we discussed first happens at $\mathcal{O}(u_0^2)$)

$$\gamma_{|\phi|^2} = \frac{2u_0}{16\pi^2}.$$

Plugging in numbers, we get, at the $N = 2$ (XY) Wilson-Fisher fixed point at $u_0^\star = \epsilon/b$,

$$\nu = \frac{1}{\Delta_{|\phi|^2}} = \frac{1}{2 - \gamma_{|\phi|^2}} \overset{D=4-\epsilon}{=} \frac{1}{2 - \frac{2u_0^\star}{16\pi^2}} = \frac{1}{2 - 2\frac{16\pi^2}{5}\frac{\epsilon}{16\pi^2}} = \frac{1}{2 - \frac{2\epsilon}{5}} \ .$$

(for the Ising fixed point the 5/2 would be replaced by $\frac{N+8}{N+2}|_{N=1} = 3$).

It is rather amazing how well one can do at estimating the answers for $D = 3$ by expanding in $\epsilon = 4 - D$, keeping the leading order correction, and setting $\epsilon = 1$. The answer from experiment and the lattice is $\nu_{D=3,N=2} \simeq 0.67$, while we find $\nu_{\epsilon=1,N=2} \simeq 0.63$. It is better than mean field theory for sure. You can do even better by Padé approximating the ϵ expansion.

One final comment about defining and renormalizing composite operators: if there are multiple operators with the same quantum numbers and the same scaling dimension, they will *mix* under renormalization. That is, in order to obtain cutoff-independent correlators of these operators, their definition must be of the form

$$\mathcal{O}^i_\Lambda = \left(Z^{-1}(\Lambda)\right)_{ij} \mathcal{O}^j_\infty$$

– there is a wavefunction renormalization *matrix*, and a matrix of anomalous dimensions

$$\gamma_{ij} = -\Lambda\partial_\Lambda \log\left(Z^{-1}(\Lambda)\right)_{ij}.$$

Operator mixing is really just the statement that correlation functions like $\langle \mathcal{O}^i \mathcal{O}^j \rangle$ are nonzero.

3.3.4 Once more with feeling (and an arbitrary number of components)

I've decided to skip this subsection in lecture. You may find it useful for problem set 5.

[Kardar, *Fields*, §5.5, 5.6] Let's derive the RG for ϕ^4 theory again, with a number of improvements:

- Instead of two components, we'll do N component fields, with $\mathcal{U} = \int d^D x u_0 \left(\phi^a \phi^a\right)^2$ (repeated indices are summed, $a = 1..N$).
- We'll show that it's not actually necessary to ever do any momentum integrals to derive the RG equations.
- We'll keep the mass perturbation in the discussion at each step; this lets us do the following:
- We'll show how to get the correlation length exponent without that annoying discussion of composite operators. (Which was still worth doing because in other contexts it is not avoidable.)

We'll now assume O(N) symmetry, $\phi^a \to R^a_b \phi^b$, with $R^t R = \mathbb{1}_{N\times N}$, and perturb about the gaussian fixed point with (euclidean) action

$$S_0[\phi] = \int_0^\Lambda \text{đ}^D k \underbrace{\phi^a(k)\phi^a(-k)}_{\equiv |\phi|^2(k)} \frac{1}{2}\left(r_0 + r_2 k^2\right).$$

The coefficient r_2 of the kinetic term is a book-keeping device that we may set to 1 if we choose. Again we break up our fields into slow and fast, and integrate out the fast modes:

$$Z_\Lambda = \int [D\phi_<] e^{-\int_0^{\Lambda/s} \text{đ}^D k |\phi_<(k)|^2 \left(\frac{r_0 + r_2 k^2}{2}\right)} Z_{0,>} \langle e^{-\mathcal{U}[\phi_<, \phi_>]} \rangle_{0,>} .$$

Again the $\langle ... \rangle_{0,>}$ means averaging over the fast modes with their Gaussian measure, and $Z_{0,>}$ is an irrelevant normalization factor, independent of the objects of our fascination, the slow modes $\phi_<$. With N components we do Wick contractions using

$$\langle \phi^a_>(q_1) \phi^b_>(q_2) \rangle_{0,>} = \frac{\delta^{ab} \not{\delta}(q_1 + q_2)}{r_0 + q_1^2 r_2}.$$

I've defined $\not{\delta}(q) \equiv (2\pi)^D \delta^D(q)$. Notice that we are now going to keep the mass perturbation r_0 in the discussion at each step. Again

$$\log \langle e^{-\mathcal{U}} \rangle_{0,>} = -\underbrace{\langle \mathcal{U} \rangle_{0,>}}_{1} + \underbrace{\frac{1}{2}\left(\langle \mathcal{U}^2 \rangle_{0,>} - \langle \mathcal{U} \rangle^2_{0,>}\right)}_{2}$$

$$1 = \langle \mathcal{U}[\phi_<, \phi_>] \rangle_{0,>} = u_0 \int \prod_{i=1}^4 \text{đ}^D k_i \not{\delta}(\sum_i k_i) \langle \prod_i (\phi_< + \phi_>)_i \rangle_{0,>}$$

Diagramatically, these 16 terms decompose as in Fig. 12.

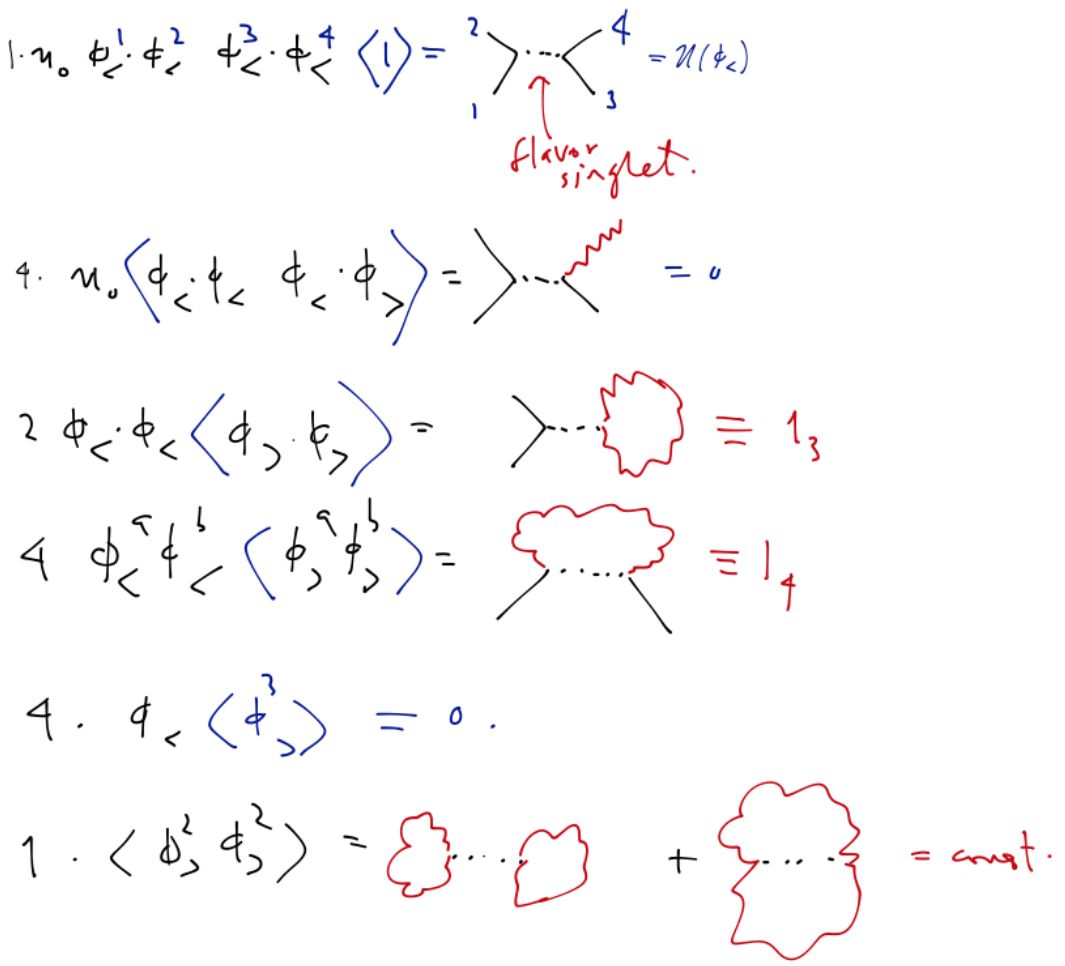


Figure 12: 1st order corrections from the quartic perturbation of the Gaussian fixed point of the O(N) model. Wiggly lines denote propagation of fast modes $\phi_>$, straight lines denote (external) slow modes $\phi_<$. A further refinement of the notation is that we split apart the 4-point vertex to indicate how the flavor indices are contracted; the dotted line denotes a direction in which no flavor flows, i.e. it represents a coupling between the two flavor singlets, $\phi^a\phi^a$ and $\phi^b\phi^b$. The numbers at left are multiplicities with which these diagrams appear. (The relative factor of 2 between 1_3 and 1_4 can be understood as arising from the fact that 1_3 has a symmetry which exchanges the fast lines but not the slow lines, while 1_4 does not.) Notice that closed loops of the wiggly lines represent factors of N, since we must sum over which flavor is propagating in the loop – the flavor of a field running in a closed loop is not determined by the external lines, just like the momentum.

The interesting terms are

$$1_3 = -u_0 \underbrace{2}_{\text{symmetry}} \underbrace{N}_{=\delta^{aa}} \int_0^{\Lambda/s} \mathrm{d}^D k |\phi_<(k)|^2 \int_{\Lambda/s}^{\Lambda} \mathrm{d}^D q \frac{1}{r_0 + r_2 q^2}$$

$$1_4 = \frac{4 \cdot 1}{2 \cdot N} 1_3$$

has a bigger symmetry factor but no closed flavor index loop. The result through $\mathcal{O}(u)$ is then

$$\boxed{r_0 \to r_0 + \delta r_0 = r_0 + 4u_0(N+2)\int_{\Lambda/s}^{\Lambda} \text{đ}^D q \frac{1}{r_0 + r_2 q^2} + \mathcal{O}(u_0^2)} \,.$$

r_2 and u are unchanged. RG step ingredients 2 (rescaling: $\tilde{q} \equiv sq$) and 3 (renormalizing: $\tilde{\phi} \equiv \zeta^{-1}\phi_<$) allow us to restore the original action; we can choose $\zeta = s^{1+D/2}$ to keep $\tilde{r}_2 = r_2$.

The second-order-in-u_0 terms are displayed in Fig. 13. The interesting part of the second

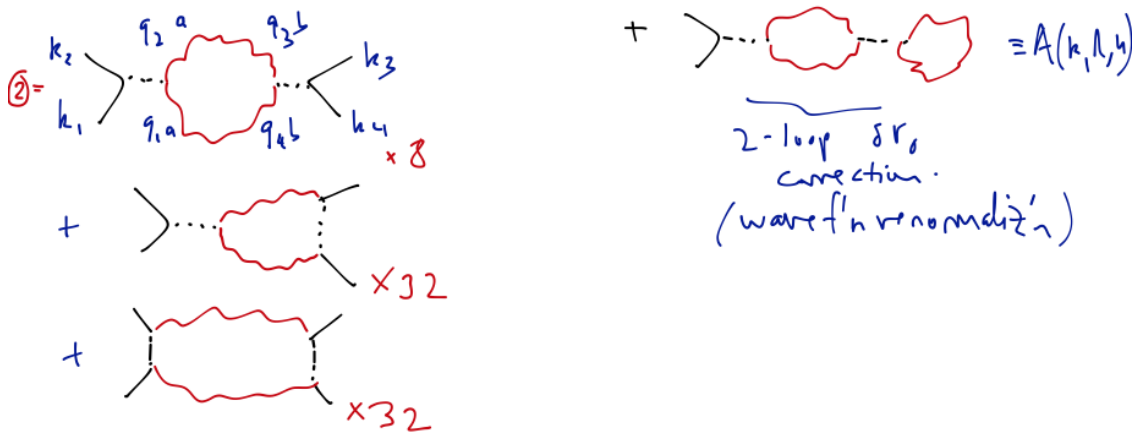


Figure 13: 2nd order corrections from the quartic perturbation of the Gaussian fixed point of the O(N) model. Notice that the diagram at right has two closed flavor loops, and hence goes like N^2, and it comes with two powers of u_0. You can convince yourself by drawing some diagrams that this pattern continues at higher orders. If you wanted to define a model with large N you should therefore consider taking a limit where $N \to \infty$, $u_0 \to 0$, holding $u_0 N$ fixed. The quantity $u_0 N$ is often called the 't Hooft coupling.

order bit

$$2 = \frac{1}{2}\langle \mathcal{U}[\phi_<, \phi_>]^2 \rangle_{0,>,\text{connected}}$$

is the correction to $\mathcal{U}[\phi_<]$. There are less interesting bits which are zero or constant or two-loop corrections to the quadratic term. The correction to the quartic term at 2nd order is

$$\delta_2 S_4[\phi_<] = u_0^2(4N+32)\int_0^{\Lambda/s} \prod_i^4 \left(\text{đ}^D k_i \phi_<(k_i)\right) \not{\delta}(\sum k_i) f(k_1 + k_2)$$

with

$$f(k_1+k_2)=\int \text{đ}^D q \frac{1}{(r_0+r_2 q^2)(r_0+r_2(k_1+k_2-q)^2)} \simeq \int \text{đ}^D q \frac{1}{(r_0+r_2 q^2)^2}\left(1+\mathcal{O}(k_1+k_2)\right)$$

– the bits that depend on the external momenta give irrelevant derivative corrections, like $\phi_<^2 \partial^2 \phi_<^2$. We ignore them.

The full result through $\mathcal{O}(u_0^2)$ is then the original action, with the parameter replacement

$$\begin{pmatrix} r_2 \\ r_0 \\ u_0 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{r}_2 \\ \tilde{r}_0 \\ \tilde{u}_0 \end{pmatrix} = \begin{pmatrix} s^{-D-2}\zeta^2(r_2+\delta r_2) \\ s^{-D}\zeta^2(r_0+\delta r_0) \\ s^{-3D}\zeta^4\left(u_0+\delta u_0\right) \end{pmatrix} + \mathcal{O}(u_0^3).$$

The shifts are:

$$\begin{cases} \delta r_2 = u_0^2 \frac{\partial_k^2 A(0)}{r_2} \\ \delta r_0 = 4u_0(N+2)\int_{\Lambda/s}^{\Lambda} \text{đ}^D q \frac{1}{r_0+r_2 q^2} - A(0)u_0^2 \\ \delta u_0 = -\frac{1}{2}u_0^2(8N+64)\int_{\Lambda/s}^{\Lambda} \text{đ}^D q \frac{1}{(r_0+r_2 q^2)^2} \end{cases} .$$

Here A is the two-loop ϕ^2 correction that we didn't compute (it contains the leading contribution to the wavefunction renormalization, $A(k)=A(0)+\frac{1}{2}k^2\partial_k^2 A(0)+...$). We can choose to keep $\tilde{r}_2 = r_2$ by setting

$$\zeta^2 = \frac{s^{D+2}}{1+u_0^2\partial_k^2 A(0)/r_2} = s^{D+2}\left(1+\mathcal{O}(u_0^2)\right).$$

Now let's make the RG step infinitesimal:

$$s = e^{\ell} \simeq 1+\delta\ell$$

$$\begin{cases} \frac{dr_0}{d\ell} = 2r_0 + \frac{4(N+2)K_D\Lambda^D}{r_0+r_2\Lambda^2}u_0 - Au_0^2 + \mathcal{O}(u_0^3) \\ \frac{du_0}{d\ell} = (4-D)u_0 - \frac{4(N+8)K_D\Lambda^D}{(r_0+r_2\Lambda^2)^2}u_0^2 + \mathcal{O}(u_0^3) \end{cases} \tag{3.18}$$

I defined $K_D \equiv \frac{\Omega_{D-1}}{(2\pi)^D}$.

To see how the previous thing arises, and how the integrals *all went away*, let's consider just the $\mathcal{O}(u_0)$ correction to the mass:

$$\begin{aligned} \tilde{r}_0 = r_0 + \delta\ell\frac{dr_0}{d\ell} &= s^2\left(r_0 + 4u(N+2)\int_{\Lambda/s}^{\Lambda}\frac{\text{đ}^D q}{r_0+r_2 q^2} + \mathcal{O}(u_0^2)\right) \\ &= (1+2\delta\ell)\left(r_0 + 4u_0(N+2)\frac{\Omega_{D-1}}{(2\pi)^D}\Lambda^D\frac{1}{r_0+r_2\Lambda^2}\delta\ell + \mathcal{O}(u_0^2)\right) \\ &= \left(2r_0 + \frac{4u_0(N+2)}{r_0+r_2\Lambda^2}K_D\Lambda^D\right)\delta\ell + \mathcal{O}(u_0^2). \end{aligned} \tag{3.19}$$

Now we are home. (3.18) has two fixed points. One is the free fixed point at the origin where nothing happens. The other (Wilson-Fisher) fixed point is at

$$\begin{cases} r_0^\star = -\frac{2u_0^\star(N+2)K_D\Lambda^D}{r_0^\star+r_2\Lambda^2} \overset{D=4-\epsilon}{=} -\frac{1}{2}\frac{N+2}{N+8}r_2\Lambda^2\epsilon + \mathcal{O}(\epsilon^2) \\ u_0^\star = \frac{(r^\star+r_2\Lambda^2)^2}{4(N+8)K_D\Lambda^D}\epsilon \overset{D=4-\epsilon}{=} \frac{1}{4}\frac{r_2^2}{(N+8)K_4}\epsilon + \mathcal{O}(\epsilon^2) \end{cases}$$

which is at positive $u_0^\star$ if $\epsilon > 0$. In the second step we keep only leading order in $\epsilon = 4 - D$.

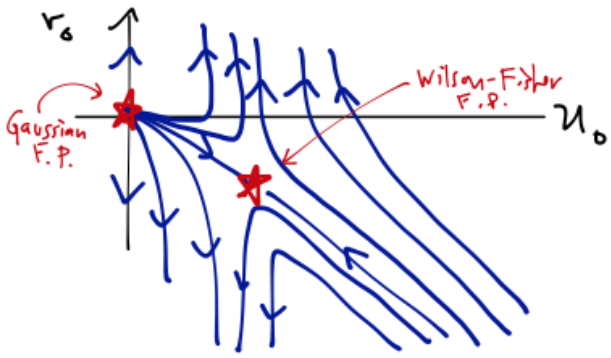


Figure 14: The ϕ^4 phase diagram, for $\epsilon > 0$.

Now we follow useful strategies for dynamical systems and *linearize* near the W-F fixed point:

$$\frac{d}{d\ell}\begin{pmatrix}\delta r_0 \\ \delta u_0\end{pmatrix} = M\begin{pmatrix}\delta r_0 \\ \delta u_0\end{pmatrix}$$

The matrix M is a 2x2 matrix whose eigenvalues describe the flows near the fixed point. It looks like

$$M = \begin{pmatrix} 2 - \frac{N+2}{N+8}\epsilon & \dots \\ \mathcal{O}(\epsilon^2) & -\epsilon \end{pmatrix}$$

Its eigenvalues (which don't care about the off-diagonal terms because the lower left entry is $\mathcal{O}(\epsilon^2)$ are

$$y_r = 2 - \frac{N+2}{N+8}\epsilon + \mathcal{O}(\epsilon^2) > 0$$

which determines the instability of the fixed point and

$$y_u = -\epsilon + \mathcal{O}(\epsilon^2) < 0 \quad \text{for } D < 4$$

which is a stable direction.

So y_r determines the correlation length exponent. Its eigenvector is δr_0 to $\mathcal{O}(\epsilon^2)$. This makes sense: r_0 is the relevant coupling which must be tuned to stay at the critical point. The correlation length can be found as follows (as we did around Eq. (3.17)). ξ is the value of $s = s_1$ at which the relevant operator has turned on by an order-1 amount, *i.e.* by setting $\xi \sim s_1$ when $1 \sim \delta r_0(s_1)$. According to the linearized RG equation, close to the fixed point, we have $\delta r_0(s) = s^{y_r} \delta r_0(0)$. Therefore

$$\xi \sim s_1^{-\frac{1}{y_r}} = (\delta r_0(0))^{-\nu} \ .$$

This last equality is the definition of the correlation length exponent (how does the correlation length scale with our deviation from the critical point $\delta r_0(0)$). Therefore

$$\nu = \frac{1}{y_r} = \left(2\left(1 - \frac{1}{2}\frac{N+2}{N+8}\epsilon\right)\right)^{-1} + \mathcal{O}(\epsilon^2) \simeq \frac{1}{2}\left(1 + \frac{N+2}{2(N+8)}\epsilon\right) + \mathcal{O}(\epsilon^2).$$

The remarkable success of setting $\epsilon = 1$ in this expansion to get answers for $D = 3$ continues. See the references for more details on this; for refinements of this estimate, see Zinn-Justin's book.

3.4 Which bits of the beta function are universal?

[Cardy, chapter 5] Some of the information in the beta functions depends on our choice of renormalization scheme and on our choice of regulator. Some of it does not, such as the topology of the fixed points, and the critical exponents associated with them. Here is a way to see that some of the data in the beta functions is also universal. It also gives a more general point of view on the epsilon expansion and why it works.

Operator product expansion (OPE). Suppose we want to understand a (vacuum) correlation function of local operators like

$$\langle \phi_i(x_1)\phi_j(x_2)\Phi\rangle$$

where $\{\Phi\}$ is a collection of other local operators at $\{r_l\}$; suppose that the two operators we've picked out are closer to each other than to any of the others:

$$|r_1 - r_2| \ll |r_{1,2} - r_l|, \ \forall l.$$

Then from the point of view of the collection Φ, $\phi_i\phi_j$ looks like a single local operator. But which one? Well, it looks like some sum over all of them:

$$\langle \phi_i(x_1)\phi_j(x_2)\Phi\rangle = \sum_k C_{ijk}(x_1 - x_2)\langle \phi_k(x_1)\Phi\rangle$$

where $\{\phi_k\}$ is some basis of local operators. For example, we can figure out the Cs by Taylor expanding:

$$\phi_j(x_2) = e^{(x_2-x_1)^\mu \frac{\partial}{\partial x_1^\mu}} \phi(x_1) = \phi(x_1) + (x_2-x_1)^\mu \partial_\mu \phi(x_1) + \cdots$$

A shorthand for this statement is the OPE

$$\phi_i(x_1)\phi_j(x_2) \sim \sum_k C_{ijk}(x_1-x_2)\phi_k(x_1)$$

which is to be understand as an operator equation: true for all states, but only up to collisions with other operator insertions (hence the $\sim$ rather than $=$).

This is an attractive concept, but is useless unless we can find a good basis. At a fixed point of the RG, it becomes much more useful, because of scale invariance. This means that we can organize our operators according to their scaling dimension. Roughly it means two wonderful simplifications:

- We can find a basis (here, for the simple case of scalar operators)

$$\langle \phi_i(x)\phi_j(0)\rangle = \frac{\delta_{ij}}{r^{2\Delta_i}} \tag{3.20}$$

  where Δ_i is the scaling dimension of ϕ_i. Then we can order the contributions to $\sum_k$ by increasing Δ_k, which means smaller contributions to $\langle\phi\phi\Phi\rangle$.

- Further, the form of C_{ijk} is fixed up to a number. Again for scalar operators,

$$\phi_i(x_1)\phi_j(x_2) \sim \sum_k \frac{c_{ijk}}{|x_1-x_2|^{\Delta_i+\Delta_j-\Delta_k}}\phi_k(x_1) \tag{3.21}$$

  where c_{ijk} is now a set of pure numbers, the *OPE coefficients* (or *structure constants*).

  The structure constants are universal data about the fixed point: they transcend perturbation theory. How do I know this? Because they can be computed from correlation functions of scaling operators *at the fixed point*: multiply the BHS of (3.21) by $\phi_k(x_3)$ and take the expectation value at the fixed point:

$$\begin{aligned}\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\rangle_\star &= \sum_{k'} \frac{c_{ijk'}}{|x_1-x_2|^{\Delta_i+\Delta_j-\Delta_k}}\langle\phi_{k'}(x_1)\phi_k(x_3)\rangle \\ &\overset{(3.20)}{=} \frac{c_{ijk}}{|x_1-x_2|^{\Delta_i+\Delta_j-\Delta_k}}\frac{1}{|x_1-x_3|^{2\Delta_k}}\end{aligned} \tag{3.22}$$

  (There is a better way to organize the RHS here, but let me not worry about that here.) The point here is that by evaluating the LHS at the fixed point, with some known positions $x_{1,2,3}$, we can extract c_{ijk}.

Confession: I (and Cardy) have used a tiny little extra assumption of *conformal invariance* to help constrain the situation here. It is difficult to have scale invariance without conformal invariance, so this is not a big loss of generality.

Conformal perturbation theory. I'll make this discussion in the Euclidean setting and we'll think about the equilibrium partition function

$$Z = \mathrm{tr} e^{-H}$$

– we set the temperature equal to 1 and include it in the couplings.

Suppose we find a fixed point of the RG, $H_\star$. (For example, it could be the gaussian fixed point of N scalar fields.) Let us study its neighborhood. (For example, we could seek out the nearby interacting Wilson-Fisher fixed point in $D < 4$ in this way.) Then

$$H = H_\star + \sum_x \sum_i g_i a^{\Delta_i} \phi_i(x)$$

where a is the short distance cutoff (*e.g.* the lattice spacing), and ϕ_i has dimensions of length$^{-\Delta_i}$ as you can check from (3.20). So g_i are de-dimensionalized couplings which we will treat as small and expand in. Then

$$\begin{aligned} Z &= \underbrace{Z_\star}_{\equiv \mathrm{tr} e^{-H_\star}} \langle e^{-\sum_x \sum_i g_i a^{\Delta_i} \phi_i(x)} \rangle_\star \\ &\overset{\sum_x \simeq \frac{1}{a^D}\int d^D r}{\simeq} Z_\star \Bigg(1 - \sum_i g_i \int \langle \phi_i(x) \rangle_\star \frac{d^D x}{a^{D-\Delta_i}} \\ &\qquad + \frac{1}{2} \sum_{ij} g_i g_j \int \frac{d^D x_1 d^D x_2}{a^{2D-\Delta_i-\Delta_j}} \langle \phi_i(x_1) \phi_j(x_2) \rangle_\star \\ &\qquad - \frac{1}{3!} \sum_{ijk} g_i g_j g_k \int\int\int \frac{\prod_{a=1}^3 d^D x_a}{a^{3D-\Delta_i-\Delta_j-\Delta_k}} \langle \phi_i(x_1) \phi_j(x_2) \phi_k(x_3) \rangle_\star + \ldots \Bigg) . \end{aligned}$$

Comments:

- We used the fact that near the fixed point, the correlation length is much larger than the lattice spacing to replace $\sum_x \simeq \frac{1}{a^D} \int d^D r$.
- There is still a UV cutoff on all the integrals – the operators can't get within a lattice spacing of each other: $|r_i - r_j| > a$.
- The integrals over space are also IR divergent; we cut this off by putting the whole story in a big box of size L. This is a physical size which should be RG-independent.

- The structure of this expansion does *not* require the initial fixed point to be a free fixed point; it merely requires us to be able to say something about the correlation functions. As we will see, the OPE structure constants c_{ijk} are quite enough to learn something.

Now let's do the RG dance. While preserving Z, we make an infinitesimal change of the cutoff:

$$a \to sa = (1+\delta\ell)a, \delta l \ll 1 \ .$$

The price for preserving Z is letting the couplings run $g_i = g_i(s)$. Where does a appear:
(1) in the integration measure factors $a^{D-\Delta_i}$.
(2) in the cutoffs on $\int dx_1 dx_2$ which enforce $|x_1 - x_2| > a$.
(3) *not* in the IR cutoff.
The leading-in-$\delta\ell$ effects of (1) and (2) are additive and so may be considered separately:

$$(1) \qquad \tilde{g}_i = (1+\delta\ell)^{D-\Delta_i} g_i \simeq g_i + (D-\Delta_i) g_i \delta\ell \equiv g_i + \delta_1 g_i$$

The effect of (2) first appears in the $\mathcal{O}(g^2)$ term, the *change* in which is

$$(2) \qquad \sum_{i,j} g_i g_j \int_{|x_1-x_2|\in(a(1+\delta\ell),a)} \int \frac{d^D x_1 d^D x_2}{a^{2D-\Delta_i-\Delta_j}} \underbrace{\langle \phi_i(x_1)\phi_j(x_2)\rangle_\star}_{=\sum_k c_{ijk}|x_1-x_2|^{\Delta_k-\Delta_i-\Delta_j}\langle\phi_k\rangle_\star}$$
$$= \sum_{ij} g_i g_j c_{ijk} \Omega_{D-1} a^{-2D+\Delta_k} \langle\phi_k\rangle_\star$$

So this correction can be absorbed by a change in g_k according to

$$\delta_2 g_k = -\frac{1}{2}\Omega_{D-1} \sum_{ij} c_{ijk} g_i g_j + \mathcal{O}(g^3)$$

where the $\mathcal{O}(g^3)$ term comes from triple collisions which we haven't considered here. Therefore we arrive at the following expression for evolution of couplings: $\frac{dg}{d\ell} = (\delta_1 g + \delta_2 g)/\delta\ell$

$$\frac{dg}{d\ell} = (D-\Delta_k) g_k - \frac{1}{2}\Omega_d \sum_{ij} c_{ijk} g_i g_j + \mathcal{O}(g^3) \ . \tag{3.23}$$

[End of Lecture 12] [37]

[37] To make the preceding discussion we considered the partition function Z. If you look carefully you will see that in fact it was not really necessary to take the expectation values $\langle\rangle_\star$ to obtain the result (3.23). Because the OPE is an operator equation, we can just consider the running of the operator e^{-H} and the calculation is identical. A reason you might consider doing this instead is that expectation values of scaling operators on the plane actually vanish $\langle\phi_i(x)\rangle_\star = 0$. However, if we consider the partition function in finite volume (say on a torus of side length L), then the expectation values of scaling operators are not zero. You can check these statements explicitly for the normal-ordered operators at the gaussian fixed point introduced below. Thanks to Sridip Pal for bringing these issues to my attention.

At $g = 0$, the linearized solution is $dg_k/g_k = (D - \Delta_k)d\ell \implies g_k \sim e^{(D-\Delta_k)\ell}$ which reproduces our understanding of relevant and irrelevant at the initial fixed point.

Let's consider the Ising model.

$$\begin{aligned} H &= -\frac{1}{2}\sum_{x,x'} J(x-x')S(x)S(x') - h\sum_x S(x) \\ &\simeq -\frac{1}{2}\sum_{x,x'} J(x-x')S(x)S(x') - h\sum_x \phi(x) + \lambda\sum_x \left(S(x)^2 - 1\right)^2 \\ &\simeq \int d^D x \left(\frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + r_0 a^{-2}\phi^2 + u_0 a^{D-4}\phi^4 + h a^{-1-D/2}\phi\right) \end{aligned} \tag{3.24}$$

In the first step I wrote a lattice model of spins $S = \pm 1$; in the second step I used the freedom imparted by universality to relax the $S = \pm 1$ constraint, and replace it with a potential which merely discourages other values of S; in the final step we took a continuum limit.

In (3.24) I've temporarily included a Zeeman-field term hS which breaks the $\phi \to -\phi$ symmetry. Setting it to zero it stays zero (*i.e.* it will not be generated by the RG) because of the symmetry. This situation is called *technically natural*.

Now, consider for example as our starting fixed point the Gaussian fixed point, with

$$H_{\star,0} \propto \int d^D x \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 .$$

Since this is quadratic in ϕ, all the correlation functions (and hence the OPEs, which we'll write below) are determined by Wick contractions using

$$\langle \phi(x_1)\phi(x_2)\rangle_{\star,0} = \frac{\mathcal{N}}{|x_1 - x_2|^{D-2}}.$$

It is convenient to rescale the couplings of the perturbing operators by $g_i \to \frac{2}{\Omega_{D-1}} g_i$ to remove the annoying $\Omega_{D-1}/2$ factor from the beta function equation. Then the RG equations (3.23) say

$$\begin{cases} \frac{dh}{d\ell} = & (1 + D/2) - \sum_{ij} c_{ijh} g_i g_j \\ \frac{dr_0}{d\ell} = & 2r_0 - \sum_{ij} c_{ijr_0} g_i g_j \\ \frac{du_0}{d\ell} = & \epsilon u_0 - \sum_{ij} c_{iju_0} g_i g_j \end{cases}$$

So we just need to know a few numbers, which we can compute by doing Wick contractions with free fields. That is: to find the beta function for g_k, we look at all the OPEs between operators in the perturbed hamiltonian (3.24) which produce g_k.

Algebra of scaling operators at the Gaussian fixed point. It is convenient to choose a basis of *normal-ordered* operators, which are defined by subtracting out their self-contractions. That is

$$\phi_n \equiv :\phi^n := \phi^n - \textit{self-contractions}$$

so that $\langle :\phi^n: \rangle = 0$, and specifically

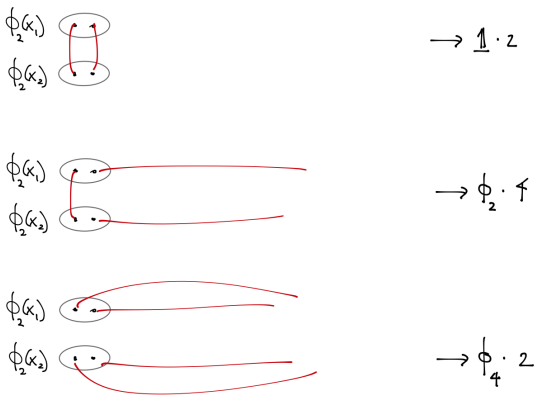
$$\phi_2 = \phi^2 - \langle\phi^2\rangle, \quad \phi_4 = \phi^4 - 3\langle\phi^2\rangle\phi^2 \ .$$

This amounts to a shift in couplings $r_0 \to r_0 + 3u\langle\phi^2\rangle_\star$. Note that the contractions $\langle\phi^2\rangle$ discussed here are defined on the plane. They are in fact quite UV sensitive and require some short-distance cutoff.

To compute their OPEs, we consider a correlator of the form above:s

$$\langle\phi_n(x_1)\phi_m(x_2)\Phi\rangle$$

We do wick contractions with the free propagator, but the form of the propagator doesn't matter for the beta function, only the combinatorial factors. If we can contract all the operators making up ϕ_n with those of ϕ_m, then what's left looks like the identity operator to Φ; that's the leading term, if it's there, since the identity has dimension 0, the lowest possible. More generally, some number of ϕs will be left over and will need to be contracted with bits of Φ to get a nonzero correlation function. For example, the contributions to $\phi_2 \cdot \phi_2$ are depicted at right.



The part of the result we'll need (if we set $h = 0$) can be written as (omitting the implied factors of $|x_1 - x_2|^{\Delta_i+\Delta_j-\Delta_k}$ necessary to restore dimensions):

$$\begin{cases} \phi_2\phi_2 & \sim 2\mathbb{1} + 4\phi_2 + \phi_4 + \cdots \\ \phi_2\phi_4 & \sim 12\phi_2 + 8\phi_4 + \cdots \\ \phi_4\phi_4 & \sim 24\mathbb{1} + 96\phi_2 + 72\phi_4 + \cdots \end{cases}$$

At $h = 0$, the result is (the $N = 1$ case of the result in §3.3.4)

$$\begin{cases} \frac{dr_0}{d\ell} = & 2r_0 - 4r_0^2 - 2\cdot 12 r_0 u_0 - 96 u_0^2 \\ \frac{du_0}{d\ell} = & \epsilon u_0 - r_0^2 - 2\cdot 8 r_0 u_0 - 72 u_0^2 \end{cases}$$

and so the $(N = 1)$ WF fixed point occurs at $u_0 = u_0^\star = \epsilon/72, r_0 = \mathcal{O}(\epsilon^2)$.

Linearizing the RG flow about the *new* fixed point,

$$\frac{dr_0}{d\ell} = 2r_0 - 24u_0^\star r_0 + \cdots$$

gives

$$\frac{dr_0}{r_0} = (2 - \frac{24}{72}\epsilon)d\ell \quad \Longrightarrow \quad r_0 \sim e^{(2-\frac{24}{72}\epsilon)\ell} \equiv \left(e^\ell\right)^{\frac{1}{\nu}}$$

which gives $\nu = \frac{1}{2} + \frac{1}{12}\epsilon + \mathcal{O}(\epsilon^2)$.

4 Effective field theory

[Some nice lecture notes on effective field theory can be found here: J. Polchinski, A. Manohar, D. B. Kaplan, H. Georgi.]

Diatribe about 'renormalizability'. Having internalized Wilson's perspective on renormalization – namely that we should include all possible operators consistent with symmetries and let the dynamics decide which are important at low energies – we are led immediately to the idea of an *effective field theory* (EFT). There is no reason to demand that a field theory that we have found to be relevant for physics in some regime should be a valid description of the world to arbitrarily short (or long!) distances. This is a happy statement: there can always be new physics that has been so far hidden from us. Rather, an EFT comes with a regime of validity, and with necessary cutoffs. As we will discuss, in a useful implementation of an EFT, the cutoff implies a small parameter in which we can expand (and hence compute).

Caring about renormalizibility is pretending to know about physics at arbitrarily short distances. Which you don't.

Even when theories are renormalizable, this apparent victory is often false. For example, QED requires only two independent counterterms (mass and charge of the electron), and is therefore by the old-fashioned definition renormalizable, but it is superseded by the electroweak theory above 80GeV. Also: the coupling in QED actually increases logarithmically at shorter distances, and ultimately reaches a *Landau pole* at SOME RIDICULOUSLY HIGH ENERGY (of order $e^{+\frac{c}{\alpha}}$ where $\alpha \sim \frac{1}{137}$ is the fine structure constant (*e.g.* at the scale of atomic physics) and c is some numerical number. Plugging in numbers gives something like 10^{330} GeV, which is quite a bit larger than the Planck scale). This is of course completely irrelevant for physics and even in principle because of the previous remark about electroweak unification. And if not because of that, because of the Planck scale. A heartbreaking historical fact is that Landau and many other smart people gave up on QFT as a whole because of this silly fantasy about QED in an unphysical regime.

We will see below that even in QFTs which are non-renormalizable in the strict sense, there is a more useful notion of renormalizability: effective field theories come with a parameter (often some ratio of mass scales), in which we may expand the action. A useful EFT requires a finite number of counterterms at each order in the expansion.

Furthermore, I claim that this is *always* the definition of renormalizability that we are using, even if we are using a theory which is renormalizable in the traditional sense, which allows us to pretend that there is no cutoff. That is, there could always be corrections of

order $\left(\frac{E}{E_{\text{new}}}\right)^n$ where E is some energy scale of physics that we are doing and E_{new} is some UV scale where new physics might come in; for large enough n, this is too small for us to have seen. The property of renormalizibility that actually matters is that we need a finite number of counterterms at each order in the expansion in $\frac{E}{E_{\text{new}}}$.

Renormalizable QFTs are in some sense *less* powerful than non-renormalizable ones – the latter have the decency to tell us when they are giving the wrong answer! That is, they tell us at what energy new physics *must* come in; with a renormalizable theory we may blithely pretend that it is valid in some ridiculously inappropriate regime like 10^{330} GeV.

Notions of EFT. There is a dichotomy in the way EFTs are used. Sometimes one knows a lot about the UV theory (*e.g.*

- electroweak gauge theory,
- QCD,
- electrons in a solid,
- water molecules

...) but it is complicated and unwieldy for the questions one wants to answer, so instead one develops an effective field theory involving just the appropriate and important dofs (*e.g.*, respectively,

- Fermi theory of weak interactions,
- chiral lagrangian (or HQET or SCET or ...),
- Landau Fermi liquid theory (or the Hubbard model or a topological field theory or ...),
- hydrodynamics (or some theory of phonons in ice or ...)

...). As you can see from the preceding lists of examples, even a single UV theory can have many different IR EFTs depending on what phase it is in, and depending on what question one wants to ask. The relationship between the pairs of theories above is always coarse-graining from the UV to the IR, though exactly what plays the role of the RG parameter can vary wildly. For example, in the example of the Fermi liquid theory, the scaling is $\omega \to 0$, and momenta scale towards the Fermi surface, not $\vec{k} = 0$.

A second situation is when one knows a description of some low-energy physics up to some UV scale, and wants to try to infer what the UV theory might be. This is a common

situation in physics! Prominent examples include: the Standard Model, and quantized Einstein gravity. Occasionally we (humans) actually learn some physics and an example of an EFT from the second category moves to the first category.

Summary of basic EFT logic. Answer the following questions:

1. what are the dofs?
2. what are the symmetries?
3. where is the cutoff on its validity?

Then write down all interactions between the dofs which preserve the symmetry in an expansion in derivatives, with higher-dimension operators suppressed by more powers of the UV scale.

[End of Lecture 13]

I must also emphasize two distinct usages of the term 'effective field theory' which are common, and which the discussion above is guilty of conflating (this (often slippery) distinction is emphasized in the review article by Georgi linked at the beginning of this subsection). The Wilsonian perspective advocated in the previous subsection produces a low-energy description of the physics which is really just a way of solving (if you can) the original model; very reductively, it's just a physically well-motivated order for doing the integrals. If you really integrate out the high energy modes exactly, you will get a non-local action for the low energy modes. This is to be contrasted with the local actions one uses in practice, by truncating the derivative expansion. It is the latter which is really the action of the effective field theory, as opposed to the *full theory*, with some of the integrals done already. The latter will give correct answers for physics below the cutoff scale, and it will give them much more easily.

Some interesting and/or important examples of EFT that we will not discuss explicitly, and where you can learn about them:

- Hydrodynamics [Kovtun]
- Fermi liquid theory [J. Polchinski, R. Shankar, *Rev. Mod. Phys.* **66** (1994) 129]
- chiral perturbation theory [D. B. Kaplan, §4]

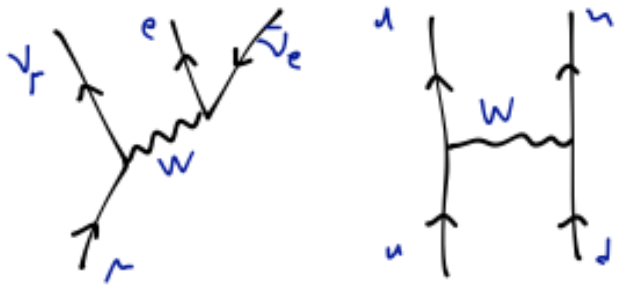
- heavy quark effective field theory [D. B. Kaplan, §1.3]
- random surface growth (KPZ) [Zee, chapter VI]
- color superconductors [D. B. Kaplan, §5]
- gravitational radiation [Goldberger, Rothstein]
- soft collinear effective theory [Becher, Stewart]
- magnets [Zee, chapter VI.5, hep-ph/9311264v1]
- effective field theory of cosmological inflation [Senatore et al, Cheung et al]
- effective field theory of dark matter direct detection [Fitzpatrick et al]

There are many others, the length of this list was limited by how long I was willing to spend digging up references. Here is a longer list.

4.1 Fermi theory of Weak Interactions

[from §5 of A. Manohar's EFT lectures] As a first example, let's think about part of the Standard Model.

$$L_{EW} \ni -\frac{\mathbf{i}g}{\sqrt{2}}\bar{\psi}_i\gamma^\mu P_L\psi_j W_\mu V_{ij} \quad + \text{terms involving } Z \text{ bosons}$$



Some things intermediate Ws can do: μ decay, $\Delta S = 1$ processes, neutron decay

If we are asking questions with external momenta less than M_W, we can integrate out W and make our lives simpler:

$$\delta S_{eff} \sim \left(\frac{\mathbf{i}g}{\sqrt{2}}\right)^2 V_{ij}V_{k\ell}^\star \int \mathrm{d}^D p \frac{-\mathbf{i}g_{\mu\nu}}{p^2 - M_W^2} \left(\bar{\psi}_i\gamma^\mu P_L\psi_j\right)(p) \left(\bar{\psi}_k\gamma^\nu P_L\psi_\ell\right)(-p)$$

(I am lying a little bit about the W propagator in that I am not explicitly projecting out the fourth polarization with the negative residue.) This is non-local at scales $p \gtrsim M_W$ (recall our discussion in §1 with the two oscillators). But for $p^2 \ll M_W^2$,

$$\frac{1}{p^2 - M_W^2} \overset{p^2 \ll M_W^2}{\simeq} -\frac{1}{M_W^2}\left(1 + \underbrace{\frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + ...}_{\text{derivative couplings}}\right) \tag{4.1}$$

$$S_F = -\frac{4G_F}{\sqrt{2}} V_{ij} V_{kl}^\star \int d^4x \left(\bar{\psi}_i \gamma^\mu P_L \psi_j\right)(x) \left(\bar{\psi}_k \gamma_\mu P_L \psi_\ell\right)(x) + \mathcal{O}\left(\frac{1}{M_W^2}\right) + \text{kinetic terms for fermions} \tag{4.2}$$

where $G_F/\sqrt{2} \equiv \frac{g^2}{8M_W^2}$ is the Fermi coupling. We can use this (Fermi's) theory to compute the amplitudes above, and it is much simpler than the full electroweak theory (for example I don't have to lie about the form of the propagator of the W-boson like I did above).

On the other hand, this theory is *not the same* as the electroweak theory; for example it is not renormalizable, while the EW theory is. Its point in life is to help facilitate the expansion in $1/M_W$. There is something about the expression (4.2) that should make you nervous, namely the big red 1 in the $1/M_W^2$ corrections: what makes up the dimensions? This becomes an issue when we ask about ...

4.2 Loops in EFT

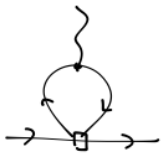
Suppose we try to define the Fermi theory S_F with a euclidean momentum cutoff $|k_E| < \Lambda$, like we've been using for most of our discussion so far. We expect that we'll have to set $\Lambda \sim M_W$. A simple example which shows that this is problematic is to ask about radiative corrections in the 4-Fermi theory to the coupling between the fermions and the Z (or the photon).

We are just trying to estimate the magnitude of this correction, so don't worry about the factors and the gamma matrices:

$$\sim I \equiv \underbrace{\frac{1}{M_W^2}}_{\propto G_F} \underbrace{\int^\Lambda \mathrm{d}^4k \frac{1}{k}\frac{1}{k} \mathrm{tr}\,(\gamma ...)}_{\sim \int^\Lambda k \mathrm{d}k \sim \Lambda^2 \sim M_W^2} \sim \mathcal{O}(1).$$

Even worse, consider what happens if we use the vertex coming from the $\left(\frac{p^2}{M_W^2}\right)^\ell$ correction

in (4.1)



$$\sim I_\ell \equiv \frac{1}{M_W^2}\int^\Lambda \mathrm{đ}^4k \frac{1}{k^2}\left(\frac{k^2}{M_W^2}\right)^\ell \sim \mathcal{O}(1)$$

– it's also unsuppressed by powers of ... well, anything. This is a problem.

Fix: A way to fix this is to use a "mass-independent subtraction scheme", such as dimensional regularization and minimal subtraction ($\overline{\text{MS}}$). The crucial feature is that the dimensionful cutoff parameter appears only inside logarithms ($\log\mu$), and not as free-standing powers (μ^2).

With such a scheme, we'd get instead

$$I \sim \frac{m^2}{M_W^2}\log\mu \qquad I_\ell \sim \left(\frac{m^2}{M_W^2}\right)^{\ell+1}\log\mu$$

where m is some mass scale *other* than the RG scale μ (like a fermion mass parameter, or an external momentum, or a dynamical scale like Λ_{QCD}).

We will give a more detailed example next. The point is that in a mass-independent scheme, the regulator doesn't produce new dimensionful things that can cancel out the factors of M_W in the denominator. It respects the 'power counting': if you see 2ℓ powers of $1/M_W$ in the coefficient of some term in the action, that's how many powers will suppress its contributions to amplitudes. This means that the EFT is like a renormalizable theory *at each order in the expansion* (here in $1/M_W$), in that there is only a finite number of allowed vertices that contribute at each order (counterterms for which need to be fixed by a renormalization condition). The insatiable appetite for counterterms is still insatiable, but it eats only a finite number at each order in the expansion. Eventually you'll get to an order in the expansion that's too small to care about, at which point the EFT will have eaten only a finite number of counterterms.

There is a price for these wonderful features of mass-independent schemes, which has two aspects:

- Heavy particles (of mass m) don't decouple when $\mu < m$. For example, in a mass-independent scheme for a gauge theory, heavy charged particles contribute to the beta function for the gauge coupling even at $\mu \ll m$.
- Perturbation theory will break down at *low* energies, when $\mu < m$; in the example just mentioned this happens because the coupling keeps running.

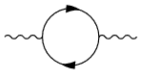
We will show both these properties very explicitly next. The solution of both these problems is to integrate out the heavy particles by hand at $\mu = m$, and make a new EFT for $\mu < m$ which simply omits that field. Processes for which we should set $\mu < m$ don't have enough energy to make the heavy particles in external states anyway. (For some situations where you should still worry about them, see Aneesh Manohar's notes linked above.)

4.2.1 Comparison of schemes, case study

The case study we will make is the contribution of a charged fermion of mass m to the running of the QED gauge coupling.

Recall that the QED Lagrangian is

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}\left(\mathbf{i}\not{D} - m\right)\psi$$

with $D_\mu = \partial_\mu - \mathbf{i}eA_\mu$. By redefining the field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ by a constant factor we can move around where the e appears, *i.e.* by writing $\tilde{A} = eA$, we can make the gauge kinetic term look like $\frac{1}{4e^2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$. This means that the charge renormalization can be seen *either* in the vacuum polarization, the correction to the photon propagator:  . I will call this diagram $\mathbf{i}\Pi_{\mu\nu}$.

So the information about the running of the coupling is encoded in the gauge field two-point function:

$$\Pi_{\mu\nu} \equiv \langle A_\mu(p)A_\nu(q)\rangle = \left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\not{\delta}(p+q)\Pi(p^2) \ .$$

The factor $P_{\mu\nu} \equiv p_\mu p_\nu - p^2 g_{\mu\nu}$ is guaranteed to be the polarization structure by the gauge invariance Ward identity: $p^\mu\langle A_\mu(p)A_\nu(q)\rangle = 0$. That is: $p^\mu P_{\mu\nu} = 0$, and there is no other symmetric tensor made from p^μ which satisfies this. This determines the correlator up to a function of p^2, which we have called $\Pi(p^2)$.

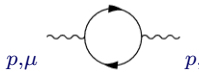
The choice of scheme shows up in our choice of renormalization condition to impose on $\Pi(p^2)$:

Mass-dependent scheme: subtract the value of the graph at $p^2 = -M^2$ (a very off-shell, euclidean, momentum). That is, we impose a renormalization condition which says

$$\Pi(p^2 = -M^2) \stackrel{!}{=} 1 \tag{4.3}$$

(which is the tree-level answer with the normalization above).

The contribution of a fermion of mass m and charge e is (factoring out the momentum-conserving delta function):



$$= -\int \mathrm{đ}^D k \mathrm{tr}\left((-\mathbf{i}e\gamma^\mu) \frac{-\mathbf{i}\left(\not{k}+m\right)}{k^2-m^2} (-\mathbf{i}e\gamma^\nu) \frac{-\mathbf{i}\left(\not{p}+\not{k}+m\right)}{(p+k)^2-m^2} \right)$$

The minus sign out front is from the fermion loop. Some boiling, which you can find in Peskin (page 247) or Zee (§III.7), reduces this to something manageable. The steps involved are: (1) a trick to combine the denominators, like the Feynman trick $\frac{1}{AB} = \int_0^1 dx \left(\frac{1}{(1-x)A+xB}\right)^2$. (2) some Dirac algebra, to turn the numerator into a polynomial in k, p. As Zee says, our job in this course is not to train to be professional integrators. The result of this boiling can be written

$$\mathbf{i}\Pi^{\mu\nu} = -e^2 \int \mathrm{đ}^D \ell \int_0^1 dx \frac{N^{\mu\nu}}{\left(\ell^2-\Delta\right)^2}$$

with $\ell = k + xp$ is a new integration variable, $\Delta \equiv m^2 - x(1-x)p^2$, and the numerator is

$$N^{\mu\nu} = 2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 - 2x(1-x)p^\mu p^\nu + g^{\mu\nu}\left(m^2 + x(1-x)p^2\right) + \text{terms linear in } \ell^\mu \quad .$$

At this point I have to point out a problem with applying the *regulator* we've been using (this is a distinct issue from the choice of RG scheme). With a euclidean momentum cutoff, the diagram  gives something of the form

$$\mathbf{i}\Pi^{\mu\nu} \propto e^2 \int^\Lambda d^4\ell_E \frac{\ell_E^2 g^{\mu\nu}}{\left(\ell_E^2+\Delta\right)^2} + \ldots \propto e^2\Lambda^2 g^{\mu\nu}$$

This is *NOT* of the form $\Pi^{\mu\nu} = P^{\mu\nu}\Pi(p^2)$; rather it produces a correction to the photon mass proportional to the cutoff. What happened? Our cutoff was not gauge invariant. Oops.

Dimensional regularization. A regulator which is gauge invariant is dimensional regularization (dim reg). I have already been writing most of the integrals in D dimensions. One small difference when we are considering this as a regulator for an integral of fixed dimension is that we don't want to violate dimensional analysis, so we should really replace

$$\int \mathrm{đ}^4\ell \longrightarrow \int \frac{d^{4-\epsilon}\ell}{\bar{\mu}^{-\epsilon}}$$

where $D = 4-\epsilon$ and $\bar{\mu}$ is an arbitrary mass scale which will appear in the regulated answers, which we put here to preserve dim'l analysis – *i.e.* the couplings in dim reg will have the same engineering dimensions they had in the unregulated theory (dimensionless couplings remain dimensionless). $\bar{\mu}$ will parametrize our RG, *i.e.* play the role of the RG scale. (It

is often called μ at this step and then suddenly replaced by something also called μ; I will instead call this $\bar{\mu}$ and relate it to the thing that ends up being called μ.)

[Zinn-Justin 4th ed page 233] Dimensionally regularized integrals can be defined systematically with a few axioms indicating how the D-dimensional integrals behave under

1. translations $\int \text{đ}^D p f(p+q) = \int \text{đ}^D p f(p)$ [38]
2. scaling $\int \text{đ}^D p f(sp) = |s|^{-D} \int \text{đ}^D p f(p)$
3. factorization $\int \text{đ}^D p \int \text{đ}^D q f(p) g(q) = \int \text{đ}^D p f(p) \int \text{đ}^D q g(q)$

The (obvious?) third axiom implies our formula (3.16) for the sphere volume as a continuous function of D.

In dim reg, the one-loop vacuum polarization correction *does* satisfy the gauge invaraince Ward identity $\Pi^{\mu\nu} = P^{\mu\nu}\delta\Pi_2$. A peek at the tables of dim reg integrals shows that $\delta\Pi_2$ is:

$$\begin{aligned}\delta\Pi_2(p^2) &\overset{\text{Peskin p. 252}}{=} -\frac{8e^2}{(4\pi)^{D/2}} \int_0^1 dx x(1-x) \frac{\Gamma(2-D/2)}{\Delta^{2-D/2}} \bar{\mu}^\epsilon \\ &\overset{D\to 4}{=} -\frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \left(\frac{2}{\epsilon} - \log\left(\frac{\Delta}{\mu^2}\right)\right)\end{aligned} \tag{4.4}$$

where we have introduced the heralded μ:

$$\mu^2 \equiv 4\pi\bar{\mu}^2 e^{-\gamma_E}$$

where γ_E is the Euler-Mascheroni constant; we define μ in this way so that, like Rosencrantz and Guildenstern, γ_E both appears and disappears from the discussion at this point. In the second line of (4.4), we expanded the Γ-function about $D = 4$; there are other singularities at other integer dimensions. It is an interesting question to ponder why the integrals have such nice behavior as a function of D. That is: they only have simple poles. A partial answer is that in order to have worse (*e.g.* essential) singularities at some D, the perturbative field theory would have to somehow fail to make sense at larger D.

Mass-dependent scheme: Now back to our discussion of schemes. I remind you that in a mass-independent scheme, we demand that the counterterm cancels $\delta\Pi_2$ when we set the external momentum to $p^2 = -M^2$, so that the whole contribution at order e^2 is :

$$0 \overset{(4.3)!}{=} \Pi_2^{(M)}(p^2 = -M^2) = \underbrace{\delta_{F^2}^{(M)}}_{\text{counterterm coefficient for } \frac{1}{4}F_{\mu\nu}F^{\mu\nu}} + \delta\Pi_2$$

[38] Note that this rule fails for the euclidean momentum cutoff.

$$\implies \Pi_2^{(M)}(p^2) = \frac{e^2}{2\pi^2}\int dx x(1-x)\log\left(\frac{m^2 - x(1-x)p^2}{m^2 + x(1-x)M^2}\right).$$

Notice that the μs go away in this scheme.

Mass-Independent scheme: This is to be contrasted with what we get in a mass-independent scheme, such as $\overline{\rm MS}$, in which Π is defined by the rule that we *subtract the* $1/\epsilon$ *pole*. This means that the counterterm is

$$\delta_{F^2}^{(\overline{\rm MS})} = -\frac{e^2}{2\pi^2}\frac{2}{\epsilon}\underbrace{\int_0^1 dx x(1-x)}_{=1/6}.$$

(Confession: I don't know how to state this in terms of a simple renormalization condition on Π_2. Also: the bar in $\overline{\rm MS}$ refers to the (not so important) distinction between $\bar{\mu}$ and μ.) The resulting vacuum polarization function is

$$\Pi_2^{(\overline{\rm MS})}(p^2) = \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\log\left(\frac{m^2 - x(1-x)p^2}{\mu^2}\right).$$

[End of Lecture 14]

Next we will talk about beta functions, and verify the claim above about the failure of decoupling. First let me say some words about what is failing. What is failing – the price we are paying for our power counting – is the basic principle of the RG, namely that physics at low energies shouldn't care about physics at high energies, except for small corrections to couplings. An informal version of this statement is: you don't need to know about nuclear physics to make toast. A more formal version is the *Appelquist-Carazzone Decoupling Theorem*, which I will not state (Phys. Rev. D11, 28565 (1975)). So it's something we must and will fix.

Beta functions. $\boxed{M}$: First in the mass-dependent scheme. Demanding that physics is independent of our made-up RG scale, we find

$$0 = M\frac{d}{dM}\Pi_2^{(M)}(p^2) = \left(M\frac{\partial}{\partial M} + \beta_e^{(M)}e\frac{\partial}{\partial e}\right)\Pi_2^{(M)}(p^2) = \left(M\frac{\partial}{\partial M} + \beta_e^{(M)}\underbrace{\cdot 2}_{\text{to this order}}\right)\Pi_2^{(M)}(p^2)$$

where I made the high-energy physics definition of the beta function[39]:

$$\beta_e^{(M)} \equiv \frac{1}{e}\left(M\partial_M e\right) = -\frac{\partial_\ell e}{e}, \quad M \equiv e^{-\ell}M_0\ .$$

[39] I've defined these beta functions to be dimensionless, *i.e.* they are $\partial_{\log M}\log(g)$; this convention is not universally used.

Here ℓ is the RG time again, it grows toward the IR. So we find

$$\beta_e^{(M)} = -\frac{1}{2}\left(\frac{e^2}{2\pi}\right)\int_0^1 dx x(1-x)\left(\frac{-2M^2x(1-x)}{m^2+M^2x(1-x)}\right)+\mathcal{O}(e^3)$$
$$\begin{cases} \overset{m\ll M}{\simeq} \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x) = \frac{e^2}{12\pi^2} \\ \\ \overset{m\gg M}{\simeq} \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\frac{M^2x(1-x)}{m^2} = \frac{e^2}{60\pi^2}\frac{M^2}{m^2} \end{cases}. \tag{4.5}$$

$$\boxed{\overline{\text{MS}}}: \quad 0 = \mu\frac{d}{d\mu}\Pi_2^{(\overline{\text{MS}})}(p^2) = \left(\mu\frac{\partial}{\partial\mu}+\beta_e^{(\overline{\text{MS}})}e\frac{\partial}{\partial e}\right)\Pi_2^{(\overline{\text{MS}})}(p^2) = \left(\mu\frac{\partial}{\partial\mu}+\beta_e^{(\overline{\text{MS}})}\underbrace{\cdot 2}_{\text{to this order}}\right)\Pi_2^{(\overline{\text{MS}})}(p^2)$$

$$\implies \beta_e^{(\overline{\text{MS}})} = -\frac{1}{2}\frac{e^2}{2\pi^2}\underbrace{\int_0^1 dx x(1-x)}_{=1/6}\underbrace{\mu\partial_\mu \log\frac{m^2-p^2x(1-x)}{\mu^2}}_{=-2}$$
$$= \frac{e^2}{12\pi^2}. \tag{4.6}$$

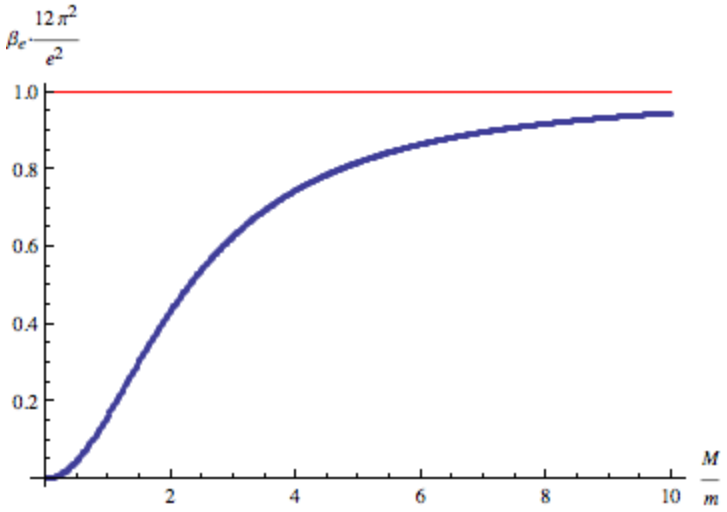
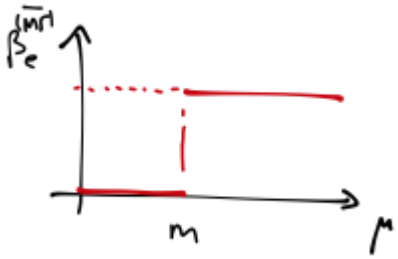


Figure 15: The blue curve is the mass-dependent-scheme beta function; at scales $M \ll m$, the mass of the heavy fermion, the fermion sensibly stops screening the charge. The red line is the $\overline{\text{MS}}$ beta function, which is just a constant, pinned at the UV value.

Also, the $\overline{\text{MS}}$ vacuum polarization behaves for small external momenta like

$$\Pi_2(p^2 \ll m^2) \simeq -\frac{e^2}{2\pi^2}\int_0^1 dx x(1-x) \quad \underbrace{\log\frac{m^2}{\mu^2}}_{\gg 1,\text{for } \mu\ll m!\text{ bad!}}$$

As I mentioned, the resolution of both these problems is simply to define a new EFT for $\mu < m$ which omits the heavy field. Then the strong coupling problem goes away and the heavy fields do decouple. The price is that we have to do this by hand, and the beta function jumps at $\mu = m$; the coupling is continuous, though.



4.3 The SM as an EFT.

Now I can elaborate on a comment I made in lecture in response to questions about the naturalness problem for the Higgs mass. I said that we have evidence against a cutoff on the Standard Model (SM) at energies less than something like 10 TeV. The evidence I had in mind was the absence of interactions of the form

$$\delta L = \frac{1}{M^2} \left(\bar{\psi} A \psi\right) \cdot \left(\bar{\psi} B \psi\right)$$

(where ψ represent various SM fermion fields and A, B can be various gamma and flavor matrices) with $M \lesssim 10$ TeV. Notice that I am talking now about interactions *other* than the electroweak interactions, which as we've just discussed, for energies above $M_W \sim 80$GeV cannot be treated as contact interactions – you can see the Ws propagate!

If such operators were present, we would have found different answers for experiments at LEP. But such operators would be present if we consider new physics in addition to the Standard Model (in most ways of doing it) at energies less than 10 TeV. For example, many interesting ways of coupling in new particles with masses that make them accessible at the LHC would have generated such operators.

A little more explicitly: the Standard Model Lagrangian L_0 contains all the renormalizable (*i.e.* engineering dimension ≤ 4) operators that you can make from its fields (though the coefficients of the dimension 4 operators do vary through quite a large range, and the coefficients of the two relevant operators – namely the identity operator which has dimension zero, and the Higgs mass, which has engineering dimension two, are strangely small, and so is the QCD θ angle).

To understand what lies beyond the Standard Model, we can use our knowledge that whatever it is, it is probably heavy (it could also just be very weakly coupled, which is a different story), with some intrinsic scale Λ_{new}, so we can integrate it out and include its

effects by corrections to the Standard Model:

$$L = L_0 + \frac{1}{\Lambda_{\rm new}}\mathcal{O}^{(5)} + \frac{1}{\Lambda_{\rm new}^2}\sum_i c_i \mathcal{O}_i^{(6)}$$

where the $\mathcal{O}$s are made of SM fields, and have the indicated engineering dimensions, and preserve the necessary symmetries of the SM.

In fact there is only one kind of operator of dimension 5:

$$\mathcal{O}^{(5)} = c_5 \epsilon_{ij} \left(\bar{L}^c\right)^i H^j \epsilon_{kl} L^k H^l$$

where $H^i = (h^+, h^0)^i$ is the $\mathrm{SU}(2)_{EW}$ Higgs doublet and $L^i = (\nu_L, e_L)^i$ is an $\mathrm{SU}(2)_{EW}$ doublet of left-handed leptons, and $\bar{L}^c \equiv L^T C$ where C is the charge conjugation matrix. (I say 'kind of operator' because we can have various flavor matrices in here.) On problem set 6 you get to see from where such an operator might arise, and what it does if you plug in the higgs vev $\langle H \rangle = (0, v)$. This term violates lepton number.

At dimension 6, there are operators that directly violate baryon number, such as

$$\epsilon_{\alpha\beta\gamma} (\bar{u}_R)^c_\alpha (u_R)_\beta \, (\bar{u}_R)^c_\gamma \, e_R.$$

You should read the above tangle of symbols as '$qqq\ell$' – it turns three quarks into a lepton. The epsilon tensor makes a color SU(3) singlet; this thing has the quantum numbers of a baryon. The long lifetime of the proton (you can feel it in your bones – see Zee p. 413) then directly constrains the scale of new physics appearing in front of this operator. [40]

There are $\sim 10^2$ dimension 6 operators that preserve baryon number, and therefore are not as tightly constrained[41]. (Those that induce flavor-changing processes in the SM are more highly constrained and must have $\Lambda_{\rm new} > 10^4$ TeV.) Two such operators are considered equivalent if they differ by something which vanishes by the tree-level SM equations of

[40] Two more comments about this:

- If we didn't know about the Standard Model, (but after we knew about QM and GR and EFT (the last of which people didn't know before the SM for some reason)) we should have made the estimate that dimension-5 Planck-scale-suppressed operators like $\frac{1}{M_{\rm Planck}} p\mathcal{O}$ would cause proton decay (into whatever $\mathcal{O}$ makes). This predicts $\Gamma_p \sim \frac{m_p^3}{M_{\rm Planck}^2} \sim 10^{-13} s^{-1}$ which is *not* consistent with our bodies not glowing. Actually it is a remarkable fact that there are no gauge-invariant operators made of SM fields of dimension less than 6 that violate baryon number. This is an *emergent* symmetry, expected to be violated by the UV completion.
- Surely nothing can prevent $\Delta L \sim \left(\frac{1}{M_{\rm Planck}}\right)^2 qqq\ell$. Happily, this is consistent with the observed proton lifetime.

[41] For an up-to-date counting of these operators, see 1008.4884; thanks to Chris Murphy for the reference.

motion. This is the right thing to do, even for off-shell calculations (like green's functions and for fields running in loops). You know this from Problem Set 2: the EOM are true as operator equations – Ward identities resulting from being free to change integration variables in the path integral[42].

4.4 Quantum Rayleigh scattering

[from hep-ph/9606222 and nucl-th/0510023] Why is the sky blue? Basically, it's because the blue light from the sun scatters in the atmosphere more than the red light, and you (I hope) only look at the scattered light.

With all the buildup, this is going to be disappointingly simple. Consider the scattering of photons off atoms at low energies. Low energy means that the photon does not have enough energy to probe the substructure of the atom – it can't excite the electrons or the nuclei. This means that the atom is just a particle, with some mass M.

The dofs are just the photon field and the field that creates an atom.

The symmetries are Lorentz invariance and charge conjugation invariance and parity. We'll use the usual redundant description of the photon which has also gauge invariance.

The cutoff is the energy ΔE that it takes to excite atomic energy levels we've left out of the discussion. We allow no inelastic scattering. This means we require

$$E_\gamma \ll \Delta E \sim \frac{\alpha}{a_0} \ll a_0^{-1} \ll M_{\text{atom}}$$

Because of this separation of scales, we can also ignore the recoil of the atom, and treat it as infinitely heavy.

Since there are no charged objects in sight – atoms are neutral – gauge invariance means the Lagrangian can depend on the field strength $F_{\mu\nu}$. Let's call the field which destroys an atom with velocity v ϕ_v. $v^\mu v_\mu = 1$ and $v_\mu = (1,0,0,0)_\mu$ in the atom's rest frame. The Lagrangian can depend on v^μ. We can write a Lagrangian for the free atoms as

$$L_{\text{atom}} = \phi_v^\dagger \mathbf{i} v^\mu \partial_\mu \phi_v \quad .$$

[42] There are a few meaningful subtleties here, as you might expect if you recall that the Ward identity is only true up to contact terms. The measure in the path integral can produce a Jacobian which renormalizes some of the couplings; the changes in source terms will drop out of S-matrix elements (recall our discussion of changing field variables in §2.4) but can change the form of Green's functions. For more information on the use of eom to eliminate redundant operators in EFT, see Arzt, hep-ph/9304230 and Georgi, "On-Shell EFT".

This action is related by a boost to the statement that the atom at rest has zero energy – in the rest frame of the atom, the eom is just $\partial_t \phi_{v=(1,\vec{0})} = 0$.

So the Lagrangian density is

$$L_{\text{Maxwell}}[A] + L_{\text{atom}}[\phi_v] + L_{\text{int}}[A, \phi_v]$$

and we must determine L_{int}. It is made from local, Hermitian, gauge-invariant, Lorentz invariant operators we can construct out of $\phi_v, F_{\mu\nu}, v_\mu, \partial_\mu$ (It can only depend on $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and not A_μ directly, by gauge invariance.). It should actually only depend on the combination $\phi_v^\dagger \phi_v$ since we will not create and destroy atoms. Therefore

$$L_{\text{int}} = c_1 \phi_v^\dagger \phi_v F_{\mu\nu} F^{\mu\nu} + c_2 \phi_v^\dagger \phi_v v^\sigma F_{\sigma\mu} v_\lambda F^{\lambda\mu} + c_3 \phi_v^\dagger \phi_v \left(v^\lambda \partial_\lambda\right) F_{\mu\nu} F^{\mu\nu} + \dots$$

$\dots$ indicates terms with more derivatives and more powers of velocity (*i.e.* an expansion in $\partial \cdot v$). Which are the most important terms at low energies? Demanding that the Maxwell term dominate, we get the power counting rules (so time and space should scale the same way):

$$[\partial_\mu] = 1, \quad [F_{\mu\nu}] = 2$$

This then implies $[\phi_v] = 3/2, [v] = 0$ and therefore

$$[c_1] = [c_2] = -3, [c_3] = -4 \ .$$

Terms with more partials are more irrelevant.

What makes up these dimensions? They must come from the length scales that we have integrated out to get this description – the size of the atom $a_0 \sim \alpha m_e$ and the energy gap between the ground state and the electronic excited states $\Delta E \sim \alpha^2 m_e$. For $E_\gamma \ll \Delta E, a_0^{-1}$, we can just keep the two leading terms.

In the rest frame of the atom, these two leading terms $c_{1,2}$ represent just the scattering of E and B respectively. To determine their coefficients one would have to do a matching calculation to a more complete theory (compute transition rates in a theory that does include extra energy levels of the atom). But a reasonable guess is just that the scale of new physics (in this case atomic physics) makes up the dimensions: $c_1 \simeq c_2 \simeq a_0^3$. (In fact the magnetic term c_2 comes with extra factor of v/c which suppresses it.) The scattering cross section then goes like $\sigma \sim c_i^2 \sim a_0^6$; dimensional analysis ($[\sigma] = -2$ is an area, $[a_0^6] = -6$) then tells us that we have to make up four powers with the only other scale around:

$$\sigma \propto E_\gamma^4 a_0^6.$$

(The factor of E_γ^2 in the amplitude arises from $\vec{E} \propto \partial_t \vec{A}$.) Blue light, which has about twice the energy of red light, is therefore scattered 16 times as much.

The leading term that we left out is the one with coefficient c_3. The size of this coefficient determines when our approximations break down. We might expect this to come from the next smallest of our neglected scales, namely ΔE. That is, we expect

$$\sigma \propto E_\gamma^4 a_0^6 \left(1 + \mathcal{O}\left(\frac{E_\gamma}{\Delta E}\right)\right).$$

The ratio in the correction terms is appreciable for UV light.

4.5 QFT of superconductors and superfluids

4.5.1 Landau-Ginzburg description of superconductors

[Zee §V.3, Weinberg (vII), chapter 21.6.] Without knowing any microscopic details about what the heck is going on inside a superconductor, we can get quite far towards understanding the phenomenology; the only thing we need to know is that charge-$2e$ bosons are condensing. These bosons are created by a complex scalar field Φ. (We will not need to know anything about Cooper pairing or any of that, as long as the boson which is condensing is a scalar.)

So the dofs involved are Φ, A_μ, and there is a *gauge* redundancy $\Phi \to e^{\mathbf{i}2\alpha(x)e}\Phi$, $A_\mu \to A_\mu + \partial_\mu \alpha$. (The third ingredient in the EFT logic is to specify the cutoff; here that is the energy where we are able to see that the theory is made of fermions, let's call it ΔE_ψ. We'll determine it below.) For field configurations that are constant in time, the free energy density (aka the euclidean Lagrangian) must take the form

$$\mathcal{F} = \frac{1}{4}F_{ij}F_{ij} + |D_i\Phi|^2 + a|\Phi|^2 + \frac{1}{2}b|\Phi|^4 + ... \tag{4.7}$$

with $D_i\Phi \equiv (\partial_i - 2e\mathbf{i}A_i)\,\Phi$. Basically this is the same as (3.3) for the O(2)-symmetric magnet, but allowing for the fact that Φ is charged.

Now, as we did above, suppose that a has a zero at some temperature $a(T) = a_1(T_c - T) + ...$, with $a_1 > 0$ (this sign is a physical expectation). For $T > T_c$, the minimum is at $\Phi = 0$. For $T < T_c$ the potential has a minimum at $\langle|\Phi|^2\rangle = -a/b \equiv \rho_0 > 0$. Notice that only the amplitude is fixed. For $T < T_c$, parametrize the field by $\Phi = \sqrt{\rho}e^{\mathbf{i}e\varphi}$ and plug back into the Lagrangian:

$$\mathcal{F} = \frac{1}{4}F_{ij}F_{ij} + (2e)^2\rho\,(\partial_i\varphi + A_i)^2 + \frac{(\partial_i\rho)^2}{4\rho} + V(\rho)$$

(Note that there is a Jacobian for this change of variables in the path integral. We can ignore it.)

We still have a gauge redundancy, which acts by $\varphi \to \varphi + \alpha(x)$. We can use it to fix $\varphi = 0$[43].

If we consider $T \ll T_c$, so that $V(\rho)$ does a good job of keeping $\rho = \rho_0 > 0$, we find:

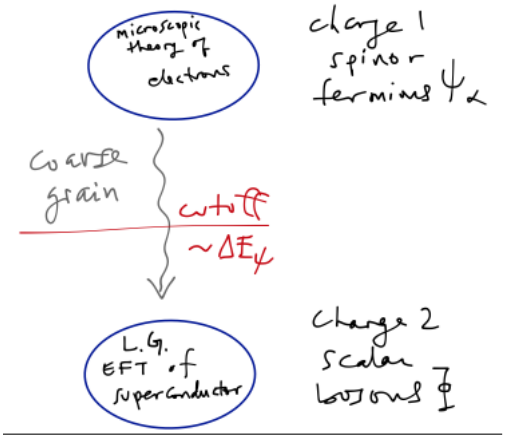
$$\mathcal{F} = \frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}m^2\,(A_i)^2 \tag{4.8}$$

[43] A fancy point: this leaves a residual $\mathbb{Z}_2$ redundancy unfixed. Gauge transformations of the form $\Phi \to e^{\mathbf{i}2e\alpha}\Phi$ with $e^{\mathbf{i}2e\alpha} = 1$ don't act on the charge-2 order parameter field. In this sense, there is a discrete gauge theory left over.

with $m^2 = 2\rho_0^2 e^2$. The photon gets a mass[44]. This is the Anderson-Higgs mechanism. A physical consequence of this that it is not possible to get a magnetic field to penetrate very far into a superconductor. In particular, imagine sticking a magnet on the surface of a superconductor filling $x > 0$; solving the equations of motion following from (4.8) with the boundary condition that $\vec{B}(x=0) = \vec{B}_0$ will show that $\vec{B}(x) = \vec{B}_0 e^{-x/\lambda}$ (it is the same as the Green's function calculation on pset 2) with $\lambda \sim 1/m$ is the *penetration depth*.

[End of Lecture 15]

Symmetry breaking by fluctuations (Coleman-Weinberg) revisited. [Zee problem IV.6.9.] What happens near the transition, when $a = 0$ in (4.7)? Quantum fluctuations can lead to symmetry breaking. This is just the kind of question we discussed earlier, when we introduced the effective potential. Here it turns out that we can trust the answer (roughly because in this scalar electrodynamics, there are two couplings: e and the quartic self-coupling b).



A feature of this example that I want you to notice: the microscopic description of real superconductor involves electrons – charge $1e$ spinor fermions, created by some fermionic operator ψ_α, $\alpha = \uparrow, \downarrow$. We are describing the low-energy physics of a system of electrons in terms of a bosonic field, which (in simple 's-wave' superconductors) is roughly related to the electron field by

$$\Phi \sim \psi_\alpha \psi_\beta \epsilon^{\alpha\beta} \ ; \tag{4.9}$$

[44] For the purposes of this footnote, let's assume that our system is relativistic, so that the form of the lagrangian including the time-derivative terms is fixed:

$$L_{\text{relativistic}} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \Phi|^2 + a|\Phi|^2 + \frac{1}{2} b |\Phi|^4 + \ldots .$$

Everything above is still true. Letting $\langle |\Phi|^2 \rangle = \rho_0$ and choosing unitary gauge $\varphi = 0$, we find

$$L_{\text{relativistic}}|_{\langle|\Phi|^2\rangle=\rho_0, \text{unitary gauge}} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \ .$$

The Proca equation (the eom for A_μ that comes from (4.8))

$$\partial_\nu F^{\mu\nu} = m^2 A^\nu$$

is the Maxwell equation with a source current $j_\mu = m^2 A_\mu$. The Bianchi identity requires $\partial^\mu A_\mu = 0$. In Maxwell theory this is called Lorentz gauge, it is a choice of gauge; here it is not a choice. It is the equation of motion for the field φ that we gauge-fixed, which must be imposed.

Φ is called a Cooper pair field. At least, the charges and the spins and the statistics work out. The details of this relationship are not the important point I wanted to emphasize. Rather I wanted to emphasize the dramatic difference in the correct choice of variables between the UV description (spinor fermions) and the IR description (scalar bosons). One reason that this is possible is that it costs a large energy to make a fermionic excitation of the superconductor. This can be understood roughly as follows: The microscopic theory of the electrons looks something like

$$S[\psi] = S_2[\psi] + \int \mathrm{d}t\mathrm{d}^d x \; u\psi^\dagger\psi\psi^\dagger\psi + h.c. \tag{4.10}$$

where

$$S_2 = \int \mathrm{d}t \int \mathrm{đ}^d k \psi_k^\dagger \left(\mathbf{i}\partial_t - \epsilon(k)\right) \psi_k.$$

Notice the strong similarity with the XY model action in §3.3 (in fact this similarity was Shankar's motivation for explaining the RG for the XY model in the (classic) paper I cited there). A mean field theory description of the condensation of Cooper pairs (4.9) is obtained by replacing the quartic term in (4.10) by expectation values:

$$\begin{aligned} S_{MFT}[\psi] &= S_2[\psi] + \int \mathrm{d}t\mathrm{d}^d x \; u\langle\psi\psi\rangle\psi^\dagger\psi^\dagger + h.c. \\ &= S_2[\psi] + \int \mathrm{d}t\mathrm{d}^d x \; u\Phi\psi^\dagger\psi^\dagger + h.c. \end{aligned} \tag{4.11}$$

So an expectation value for Φ is a mass for the fermions. It is a funny kind of symmetry-breaking mass, but if you diagonalize the quadratic operator in (4.11) (actually it is done below) you will find that it costs an energy of order $\Delta E_\psi = u\langle\Phi\rangle$ to excite a fermion. That's the cutoff on the LG EFT.

A general lesson from this example is: the useful degrees of freedom at low energies can be very different from the microscopic dofs.

4.5.2 Lightning discussion of BCS.

I am sure that some of you are nervous about the step from $S[\psi]$ to $S_{MFT}[\psi]$ above. To make ourselves feel better about it, I will say a few more words about the steps from the microscopic model of electrons (4.10) to the LG theory of Cooper pairs (these steps were taken by Bardeen, Cooper and Schreiffer (BCS)).

First let me describe a useful trick called *Hubbard-Stratonovich transformation* or *completing the square*. It is a ubiquitous stategem in theoretical physics, and is sometimes even useful.

It begins with the following observation about 0+0 dimensional field theory:

$$e^{-\mathbf{i}ux^4} = \sqrt{2\pi u}\int_{-\infty}^{\infty} \mathrm{d}\sigma\ e^{-\frac{1}{\mathbf{i}u}\sigma^2 - 2\mathbf{i}x^2\sigma}\ . \tag{4.12}$$

At the cost of introducing an extra field σ, we can turn a quartic term in x into a quadratic term in x. The RHS of (4.12) is gaussian in x and we know how to integrate it over x. (The version with **i** is relevant for the real-time integral.)

Notice the weird extra factor of **i** lurking in (4.12). This can be understood as arising because we are trying to use a scalar field σ, to mediate a repulsive interaction (which it is, for positive u) (see Zee p. 193, 2nd Ed).

Actually, we'll need a complex H-S field:

$$e^{-\mathbf{i}ux^2\bar{x}^2} = 2\pi u^2\int_{-\infty}^{\infty}\mathrm{d}\sigma\int_{-\infty}^{\infty}\mathrm{d}\bar{\sigma}\ e^{-\frac{1}{\mathbf{i}u}|\sigma|^2 - \mathbf{i}x^2\bar{\sigma} - \mathbf{i}\bar{x}^2\sigma}\ . \tag{4.13}$$

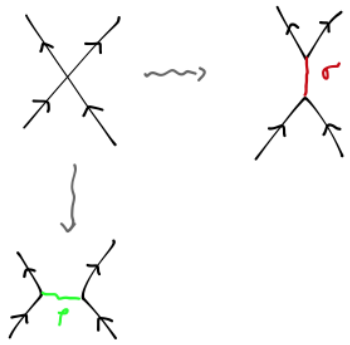
(The field-independent prefactor is, as usual, not important for path integrals.)

We can use a field theory generalization of (4.13) to 'decouple' the 4-fermion interaction in (4.10):

$$Z = \int [D\psi D\psi^\dagger] e^{\mathbf{i}S[\psi]} = \int [D\psi D\psi^\dagger D\sigma D\sigma^\dagger] e^{\mathbf{i}S_2[\psi] + \mathbf{i}\int d^Dx(\bar{\sigma}\psi\psi + h.c.) - \int \mathrm{d}^Dx \frac{|\sigma|^2(x)}{\mathbf{i}u}}\ . \tag{4.14}$$

The point of this is that now the fermion integral is gaussian. At the saddle point of the σ integral (which is exact because it is gaussian), σ is the Cooper pair field, $\sigma_{\text{saddle}} = u\psi\psi$.

Notice that we made a choice here about in which 'channel' to make the decoupling – we could have instead introduces a different auxiliary field ρ and written $S[\rho,\psi] = \int \rho\psi^\dagger\psi + \int \frac{\rho^2}{2u}$, which would break up the 4-fermion interaction in the t-channel (as an interaction of the fermion density $\psi^\dagger\psi$) instead of the s (BCS) channel (as an interaction of Cooper pairs ψ^2). At this stage both are correct, but they lead to different mean-field approximations below. That the BCS mean field theory wins is a consequence of the RG.



How can you resist doing the fermion integral in (4.14)? Let's study the case where the single-fermion dispersion is $\epsilon(k) = \frac{\vec{k}^2}{2m} - \mu$.

$$I_\psi[\sigma] \equiv \int [D\psi D\psi^\dagger] e^{\mathbf{i}\int \mathrm{d}t\mathrm{d}^dx\left(\psi^\dagger\left(\frac{\nabla^2}{2m} - \mu\right)\psi + \psi\bar{\sigma}\psi + \bar{\psi}\bar{\psi}\sigma\right)}$$

The action here can be written as the integral of

$$L = \begin{pmatrix} \bar\psi & \psi \end{pmatrix} \begin{pmatrix} \mathbf{i}\partial_t - \epsilon(-\mathbf{i}\nabla) & \sigma \\ \bar\sigma & -(\mathbf{i}\partial_t - \epsilon(-\mathbf{i}\nabla)) \end{pmatrix} \begin{pmatrix} \psi \\ \bar\psi \end{pmatrix} \equiv \begin{pmatrix} \bar\psi & \psi \end{pmatrix} M \begin{pmatrix} \psi \\ \bar\psi \end{pmatrix}$$

so the integral is

$$I_\psi[\sigma] = \det M = e^{\operatorname{tr}\log M(\sigma)}.$$

The matrix M is diagonal in momentum space, and the integral remaining to be done is

$$\int [D\sigma D\sigma^\dagger] e^{-\int \mathrm{d}^D x \frac{|\sigma(x)|^2}{2\mathbf{i}u} + \int đ^D k \log\left(\omega^2 - \epsilon_k^2 - |\sigma_k|^2\right)}.$$

It is often possible to do this integral by saddle point. This can justified, for example, by the largeness of the volume of the Fermi surface, $\{k|\epsilon(k) = \mu\}$, or by large N number of species of fermions. The result is an equation which determines σ, which as we saw earlier determines the fermion gap.

$$0 = \frac{\delta \text{exponent}}{\delta\bar\sigma} = \mathbf{i}\frac{\sigma}{2u} + \int đ\omega đ^d k \frac{2\sigma}{\omega^2 - \epsilon_k^2 - |\sigma|^2 + \mathbf{i}\epsilon} \ .$$

We can do the frequency integral by residues:

$$\int đ\omega \frac{1}{\omega^2 - \epsilon_k^2 - |\sigma|^2 + \mathbf{i}\epsilon} = \frac{1}{2\pi} 2\pi\mathbf{i} \frac{1}{2\sqrt{\epsilon_k^2 + |\sigma|^2}}.$$

The resulting equation is naturally called the *gap equation*:

$$1 = -2u \int đ^d p' \frac{1}{\sqrt{\epsilon(p')^2 + |\sigma|^2}} \tag{4.15}$$

which you can imagine solving self-consistently for σ. Plugging back into the action (4.14) says that σ determines the energy cost to have electrons around; more precisely, σ is the energy required to break a Cooper pair.

Comments:

- If we hadn't restricted to a delta-function 4-fermion interaction $u(p, p') = u_0$ at the outset, we would have found a more general equation like

$$\sigma(\vec p) = -\frac{1}{2} \int đ^d p' \frac{u(p, p')\sigma(\vec{p'})}{\sqrt{\epsilon(p')^2 + |\sigma(p')|^2}} \ .$$

- Notice that a solution of (4.15) requires $u < 0$, an *attractive* interaction. Superconductivity happens because the u that appears here is not the bare interaction between electrons, which is certainly repulsive (and long-ranged). This is where the phonons come in in the BCS discussion.

- I haven't included here effects of the fluctuations of the fermions. In fact, they make the four-fermion interaction which leads to Cooper pairing marginally *relevant*. This breaks the degeneracy in deciding how to split up the $\psi\psi\psi^\dagger\psi^\dagger$ into *e.g.* $\psi\psi\sigma$ or $\psi^\dagger\psi\rho$. BCS wins. This is explained beautifully in Polchinski, lecture 2, and R. Shankar. I will try to summarize the EFT framework for understanding this in §4.6.

- A conservative perspective on the preceding calculation is that we have made a variational ansatz for the groundstate wavefunction, and the equation we solve for σ is minimizing the variational energy – finding the best wavefunction within the ansatz.

- I've tried to give the most efficient introduction I could here. I left out any possibility of k-dependence or spin dependence of the interactions or the pair field, and I've conflated the pair field with the gap. In particular, I've been sloppy about the dependence on k of σ above.

- You will study a very closely related manipulation on the problem set, in an example where the saddle point is justified by large N.

4.5.3 Non-relativistic scalar fields

[Zee §III.5, V.1, Kaplan nucl-th/0510023 §1.2.1] In the previous discussion of the EFT for a superconductor, I just wrote the free energy, and so we didn't have to think about whether the complex scalar in question was relativistic or not.

It is not. In real superconductors, at least. How should we think about a non-relativistic field? A simple answer comes from realizing that a relativistic field which can make a boson of mass m can certainly make a boson of mass m which is moving slowly, with $v \ll c$. By taking a limit of the relativistic model, then, we can make a description which is useful for describing the interactions of an indefinite number of bosons moving slowly in some Lorentz frame. A situation that calls for such a description is a large collection of ^{4}He atoms.

Non-relativistic limit of a relativistic scalar field. A non-relativistic particle in a relativistic theory (like the ϕ^4 theory that we've been spending time with) has energy

$$E = \sqrt{p^2 + m^2} \overset{\text{if } v \ll c}{=} m + \frac{p^2}{2m} + ...$$

This means that the field that creates and annihilates it looks like

$$\phi(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}}} \left(a_{\vec{k}} e^{\mathbf{i}E_{\vec{k}}t - \mathbf{i}\vec{k}\cdot\vec{x}} + h.c. \right)$$

In particular, we have

$$\dot{\phi}^2 \simeq m^2\phi^2$$

and the BHS of this equation is large. To remove this large number let's change variables:

$$\phi(x,t) \equiv \frac{1}{\sqrt{2m}}\left(e^{-\mathbf{i}mt}\underbrace{\Phi(x,t)}_{\text{complex},\dot{\Phi}\ll m\Phi} + h.c.\right) .$$

Notice that Φ is complex, even if ϕ is real.

Let's think about the action governing this NR sector of the theory. We can drop terms with unequal numbers of Φ and $\Phi^\star$ since such terms would come with a factor of $e^{\mathbf{i}mt}$ which gives zero when integrated over time. Starting from $(\partial\phi)^2 - m^2\phi^2 - \lambda\phi^4$ we get:

$$L_{\text{real time}} = \Phi^\star\left(\mathbf{i}\partial_t + \frac{\vec{\nabla}^2}{2m}\right)\Phi - g^2\left(\Phi^\star\Phi\right)^2 + ... \tag{4.16}$$

with $g^2 = \frac{\lambda}{4m^2}$.

Notice that Φ is a complex field and its action has a U(1) symmetry, $\Phi \to e^{\mathbf{i}\alpha}\Phi$, even though the full theory did not. The associated conserved charge is the number of particles:

$$j_0 = \Phi^\star\Phi, j_i = \frac{\mathbf{i}}{2m}\left(\Phi^\star\partial_i\Phi - \partial_i\Phi^\star\Phi\right), \quad \partial_t j_0 - \nabla\cdot\vec{j} = 0 .$$

Notice that the 'mass term' $\Phi^\star\Phi$ is then actually the chemical potential term, which encourages a nonzero density of particles to be present.

This is another example of an *emergent* symmetry (like baryon number in the SM): a symmetry of an EFT that is not a symmetry of the microscopic theory. The ... in (4.16) include terms which break this symmetry, but they are irrelevant.

To see more precisely what we mean by irrelevant, let's think about scaling. To keep this kinetic term fixed we must scale time and space differently:

$$x \to \tilde{x} = sx,\ t \to \tilde{t} = s^2 t,\ \Phi \to \tilde{\Phi}(\tilde{x},\tilde{t}) = \zeta\Phi(sx, s^2t) .$$

A fixed point with this scaling rule has dynamical exponent $z = 2$. The scaling of the bare action (with no mode elimination step) is

$$S_E^{(0)} = \int \underbrace{dtd^d\vec{x}}_{=s^{d+z}d\tilde{t}d^d\tilde{x}}\left(\Phi^\star\left(sx,s^2t\right)\underbrace{\left(\partial_t - \frac{\vec{\nabla}^2}{2m}\right)}_{=s^{-2}\left(\tilde{\partial}_t - \frac{\tilde{\vec{\nabla}}^2}{2m}\right)}\Phi(sx,s^2t) - g^2\left(\Phi^\star\Phi(sx,s^2t)\right)^2 + ...\right)$$

$$= \underbrace{s^{d+z-2}\zeta^{-2}}_{\stackrel{!}{=}1 \implies \zeta = s^{-3/2}} \int \mathrm{d}\tilde{t}\mathrm{d}^d\tilde{x} \left(\tilde{\Phi}^\star \left(\tilde{\partial}_t - \frac{\tilde{\vec{\nabla}}^2}{2m} \right) \tilde{\Phi} - \zeta^{-2} g^2 \left(\tilde{\Phi}^\star \tilde{\Phi}(\tilde{x}, \tilde{t}) \right)^2 + ... \right) \tag{4.17}$$

From this we learn that $\tilde{g} = s^{-3+2=-1} g \to 0$ in the IR – the quartic term is irrelevant in $D = d + 1 = 3 + 1$ with nonrelativistic scaling! Where does it become marginal? Do pset 5 and think about the delta function problem in pset 1.

[End of Lecture 16]

Number and phase angle. In the NR theory, the canonical momentum for Φ is just $\frac{\partial L}{\partial \dot{\Phi}} \sim \Phi^\star$, with no derivatives. This statement becomes more shocking if we change variables to $\Phi = \sqrt{\rho} e^{\mathbf{i}\theta}$ (which would be useful *e.g.* if we knew ρ didn't want to be zero); the action density is

$$L = \frac{\mathbf{i}}{2}\partial_t \rho - \rho \partial_t \theta - \frac{1}{2m}\left(\rho \left(\nabla\theta\right)^2 + \frac{1}{4\rho}\left(\nabla\rho\right)^2 \right) - g^2\rho^2. \tag{4.18}$$

The first term is a total derivative. The second term says that the canonical momentum for the phase variable θ is $\rho = \Phi^\star\Phi = j_0$, the particle number density. Quantumly, then:

$$[\hat{\rho}(\vec{x}, t), \hat{\varphi}(\vec{x}', t)] = \mathbf{i}\delta^d(\vec{x} - \vec{x}').$$

Number and phase are canonically conjugate variables. If we fix the phase, the amplitude is maximally uncertain.

If we integrate over space, $N \equiv \int d^d x \rho(\vec{x}, t)$ gives the total number of particles, which is time independent, and satisfies $[N, \theta] = \mathbf{i}$.

This relation explains why there's no Higgs boson in most non-relativistic superconductors and superfluids (in the absence of some extra assumption of particle-hole symmetry). In the NR theory with first order time derivative, the would-be amplitude mode which oscillates about the minimum of $V(\rho)$ is actually just the conjugate momentum for the goldstone boson!

4.5.4 Superfluids.

[Zee §V.1] Let me amplify the previous remark. A superconductor is just a superfluid coupled to an external U(1) gauge field, so we've already understood something about superfluids.

The effective field theory has the basic lagrangian (4.18), with $\langle\rho\rangle = \bar{\rho} \neq 0$. This nonzero density can be accomplished by adding an appropriate chemical potential to (4.18); up to

an uninteresting constant, this is

$$L = \frac{\mathbf{i}}{2}\partial_t\rho - \rho\partial_t\theta - \frac{1}{2m}\left(\rho\left(\nabla\theta\right)^2 + \frac{1}{4\rho}\left(\nabla\rho\right)^2\right) - g^2\left(\rho-\bar{\rho}\right)^2.$$

Expand around such a condensed state in small fluctuations $\sqrt{\rho} = \sqrt{\bar{\rho}} + h$, $h \ll \sqrt{\bar{\rho}}$:

$$L = -2\sqrt{\bar{\rho}}h\partial_t\theta - \frac{\bar{\rho}}{2m}\left(\vec{\nabla}\theta\right)^2 - \frac{1}{2m}\left(\vec{\nabla}h\right)^2 - 4g^2\bar{\rho}h^2 + ...$$

Notice that h, the fluctuation of the amplitude mode, is playing the role of the canonical momentum of the goldstone mode θ. The effects of the fluctuations can be incorporated by doing the gaussian integral over h (What suppresses self-interactions of h?), and the result is

$$\begin{aligned} L &= \bar{\rho}\partial_t\theta\frac{1}{4g^2\bar{\rho} - \frac{\nabla^2}{2m}}\bar{\rho}\partial_t\theta - \frac{\bar{\rho}}{2m}\left(\vec{\nabla}\theta\right)^2 \\ &= \frac{1}{4g^2}\left(\partial_t\theta\right)^2 - \frac{\bar{\rho}}{2m}\left(\nabla\theta\right)^2 + ... \end{aligned} \tag{4.19}$$

where in the second line we are expanding in the small wavenumber k of the modes, that is, we are constructing an action for Goldstone modes whose wavenumber is $k \ll \sqrt{9g^2\bar{\rho}m}$ so we can ignore higher gradient terms.

The linearly dispersing mode in this superfluid that we have found, sometimes called the phonon, has dispersion relation

$$\omega^2 = \frac{2g^2\bar{\rho}}{m}\vec{k}^2.$$

This mode has an emergent Lorentz symmetry with a lightcone with velocity $v_c = g\sqrt{2\bar{\rho}/m}$. The fact that the sound velocity involves g – which determined the steepness of the walls of the wine-bottle potential – is a consequence of the non-relativistic dispersion of the bosons. In the relativistic theory, we have $L = \partial_\mu\Phi^\star\partial^\mu\Phi - g\left(\Phi^\star\Phi - v^2\right)^2$ and we can take $g \to \infty$ fixing v and still get a linearly dispersing mode by plugging in $\Phi = e^{\mathbf{i}\theta}v$.

The importance of the linearly dispersing phonon mode of the superfluid is that there is *no other* low energy excitation of the fluid. With a classical pile of (*e.g.* non interacting) bosons, a chunk of moving fluid can donate some small momentum $\vec{k}$ to a single boson at energy cost $\frac{(\hbar\vec{k})^2}{2m}$. A quadratic dispersion means more modes at small k than a linear one (the density of states is $N(E) \propto k^{D-1}\frac{dk}{dE}$). With only a linearly dispersing mode at low energies, there is a critical velocity below which a non-relativistic chunk of fluid cannot give up any momentum [Landau]: conserving momentum $M\vec{v} = M\vec{v}' + \hbar\vec{k}$ says the change in energy (which must be negative for this to happen on its own) is

$$\frac{1}{2}M(v')^2 + \hbar\omega(k) - \frac{1}{2}Mv^2 = \hbar kv + \frac{(\hbar k)^2}{2m} + \hbar\omega(k) = (-v+v_c)k + \frac{(\hbar k)^2}{2m}.$$

For small k, this is only negative when $v > v_c$.

You can ask: an ordinary liquid also has a linearly dispersing sound mode; why doesn't Landau's argument mean that it has superfluid flow? The answer is that it has *other* modes with softer dispersion (so more contribution at low energies), in particular diffusion modes, with $\omega \propto k^2$ (there is an important factor of $\mathbf{i}$ in there).

The Goldstone boson has a compact target space, $\theta(x) \equiv \theta(x) + 2\pi$, since, after all, it is the phase of the boson field. This is significant because it means that as the phase wanders around in space, it can come back to its initial value after going around the circle – such a loop encloses a *vortex*. Somewhere inside, we must have $\Phi = 0$. There is much more to say about this.

4.6 Effective field theory of Fermi surfaces

[Polchinski, lecture 2, and R. Shankar] Electrically conducting solids are a remarkable phenomenon. An arbitrarily small electric field $\vec{E}$ leads to a nonzero current $\vec{j} = \sigma\vec{E}$. This means that there must be gapless modes with energies much less than the natural cutoff scale in the problem.

Scales involved: The Planck scale of solid state physics (made by the logic by which Planck made his quantum gravity energy scale, namely by making a quantity with dimensions of energy out of the available constants) is

$$E_0 = \frac{1}{2}\frac{e^4 m}{\hbar^2} = \frac{1}{2}\frac{e^2}{a_0} \sim 13\text{eV}$$

(where $m \equiv m_e$ is the electron mass and the factor of 2 is an abuse of outside information) which is the energy scale of *chemistry*. Chemistry is to solids as the melting of spacetime is to particle physics. There are other scales involved however. In particular a solid involves a lattice of nuclei, each with $M \gg m$ (approximately the proton mass). So m/M is a useful small parameter which controls the coupling between the electrons and the lattice vibrations. Also, the actual speed of light $c \gg v_F$ can generally also be treated as ∞ to first approximation. v_F/c suppresses spin orbit couplings (though large atomic numbers enhance them: $\lambda_{\text{SO}} \propto Zv_F/c$).

Let us attempt to construct a Wilsonian-natural effective field theory of this phenomenon. The answer is called Landau Fermi Liquid Theory. What are the right low-energy degrees of freedom? Let's make a guess that they are like electrons – fermions with spin and electric charge. They will not have exactly the properties of free electrons, since they must incorporate the effects of interactions with all their friends. The 'dressed' electrons are called quasielectrons, or more generally quasiparticles.

Given the strong interactions between so many particles, why should the dofs have anything at all to do with electrons? Landau's motivation for this description (which is not always correct) is that we can imagine starting from the free theory and adiabatically turning up the interactions. If we don't encounter any phase transition along the way, we can follow each state of the free theory, and use the same labels in the interacting theory.

We will show that there is a nearly-RG-stable fixed point describing gapless quasielectrons. Notice that we are not trying to match this description directly to some microscopic lattice model of a solid; rather we will do bottom-up effective field theory.

Having guessed the necessary dofs, let's try to write an action for them consistent with the

symmetries. A good starting point is the free theory:

$$S_{\text{free}}[\psi] = \int dt\, \text{đ}^d p \left(\mathbf{i}\psi_\sigma^\dagger(p)\partial_t \psi_\sigma(p) - (\epsilon(p) - \epsilon_F)\, \psi_\sigma^\dagger(p)\psi_\sigma(p) \right)$$

where σ is a spin index, ϵ_F is the Fermi energy (zero-temperature chemical potential), and $\epsilon(p)$ is the single-particle dispersion relation. For non-interacting non-relativistic electrons in free space, we have $\epsilon(p) = \frac{p^2}{2m}$. It will be useful to leave this as a general function of p. [45] [46]

The groundstate is the filled Fermi sea:

$$|\text{gs}\rangle = \prod_{p|\epsilon(p)<\epsilon_F} \psi_p^\dagger |0\rangle, \quad \psi_p|0\rangle = 0, \quad \forall p.$$

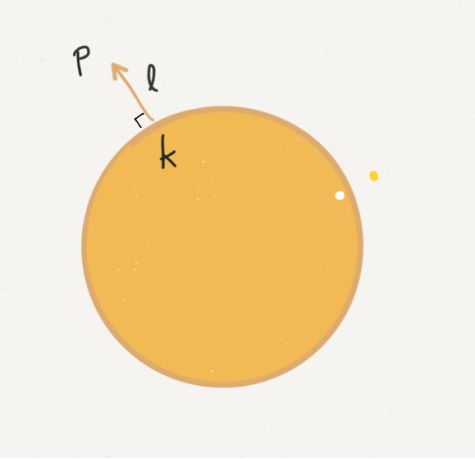
(If you don't like continuous products, put the system in a box so that p is a discrete label.) The Fermi surface is the set of points in momentum space at the boundary of the filled states:

$$\text{FS} \equiv \{p|\epsilon(p) = \epsilon_F\}.$$

The low-lying excitations are made by adding an electron just above the FS or removing an electron (creating a hole) just below.

We would like to define a scaling transformation which focuses on the low-energy excitations. We scale energies by a factor $E \to bE, b < 1$. In relativistic QFT, $\vec{p}$ scales like E, toward zero, $\vec{p} \to b\vec{p}$, since all the low-energy stuff is near $\vec{p} = 0$. Here the situation is much more interesting because the low-energy stuff is on the FS.

One way to implement this is to introduce a hierarchical labeling of points in momentum space, by breaking the momentum space into *patches* around the FS. (An analogous strategy of labeling is also used in heavy quark EFT and in SCET.)



We'll use a slightly different strategy, following Polchinski. To specify a point $\vec{p}$, we pick the nearest point $\vec{k}$ on the FS, $\epsilon(\vec{k}) = \epsilon_F$ (draw a line perpendicular to the FS from $\vec{p}$), and let

$$\vec{p} = \vec{k} + \vec{\ell}.$$

[45] Notice that we are assuming translation invariance. I am not saying anything at the moment about whether translation invariance is discrete (the ions make a periodic potential) or continuous.

[46] We have chosen the normalization of ψ to fix the coefficient of the ∂_t term (this rescaling may depend on p).

So $d-1$ of the components are determined by $\vec{k}$ and one is determined by ℓ. (Clearly there are some exceptional cases if the FS gets too wiggly. Ignore these for now.)

$$\epsilon(p) - \epsilon_F = \ell v_F(\vec{k}) + \mathcal{O}(\ell^2), \qquad v_F \equiv \partial_p \epsilon|_{p=k}.$$

So a scaling rule which accomplishes our goal of focusing on the FS is

$$E \to bE, \quad \vec{k} \to \vec{k}, \quad \vec{l} \to b\vec{\ell}.$$

This implies

$$dt \to b^{-1}dt, \quad d^{d-1}\vec{k} \to d^{d-1}\vec{k}, \quad d\vec{\ell} \to bd\vec{\ell}, \quad \partial_t \to b\partial_t$$

$$S_{\text{free}} = \int \underbrace{dt\ d^{d-1}\vec{k}\ d\vec{\ell}}_{\sim b^0} \left(\mathbf{i}\psi^\dagger(p) \underbrace{\partial_t}_{\sim b^1} \psi(p) - \underbrace{\ell v_F(k)}_{\sim b^1} \psi^\dagger(p)\psi(p) \right)$$

In order to make this go like b^0 we require $\psi \to b^{-\frac{1}{2}}\psi$ near the free fixed point.

Next we will play the EFT game. To do so we must enumerate the symmetries we demand of our EFT:

1. Particle number, $\psi \to e^{\mathbf{i}\theta}\psi$
2. Spatial symmetries: either (a) continuous translation invariance and rotation invariance (as for *e.g.* liquid ^{3}He) or (b) lattice symmetries. This means that momentum space is periodically identified, roughly $p \simeq p + 2\pi/a$ where a is the lattice spacing (the set of independent momenta is called the Brillouin zone (BZ)) and p is only conserved modulo an inverse lattice vector $2\pi/a$; the momentum There can also be some remnant of rotation invariance preserved by the lattice. Case (b) reduces to case (a) if the Fermi surface does not go near the edges of the BZ.
3. Spin rotation symmetry, $\mathsf{SU}(n)$ if $\sigma = 1..n$. In the limit with $c \to \infty$, this is an internal symmetry, independent of rotations.
4. Let's assume that $\epsilon(p) = \epsilon(-p)$, which is a consequence of *e.g.* parity invariance.

Now we enumerate all terms analytic in ψ (since we are assuming that there are no other low-energy operators integrating out which is the only way to get non-analytic terms in ψ) and consistent with the symmetries; we can order them by the number of fermion operators involved. Particle number symmetry means every ψ comes with a $\psi^\dagger$. The possible **quadratic terms** are:

$$\int \underbrace{dt\ d^{d-1}\vec{k}\ d\vec{\ell}}_{\sim b^0}\ \mu(k)\ \underbrace{\psi^\dagger_\sigma(p)\psi_\sigma(p)}_{\sim b^{-1}} \sim b^{-1}$$

is relevant. This is like a mass term. But don't panic: it just shifts the FS around. The *existence* of a Fermi surface is Wilson-natural; any precise location or shape (modulo something enforced by symmetries, like roundness) is not.

Adding one extra ∂_t or factor of ℓ costs a b^1 and makes the operator marginal; those terms are already present in S_{free}. Adding more than one makes it irrelevant.

Quartic terms:

$$S_4 = \int \underbrace{dt \prod_{i=1}^{4} d^{d-1}\vec{k}_i d\vec{\ell}_i}_{\sim b^{-1+4-4/2}} u(4\cdots 1)\psi^\dagger_\sigma(p_1)\psi_\sigma(p_3)\psi^\dagger_{\sigma'}(p_2)\psi_{\sigma'}(p_4)\delta^d(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$

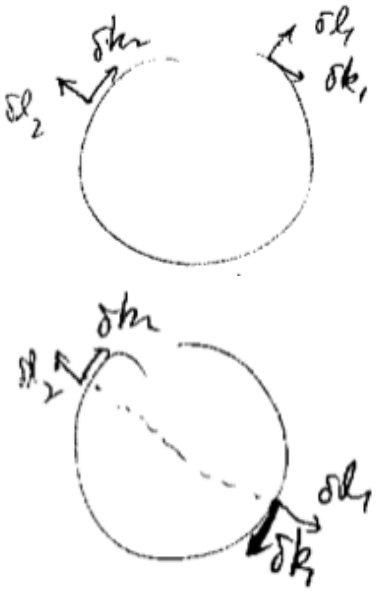
Note the similarity with the discussion of the XY model in §3.3. The minus signs on $p_{3,4}$ is because $\psi(p)$ *removes* a particle with momentum p. We assume u depends only on k, σ, so does not scale – this will give the most relevant piece. How does the delta function scale?

$$\delta^d\left(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4\right) = \delta^d\left(k_1+k_2-k_3-k_4+\ell_1+\ell_2-\ell_3-\ell_4\right) \overset{?}{\simeq} \delta^d\left(k_1+k_2-k_3-k_4\right)$$

In the last (questioned) step, we used the fact that $\ell \ll k$ to ignore the contributions of the ℓs. If this is correct then the delta function does not scale (since ks do not), and $S_4 \sim b^1$ is irrelevant (and quartic interactions with derivatives are moreso). If this were correct, the free-fixed point would be exactly stable.

[End of Lecture 17]

There are two important subtleties: (1) there exist phonons. (2) the questioned equality above is questionable because of kinematics of the Fermi surface. We will address these two issues in reverse order.



The **kinematic subtlety** in the treatment of the scaling of $\delta(p_1+p_2-p_3-p_4)$ arises because of the geometry of the Fermi surface. Consider scattering between two points on the FS, where (in the labeling convention above)

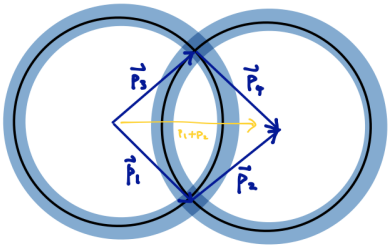
$$p_3 = p_1 + \delta k_1 + \delta \ell_1, \quad p_4 = p_2 + \delta k_2 + \delta \ell_2,$$

in which case the momentum delta function is

$$\delta^d(p_1+p_2-p_3-p_4) = \delta^d(\delta k_1 + \delta \ell_1 + \delta k_2 + \delta \ell_2).$$

For generic choices of the two points $p_{1,2}$ (top figure at left), δk_1 and δk_2 are linearly independent and the $\delta\ell$s can

indeed be ignored as we did above. However, for two points with $p_1 = -p_2$ (they are called *nested*, as depicted in the bottom figure at left), then one component of $\delta k_1 + \delta k_2$ is automatically zero, revealing the tiny $\delta\ell$s to the force of (one component of) the delta function. In this case, $\delta(\ell)$ scales like b^{-1}, and for this particular kinematic configuration the four-fermion interaction is (classically) marginal. Classically marginal means quantum mechanics has a chance to make a big difference.



A useful visualization is at right ($d = 2$ with a round FS is shown; this is what's depicted on the cover of the famous book by Abrikosov-Gorkov-Dzyaloshinski): the blue circles have radius k_F; the yellow vector is the sum of the two initial momenta $p_1 + p_2$, both of which are on the FS; the condition that $p_3 + p_4$, each also on the FS, add up to the same vector means that p_3 must lie on the intersection of the two circles (spheres in $d > 2$). But when $p_1 + p_2 = 0$, the two circles are on top of each other so they intersect everywhere! Comments:

1. We assumed that both p_1 and $-p_2$ were actually *on* the FS. This is automatic if $\epsilon(p) = \epsilon(-p)$, *i.e.* if ϵ is only a function of p^2.

2. This discussion works for any $d > 1$.

3. **Forward scattering.** There is a similar phenomenon for the case where $p_1 = p_3$ (and hence $p_2 = p_4$). This is called *forward scattering* because the final momenta are the same as the initial momenta. (We could just as well take $p_1 = p_4$ (and hence $p_2 = p_3$).) In this case too the delta function will constrain the ℓs and will therefore scale.

The tree-level-marginal 4-Fermi interactions at special kinematics leads to a *family* of fixed points labelled by 'Landau parameters'. In fact there is whole *function*s worth of fixed points. In 2d, the fixed point manifold is parametrized by the forward-scattering function

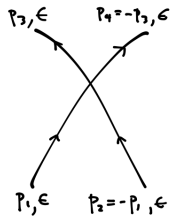
$$F(\theta_1, \theta_2) \equiv u(\theta_4 = \theta_2, \theta_3 = \theta_1, \theta_2, \theta_1)$$

(Fermi statistics implies that $u(\theta_4 = \theta_1, \theta_3 = \theta_2, \theta_2, \theta_1) = -F(\theta_1, \theta_2)$.) and the BCS-channel interaction:

$$V(\theta_1, \theta_3) = u(\theta_4 = -\theta_3, \theta_3, \theta_2 = -\theta_1, \theta_1).$$

Now let's think about what decision the fluctuations make about the fate of the nested interactions. The first claim, which I will not justify here, is that F is not renormalized at

one loop. The interesting bit is the renormalization of the BCS interaction:



The electron propagator, obtained by inverting the kinetic operator S_{free}, is

$$G(\epsilon, p = k + l) = \frac{1}{\epsilon(1 + \mathbf{i}\eta) - v_F(k)\ell + \mathcal{O}(\ell)^2}$$

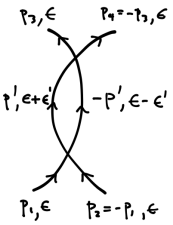
where I used $\eta \equiv 0^+$ for the infinitesimal specifying the contour prescription. (To understand the contour prescription for the *hole* propagator, it is useful to begin with

$$G(t, p) = \langle \epsilon_F | c_p^\dagger(t) c_p(0) | \epsilon_F \rangle, \quad c_p^\dagger(t) \equiv e^{-\mathbf{i}\mathbf{H}t} c_p^\dagger e^{\mathbf{i}\mathbf{H}t}$$

and use the free-fermion fact $[\mathbf{H}, c_p^\dagger] = \epsilon_p c_p^\dagger$.)

Let's assume rotation invariance. Then $V(\theta_3, \theta_1) = V(\theta_3 - \theta_1)$, $V_l = \int \mathrm{d}\theta e^{\mathbf{i}l\theta} V(\theta)$. Different angular momentum sectors decouple from each other at one loop.

We will focus on the s-wave bit of the interaction, so V is independent of momentum. We will integrate out just a shell in energy (depicted by the blue shaded shell in the Fermi surface figures) The interesting contribution comes from the following diagram:

$\delta^{(1)}V =$ 

$$\begin{aligned}
&= \mathbf{i}V^2 \int_{b\epsilon_0}^{\epsilon_0} \frac{d\epsilon' d^{d-1}k' d\ell'}{(2\pi)^{d+1}} \frac{1}{(\epsilon + \epsilon' - v_F(k')\ell')\,(\epsilon - \epsilon' - v_F(k')\ell')} \\
&\text{do } \textstyle\int d\ell' \text{ by residues} \qquad = \mathbf{i}V^2 \int \frac{d\epsilon' d^{d-1}k'}{(2\pi)^{d+1}} \frac{1}{v_F(k')} \left(\underbrace{\epsilon - \epsilon' - (\epsilon + \epsilon')}_{=-2\epsilon'} \right)^{-1} \\
&= -V^2 \underbrace{\int_{b\epsilon_0}^{\epsilon_0} \frac{d\epsilon'}{\epsilon'}}_{=\log(1/b)} \underbrace{\int \frac{d^{d-1}k'}{(2\pi)^d v_F(k')}}_{\text{dos at FS}}
\end{aligned} \tag{4.20}$$

Between the first and second lines, we did the ℓ' integral by residues. The crucial point is that we are interested in external energies $\epsilon \sim 0$, but we are integrating out a shell near the cutoff, so $|\epsilon'| > |\epsilon|$ and the sign of $\epsilon + \epsilon'$ is opposite that of $\epsilon - \epsilon'$; therefore there is a pole on either side of the real ℓ axis and we get the same answer by closing the contour either way. On one side the pole is at $\ell' = \frac{1}{v_F(k')}(\epsilon + \epsilon')$. (In the t-channel diagram (what Shankar calls ZS), the poles are on the *same* side and it therefore does not renormalize the four-fermion interaction.)

The result to one-loop is then

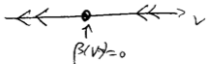
$$V(b) = V - V^2 N \log(1/b) + \mathcal{O}(V^3)$$

with $N \equiv \int \frac{d^{d-1}k'}{(2\pi)^d v_F(k')}$ is the density of states at the Fermi surface. From this we derive the beta function

$$b\frac{d}{db}V(b) = \beta_V = NV^2(b) + \mathcal{O}(V^3)$$

and the solution of the flow equation at $E = bE_1$ is

$$V(E) = \frac{V_1}{1 + NV_1 \log(E_1/E)} \begin{cases} \to 0 & \text{in IR for } V_1 > 0 \text{ (repulsive)} \\ \to -\infty & \text{in IR for } V_1 < 0 \text{ (attractive)} \end{cases} \tag{4.21}$$

There is therefore a very significant dichotomy depending on the sign of the coupling at the microscopic scale E_1, as in this phase diagram: 

The conclusion is that if the interaction starts attractive at some scale it flows to large attractive values. The thing that is decided by our perturbative analysis is that (if $V(E_1) > 0$) the decoupling we did with σ ('the BCS channel') wins over the decoupling with ρ ('the particle-hole channel'). What happens at $V \to -\infty$? Here we need non-perturbative physics.

The non-perturbative physics is in general hard, but we've already done what we can in §4.5.2.

The remaining question is: Who is V_1 and why would it be attractive (given that Coulomb interactions between electrons, while screened and therefore short-ranged, are repulsive)? The answer is:

Phonons. The lattice of positions taken by the ions making up a crystalline solid spontaneously break many spacetime symmetries of their governing Hamiltonian. This implies a collection of gapless Goldstone modes in any low-energy effective theory of such a solid[47]. The Goldstone theorem is satisfied by including a field

$$\vec{D} \propto \text{(local) displacement } \delta\vec{r} \text{ of ions from their equilibrium positions}$$

Most microscopically we have a bunch of coupled springs:

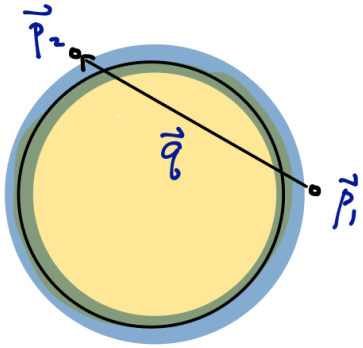
$$L_{\text{ions}} \sim \frac{1}{2}M\left(\delta\dot{\vec{r}}\right)^2 - k_{ij}\delta r^i \delta r^j + \ldots$$

[47]Note that there is a subtlety in counting Goldstone modes from spontaneously broken spacetime symmetries: there are more symmetry generators than Goldstones. Basically it's because the associated currents differ only by functions of spacetime; but a localized Goldstone particle is anyway made by a current times a function of spacetime, so you can't sharply distinguish the resulting particles. Some useful references on this subject are Low-Manohar and most recently Watanabe-Murayama.

with spring constants k independent of the nuclear mass M. It is useful to introduce a canonically normalized field in terms of which the action is

$$S[\vec{D} = (M)^{1/2}\,\delta\vec{r}] = \frac{1}{2}\int dtd^dq\left(\partial_t D_i(q)\partial_t D_i(-q) - \omega_{ij}^2(q)D_i(q)D_j(-q)\right).$$

Here $\omega^2 \propto M^{-1}$. Their status as Goldstones means that the eigenvalues of $\omega_{ij}^2(q) \sim |q|^2$ at small q: moving everyone by the same amount does not change the energy. This also constrains the coupling of these modes to the electrons: they can only couple through derivative interactions.



For purposes of their interactions with the electrons, a nonzero q which keeps the e^- on the FS must scale like $q \sim b^0$. Therefore

$$dtd^dq\,(\partial_t D)^2 \sim b^{+1+2[D]} \quad \Longrightarrow \quad D \sim b^{-\frac{1}{2}}$$

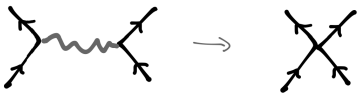
and the restoring force $dtdqD^2\omega^2(q) \sim b^{-2}$ is relevant, and dominates over the ∂_t^2 term for

$$E < E_D = \sqrt{\frac{m}{M}}E_0 \quad \text{the Debye energy.}$$

This means that phonons mediate *static* interactions below E_D – we can ignore retardation effects, and their effects on the electrons can be fully incorporated by the four-fermion interaction we used above (with some $\vec{k}$ dependence). How do they couple to the electrons?

$$\begin{aligned} S_{\rm int}[D,\psi] &= \int dtq^3qd^2k_1d\ell_1d^2k_2d\ell_2\; M^{-\frac{1}{2}}g_i(q,k_1,k_2)D_i(q)\psi_\sigma^\dagger(p_1)\psi_\sigma(p_2)\delta^3(p_1-p_2-q) \\ &\sim b^{-1+1+1-3/2} = b^{-1/2} \end{aligned} \tag{4.22}$$

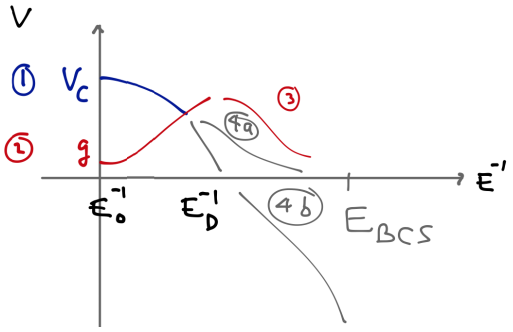
– here we took the delta function to scale like b^0 as above. This is relevant when we use the $\dot{D}^2$ scaling for the phonons; when the restoring force dominates we should scale D differently and this is irrelevant for generic kinematics. This is consistent with our previous analysis of the four-fermion interaction.

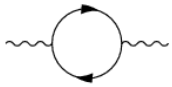
The summary of this discussion is: phonons do not destroy the Fermi surface, but they *do* produce an attractive contribution to the 4-fermion interaction, which is relevant in some range of scales (above the Debye energy). Below the Debye energy, it amounts to an addition to V that goes like $-g^2$: 

Notice that the scale at which the coupling V becomes strong ($V(E_{\rm BCS}) \equiv 1$ in (4.21)) is

$$E_{\rm BCS} \sim E_D e^{-\frac{1}{NV_D}}.$$

Two comments about this: First, it is non-perturbative in the interaction V_D. Second, it provides some verification of the role of phonons, since $E_D \sim M^{-1/2}$ can be varied by studying the same material with different isotopes and studying how the critical superconducting temperature ($\sim E_{\rm BCS}$) scales with the nuclear mass.



Here's the narrative, proceeding as a function of decreasing energy scale, beginning at E_0, the Planck scale of solids: (1) Electrons repel each other by the Coulomb interaction. However, in a metal, this interaction is *screened* by processes like this:  (the intermediate state is an electron-hole pair) and is short-ranged. It is still repulsive, however. As we coarse-grain more and more, we see more and more electron-hole pairs and the force weakens. (2) While this is happening, the electron-phonon interaction is relevant and growing. This adds an attractive bit to V. This lasts until E_D. (3) At E_D the restoring force term in the phonon lagrangian dominates (for the purposes of their interactions with the electrons) and we can integrate them out. (4) What happens next depends on the sign of $V(E_D)$. If it's positive, V flows harmlessly to zero. If it's negative, it becomes moreso until we exit the perturbative analysis at $E_{\rm BCS}$, and vindicate our choice of Hubbard-Stratonovich channel above.

Further brief comments, for which I refer you to Shankar:

1. Putting back the possible angular dependence of the BCS interaction, the result at one loop is
$$\frac{dV(\theta_1-\theta_3)}{d\ell} = -\frac{1}{8\pi^2}\int_0^{2\pi} đ\theta V(\theta_1-\theta)V(\theta-\theta_3)$$
or in terms of angular momentum components,
$$\frac{dV_l}{d\ell} = -\frac{V_l^2}{4\pi}.$$

2. This example is interesting and novel in that it is a (family of) fixed point(s) characterized by a dimensionful quantity, namely k_F. This leads to a phenomenon called *hyperscaling violation* where thermodynamic quantities need not have their naive scaling with temperature.

3. The one loop analysis gives the right answer to all loops in the limit that $N \equiv k_F/\Lambda \gg 1$, where Λ is the UV cutoff on the momentum.

4. The forward scattering interaction (for any choice of function $F(\theta_{13})$) is not renormalized at one loop. This means it is exactly marginal at leading order in N.

5. Like in ϕ^4 theory, the sunrise diagram at two loops is the first appearance of wave-function renormalization. In the context of the Fermi liquid theory, this leads to the renormalization of the effective mass which is called $m^\star$.

Another consequence of the FS kinematics which I should emphasize more: it allows the quasiparticle to be stable. The leading contribution to the decay rate of a one-quasiparticle state with momentum k can be obtained applying the optical theorem to the following process.

The intermediate state is two electrons with momenta $k'+q$ and $k-q$, and one *hole* with momentum k'. The hole propagator has the opposite $\mathbf{i}\eta$ prescription. After doing the frequency integrals by residues, we get

$$\Sigma(k,\epsilon) = \int đq\, đk' \frac{|u_q|^2}{D - \mathbf{i}\eta}$$

$$D \equiv \epsilon_k(1+\mathbf{i}\eta) + \epsilon_{k'}(1-\mathbf{i}\eta) - \epsilon_{k'+q}(1+\mathbf{i}\eta) - \epsilon_{k-q}(1+\mathbf{i}\eta)$$

(Notice that this is the eyeball diagram which gives the lowest-order contribution to the wavefunction renormalization of a field with quartic interactions.) By the optical theorem, its imaginary part is the (leading contribution to the) inverse-lifetime of the quasiparticle state with fixed k:

$$\tau^{-1}(k) = \mathrm{Im}\,\Sigma(k,\epsilon) = \pi \int đq\, đk' \delta(D) |u_q|^2 f(-\epsilon_{k'}) f(\epsilon_{k'+q}) f(\epsilon_{k-q})$$

where

$$f(\epsilon) = \lim_{T\to 0} \frac{1}{e^{\frac{\epsilon-\epsilon_F}{T}}+1} = \theta(\epsilon < \epsilon_F)$$

is the Fermi function. This is just the demand that a particle can only scatter into an empty state and a hole can only scatter into a filled state. These constraints imply that all the energies are near the Fermi energy: both $\epsilon_{k'+q}$ and $\epsilon_{k'}$ lie in a shell of radius ϵ about the FS; the answer is proportional to the density of possible final states, which is thus

$$\tau^{-1} \propto \left(\frac{\epsilon}{\epsilon_F}\right)^2 .$$

So the width of the quasiparticle resonance is

$$\tau^{-1} \propto \epsilon^2 \ll \epsilon$$

much smaller than its frequency – it is a sharp resonance, a well-defined particle.

[End of Lecture 18]

5 Roles of topology in QFT

Topology means the study of quantities which can't vary smoothly, but can only jump. Like quantities which must be integers. But the Wilson RG is a smooth process. Therefore topological information in a QFT is something the RG can't wash away – information which is RG invariant. We'll study two examples.

5.1 Anomalies

[Zee §IV.7; Polyakov, *Gauge Fields and Strings*, §6.3; K. Fujikawa, *Phys. Rev. Lett.* **42** (1979) 1195; Argyres, 1996 lectures on supersymmetry §14.3; Peskin, chapter 19]

Suppose we have in our hands a classical field theory in the continuum which has some symmetry. Is there a well-defined QFT whose classical limit produces this classical field theory and preserves that symmetry? The path integral construction of QFT offers some insight here. The path integral involves two ingredients: (1) an action, which is shared with the classical field theory, and (2) a path integral measure. It is possible that the action is invariant but the measure is not. This is called an anomaly. It means that the symmetry is broken, and its current conservation is violated by a known amount, and this often has many other consequences that can be understood by humans.

Notice that here I am speaking about actual, global symmetries. I am not talking about gauge redundancies. If you think that two field configurations are equivalent but the path integral tells you that they would give different contributions, you are doing something wrong. An anomaly in a 'gauge symmetry' means that the system has more degrees of freedom than you thought. (In particular, it does not mean that the world is inconsistent. For a clear discussion of this, please see Preskill, 1990.)

We have already seen a dramatic example of an anomaly: the violation of classical scale invariance (*e.g.* in massless ϕ^4 theory, or in massless QED) by quantum effects.

Notice that the name 'anomaly' betrays the bias that we construct a QFT by starting with a continuum action for a classical field theory; you would never imagine that *e.g.* scale invariance was an exact symmetry if you started from a well-defined quantum lattice model.

The example we will focus on here is the *chiral anomaly*. This is an equation for the violation of the chiral (aka axial) current for fermions coupled to a background gauge field. The chiral anomaly was first discovered in perturbation theory, by computing a certain Feynman diagram with a triangle; the calculation was motivated by the experimental observation of

$\pi^0 \to \gamma\gamma$, which would vanish if the chiral current were conserved.

I will outline a derivation of this effect which is more illuminating than the triangle diagram. It shows that the one-loop result is exact – there are no other corrections. It shows that the quantity on the right hand side of the continuity equation for the would-be current integrates to an integer. It gives a proof of the *index theorem*, relating numbers of solutions to the Dirac equation in a background field configuration to a certain integral of field strengths. It butters your toast.

5.1.1 Chiral anomaly

Chiral symmetries. In even-dimensional spacetimes, the Dirac representation of SO($D-1,1$) is reducible. This is because

$$\gamma^5 \equiv \prod_{\mu=0}^{D-1} \gamma^\mu \neq 1, \quad \text{satisfies} \quad \{\gamma^5, \gamma^\mu\} = 0, \forall \mu$$

which means that γ^5 commutes with the Lorentz generators

$$[\gamma^5, \Sigma^{\mu\nu}] = 0, \qquad \Sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu].$$

A left- or right-handed Weyl spinor is an irreducible representation of SO($D-1,1$), $\psi_{L/R} \equiv \frac{1}{2}(1 \pm \gamma^5)\psi$. This allows the possibility that the L and R spinors can transform differently under a symmetry; such a symmetry is a chiral symmetry.

Note that in $D = 4k$ dimensions, if ψ_L is a left-handed spinor in representation $\mathbf{r}$ of some group G, then its image under CPT, $\psi_L^{CPT}(t,\vec{x}) \equiv \mathbf{i}\gamma^0 (\psi_L(-t,-\vec{x}))^\star$, is right-handed and transforms in representation $\bar{\mathbf{r}}$ of G. Therefore chiral symmetries arise when the Weyl fermions transform in *complex representations* of the symmetry group, where $\bar{\mathbf{r}} \neq \mathbf{r}$. (In $D = 4k+2$, CPT maps left-handed fields to left-handed fields. For more detail on discrete symmetries and Dirac fields, see Peskin §3.6.)

Some more explicit words about chiral fermions in $D = 3+1$, mostly notation. Recall Peskin's *Weyl* basis of gamma matrices in 3+1 dimensions, in which γ^5 is diagonal:

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\boldsymbol{\sigma}}^\mu \\ \boldsymbol{\sigma}^\mu & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^\mu \equiv (\mathbb{1}, \vec{\boldsymbol{\sigma}})^\mu, \quad \bar{\boldsymbol{\sigma}}^\mu \equiv (\mathbb{1}, -\vec{\boldsymbol{\sigma}})^\mu, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This makes the reducibility of the Dirac representation of SO(3,1) manifest, since the Lorentz generators are $\propto [\gamma^\mu, \gamma^\nu]$ block diagonal in this basis. The gammas are a map from the $(1, \mathbf{2_R})$

representation to the $(\mathbf{2_L}, \mathbf{1})$ representation. It is sometimes useful to denote the $\mathbf{2_R}$ indices by $\alpha, \beta = 1, 2$ and the $\mathbf{2_L}$ indices by $\dot{\alpha}, \dot{\beta} = 1, 2$. Then we can define two-component Weyl spinors $\psi_{L/R} = P_{L/R}\psi \equiv \frac{1}{2}\left(1 \pm \gamma^5\right)\psi$ by simply forgetting about the other two components. The conjugate of a L spinor $\chi = \psi_L$ (L means $\gamma^5\chi = \chi$) is right-handed:

$$\bar{\chi} = \chi^\dagger\gamma^0, \quad \bar{\chi}\gamma^5 = \chi^\dagger\gamma^0\gamma^5 = -\chi^\dagger\gamma^5\gamma^0 = -\chi^\dagger\gamma^0 = -\bar{\chi}.$$

We can represent any system of Dirac fermions in terms of a collection of twice as many Weyl fermions.

For a continuous symmetry G, we can be more explicit about the meaning of a complex representation. The statement that ψ is in representation $\mathbf{r}$ means that its transformation law is

$$\delta\psi_a = \mathbf{i}\epsilon^A \left(t^A_{\mathbf{r}}\right)_{ab} \psi_b$$

where $t^A, A = 1..\dim G$ are generators of G in representation $\mathbf{r}$; for a compact lie group G, we may take the t^A to be Hermitian. The conjugate representation, by definition is the one with which you can make a singlet of G – it's the way $\psi^{\star T}$ transforms:

$$\delta\psi^{\star T}_a = -\mathbf{i}\epsilon^A \left(t^A_{\mathbf{r}}\right)^T_{ab} \psi^{\star T}_b \ .$$

So:

$$t^A_{\bar{\mathbf{r}}} = -\left(t^A_{\mathbf{r}}\right)^T .$$

The condition for a complex representation is that this is different from $t^A_{\mathbf{r}}$ (actually we have to allow for relabelling of the generators). The simplest case is $G = \mathrm{U}(1)$, where t is just a number indicating the charge. In that case, any nonzero charge gives a complex representation.

Consider the effective action produced by integrating out Dirac fermions coupled to a *background* gauge field (the gauge field is just going to sit there for this whole calculation):

$$e^{\mathbf{i}S_{\text{eff}}[A]} \equiv \int [D\psi D\bar{\psi}]\ e^{\mathbf{i}S[\psi,\bar{\psi},A]} \ .$$

We must specify how the fermions coupled to the gauge field. The simplest example is if A is a $U(1)$ gauge field and ψ is minimally coupled:

$$S[\psi, \bar{\psi}, A] = \int \mathrm{d}^D x \bar{\psi}\mathbf{i}\not{D}\psi, \qquad \not{D}\psi \equiv \gamma^\mu \left(\partial_\mu + \mathbf{i}A_\mu\right)\psi.$$

We will focus on this example, but you could imagine instead that A_μ is a non-Abelian gauge field for the group G, and ψ is in a representation R, with gauge generators $T^A(R)$ ($A = 1...\dim G$), so the coupling would be

$$\bar{\psi}\mathbf{i}\not{D}\psi = \bar{\psi}_a\gamma^\mu \left(\partial_\mu\delta_{ab} + \mathbf{i}A^A_\mu T^A(R)_{ab}\right)\psi_b \ . \tag{5.1}$$

Much of the discussion below applies for any even D.

In the absence of a mass term, the action (in the Weyl basis) involves no coupling between L and R:

$$S[\psi,\bar\psi,A]=\int \mathrm{d}^D x\left(\psi_L^\dagger \mathbf{i}\sigma^\mu D_\mu\psi_L+\psi_R^\dagger \mathbf{i}\bar\sigma^\mu D_\mu\psi_R\right)$$

and therefore is invariant under the global chiral rotation

$$\psi\to e^{\mathbf{i}\alpha\gamma^5}\psi,\quad \psi^\dagger\to\psi^\dagger e^{-\mathbf{i}\alpha\gamma^5},\quad \bar\psi\to\bar\psi e^{+\mathbf{i}\alpha\gamma^5}.\quad \text{That is: } \psi_L\to e^{\mathbf{i}\alpha}\psi_L,\quad \psi_R\to e^{-\mathbf{i}\alpha}\psi_R.$$

(The mass term couples the two components

$$L_m=\bar\psi\left(\mathrm{Re}m+\mathrm{Im}m\gamma^5\right)\psi=m\psi_L^\dagger\psi_R+h.c.;$$

notice that the mass parameter is complex.) The associated Noether current is $j_\mu^5=\bar\psi\bar\gamma^5\gamma_\mu\psi$, and it seems like we should have $\partial^\mu j_\mu^5\stackrel{?}{=}0$. This follows from the massless (classical) Dirac equation $0=\gamma^\mu\partial_\mu\psi$. (With the mass term, we would have instead $\partial^\mu j_\mu^5\stackrel{?}{=}2\mathbf{i}\bar\psi\left(\mathrm{Re}m\gamma^5+\mathrm{Im}m\right)\psi$.)

Notice that there is another current $j^\mu=\bar\psi\gamma^\mu\psi$. j^μ is the current which is coupled to the gauge field, $L\ni A_\mu j^\mu$. The conservation of this current is required for gauge invariance of the effective action

$$S_{\text{eff}}[A_\mu]\stackrel{!}{=}S_{\text{eff}}[A_\mu+\partial_\mu\lambda]\sim\log\langle e^{\mathbf{i}\int\lambda(x)\partial_\mu j^\mu}\rangle+S_{\text{eff}}[A_\mu].$$

No matter what happens we can't find an anomaly in j^μ. The anomalous one is the other one, the *axial current*.

To derive the conservation law we can use the Noether method. This amounts to substituting $\psi'(x)\equiv e^{\mathbf{i}\alpha(x)\gamma^5}\psi(x)$ into the action:

$$S_F[\psi']=\int \mathrm{d}^D x\bar\psi e^{+\mathbf{i}\alpha\gamma^5}\mathbf{i}\not{D}e^{\mathbf{i}\alpha\gamma^5}\psi=\int \mathrm{d}^D x\left(\bar\psi\mathbf{i}\not{D}\psi+\bar\psi\mathbf{i}\gamma^5\left(\not{\partial}\alpha\right)\psi\right)\stackrel{\text{IBP}}{=}S_F[\psi]-\mathbf{i}\int\alpha(x)\partial^\mu\mathrm{tr}\bar\psi\gamma^5\gamma_\mu\psi.$$

Then we can completely get rid of $\alpha(x)$ if we can change integration variables, *i.e.* if $[D\psi']\stackrel{?}{=}[D\psi]$. Usually this is true, but here we pick up an interesting Jacobian.

Claim:

$$e^{\mathbf{i}S_{\text{eff}}[A]}=\int[D\psi' D\bar\psi']e^{\mathbf{i}S_F[\psi']}=\int[D\psi D\bar\psi]e^{\mathbf{i}S_F[\psi]+\int\mathrm{d}^D x\alpha(x)\left(\partial_\mu j_5^\mu-\mathcal{A}(x)\right)}$$

where

$$\mathcal{A}(x)=\sum_n\mathrm{tr}\bar\xi_n\gamma^5\xi_n \tag{5.2}$$

where ξ_n are a basis of eigenspinors of the Dirac operator. The contribution to $\mathcal{A}$ can be attributed to zeromodes of the Dirac operator.

The expression above is actually independent of α. Usually α would multiply the divergence of the current and this would imply current conservation. Here this implies that instead of current conservation we have a specific violation of the current:

$$\partial^\mu j^5_\mu = \mathcal{A}(x).$$

What is the anomaly. [Polyakov §6.3] An alternative useful (perhaps more efficient) perspective is that the anomaly arises from trying to define the axial current operator, which after all is a composite operator. Thus we should try to compute

$$\langle \partial_\mu j^\mu_5 \rangle = \partial_\mu \langle \bar\psi(x)\gamma^\mu\gamma^5\psi(x)\rangle$$

– the coincident operators on the RHS need to be regulated.

Consider Dirac fermions coupled to a *background* gauge field configuration $A_\mu(x)$, with action

$$S = \int \mathrm{d}^D x \bar\psi \left(\mathbf{i}\gamma^\mu \left(\partial_\mu + \mathbf{i}A_\mu\right)\right)\psi.$$

For a while the discussion works in any even dimension, where $\gamma^5 = \prod_{\mu=0}^{D-1}\gamma^\mu$ satisfies $\{\gamma^\mu,\gamma^5\} = 0$ and is not the identity. (The discussion that follows actually works also for non-Abelian gauge fields.) The classical Dirac equation immediately implies that the axial current is conserved

$$\partial_\mu \left(\mathbf{i}\bar\psi\gamma^\mu\gamma^5\psi\right) \stackrel{?}{=} 0.$$

Consider, on the other hand, the expectation value

$$\begin{aligned} J^5_\mu \equiv \langle \mathbf{i}\bar\psi(x)\gamma_\mu\gamma^5\psi(x)\rangle &\equiv Z^{-1}[A]\int [D\psi D\bar\psi] e^{-S_F[\psi]} j^5_\mu \\ &= \quad + \quad + \quad + \cdots \\ &= -\mathbf{i}\mathrm{Tr}\ {}_\gamma\gamma_\mu\gamma^5 G^{[A]}(x,x) \end{aligned} \tag{5.3}$$

where G is the Green's function of the Dirac operator in the gauge field background (and the figure is from Polyakov's book). We can construct it out of eigenfunctions of $\mathbf{i}\not{D}$:

$$\mathbf{i}\not{D}\xi_n(x) = \epsilon_n\xi_n(x), \qquad \bar\xi_n(x)\mathbf{i}\gamma^\mu\left(-\overleftarrow{\partial}_\mu + \mathbf{i}A_\mu\right) = \epsilon_n\bar\xi_n \tag{5.4}$$

in terms of which[48]

$$G(x,x') = \sum_n \frac{1}{\epsilon_n}\xi_n(x)\bar\xi_n(x'). \tag{5.5}$$

[48] Actually, this step is full of danger. (Polyakov has done it to me again. Thanks to Sridip Pal for discussions of this point.) See §5.1.2 below.

(I am suppressing spinor indices all over the place, note that here we are taking the outer product of the spinors.)

We want to define the coincidence limit, as $x' \to x$. The problem with this limit arises from the large $|\epsilon_n|$ eigenvalues; the contributions of such short-wavelength modes are local and most of them can be absorbed in renormalization of couplings. It should not (and does not) matter how we regulate them, but we must pick a regulator. A convenient choice here is heat-kernel regulator:

$$G_s(x,x') \equiv \sum_n e^{-s\epsilon_n^2} \frac{1}{\epsilon_n} \xi_n(x) \bar{\xi}_n(x')$$

and

$$J^5_\mu(x) = \sum_n e^{-s\epsilon_n^2} \frac{1}{\epsilon_n} \bar{\xi}_n(x) \gamma^5 \gamma_\mu \xi_n(x) \ .$$

The anomaly is

$$\partial^\mu J^5_\mu = \partial^\mu \langle j^5_\mu \rangle = \sum_n \mathbf{i}\partial^\mu \left(\bar{\xi}_n \gamma_\mu \gamma^5 \xi_n \right) \frac{e^{-s\epsilon_n^2}}{\epsilon_n}.$$

The definition (5.4) says

$$\mathbf{i}\partial^\mu \left(\bar{\xi}_n \gamma_\mu \gamma^5 \xi_n \right) = -2\epsilon_n \bar{\xi}_n \gamma_5 \xi_n$$

using $\{\gamma^5, \gamma^\mu\} = 0$. (Notice that the story would deviate dramatically here if we were studying the vector current which lacks the γ^5.) This gives

$$\partial^\mu J^5_\mu = 2\mathrm{Tr}\ {}_\alpha\gamma^5 e^{-s\left(\mathbf{i}\not{D}\right)^2}$$

with

$$(\mathbf{i}\not{D})^2 = -\left(\gamma_\mu \left(\partial_\mu + \mathbf{i}A_\mu\right)\right)^2 = -\left(\partial_\mu + A_\mu\right)^2 - \frac{\mathbf{i}}{2}\Sigma_{\mu\nu}F^{\mu\nu}$$

where $\Sigma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ is the spin Lorentz generator. This is (5.2), now better defined by the heat kernel regulator. We've shown that in any even dimension,

$$\partial^\mu \langle j^5_\mu(x) \rangle = 2\mathrm{Tr}\ {}_\alpha\gamma^5 e^{s\not{D}^2} \tag{5.6}$$

This can now be expanded in small s, which amounts to an expansion in powers of A, F. If there is no background field, $A = 0$, we get

$$\langle x | e^{-s\left(\mathbf{i}\not{\partial}\right)^2} | x \rangle = \int \mathrm{d}^D p \ e^{-sp^2} = \underbrace{K_D}_{=\frac{\Omega_{D-1}}{(2\pi)^D} \text{ as before}} \frac{1}{s^{D-2}} \overset{D=4}{=} \frac{1}{16\pi^2 s^2}. \tag{5.7}$$

This term will renormalize the charge density

$$\rho(x) = \langle \psi^\dagger \psi(x) \rangle = \mathrm{tr}\gamma^0 G(x,x),$$

for which we must add a counterterm (in fact, it is accounted for by the counterterm for the gauge field kinetic term, *i.e.* the running of the gauge coupling). But it will not affect the axial current conservation which is proportional to

$$\mathrm{tr}\left(\gamma^5 G(x,x)\right)|_{A=0} \propto \mathrm{tr}\gamma^5 = 0.$$

Similarly, bringing down more powers of $(\partial + A)^2$ doesn't give something nonzero since the γ^5 remains.

In $D = 4$, the first term from expanding $\Sigma_{\mu\nu}F^{\mu\nu}$ is still zero from the spinor trace. (Not so in $D = 2$.) The first nonzero term comes from the next term:

$$\mathrm{tr}\left(\gamma_5 e^{-s\left(\mathbf{i}\not{D}\right)^2}\right)_{xx} = \underbrace{\langle x|e^{-s(\mathbf{i}D)^2}|x\rangle}_{\overset{(5.7)}{=}\frac{1}{16\pi^2 s^2}+\mathcal{O}(s^{-1})} \cdot \frac{s^2}{8} \cdot (\mathbf{i}^2)\underbrace{\mathrm{tr}\left(\gamma^5\Sigma^{\mu\nu}\Sigma^{\rho\lambda}\right)}_{=4\epsilon^{\mu\nu\rho\lambda}} \cdot \underbrace{\mathrm{tr}_c}_{\text{color}}(F_{\mu\nu}F_{\rho\lambda}) + \mathcal{O}(s^1)\ .$$

In the abelian case, just ignore the trace over color indices, tr_c. The terms that go like positive powers of s go away in the continuum limit. Therefore

$$\partial_\mu J_5^\mu = -2 \cdot \frac{1}{16\pi s^2} \cdot \frac{s^2}{8} \cdot 4\epsilon^{\mu\nu\rho\lambda}\mathrm{tr}_c F_{\mu\nu}F_{\rho\lambda} + \mathcal{O}(s^1) = -\frac{1}{8\pi^2}\mathrm{tr}F_{\mu\nu}\,(\star F)^{\mu\nu}\,. \tag{5.8}$$

(Here $(\star F)^{\mu\nu} \equiv \frac{1}{8}\epsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}$.) This is the chiral anomaly formula. It can also be usefully written as:

$$\partial_\mu J_5^\mu = -\frac{1}{8\pi^2}\mathrm{tr}F\wedge F = -\frac{1}{32\pi^2}\vec{E}\cdot\vec{B}.$$

- This object on the RHS is a total derivative. In the abelian case it is

  $$F\wedge F = \mathrm{d}\left(A\wedge F\right)\ .$$

  Its integral over spacetime is a topological (in fact $16\pi^2$ times an integer) characterizing the gauge field configuration. How do I know it is an integer? The anomaly formula! The change in the number of left-handed fermions minus the number of right-handed fermions during some time interval is:

  $$\Delta Q_A \equiv \Delta\left(N_L - N_R\right) = \int \mathrm{d}t\partial_t J_0^5 = \int_{M_4}\partial^\mu J_\mu^5 = 2\int_{M_4}\frac{F\wedge F}{16\pi^2}$$

  where M_4 is the spacetime region under consideration. If nothing is going on at the boundaries of this spacetime region (*i.e.* the fields go to the vacuum, or there is no boundary, so that no fermions are entering or leaving), we can conclude that the RHS is an integer.

- Look back at the diagrams in (5.3). Which term in that expansion gave the nonzero contribution to the axial current violation? In $D = 4$ it is the diagram with three current insertions, the ABJ triangle diagram. So in fact we did end up computing the triangle diagram. But this calculation also shows that nothing else contributes, even non-perturbatively.

- We chose a particular regulator above. The answer we got did not depend on the cutoff; in fact whatever regulator we used, we would get this answer.

[End of Lecture 19]

- Consider what happens if we redo this calculation in other dimensions. We only consider even dimensions because in odd dimensions there is no analog of γ^5 – the Dirac spinor representation is irreducible. In $2n$ dimensions, we need n powers of F to soak up the indices on the epsilon tensor.

- If we had kept the non-abelian structure in (5.1) through the whole calculation, the only difference is that the trace in (5.8) would have included a trace over representations of the gauge group; and we could have considered also a non-abelian flavor transformation

$$\psi_I \to \left(e^{\mathbf{i}\gamma^5 g^a \tau^a}\right)_{IJ} \psi_J$$

for some flavor rotation generator τ^a. Then we would have found:

$$\partial^\mu j^{5a}_\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\lambda} F^A_{\mu\nu} F^B_{\rho\lambda} \mathrm{tr}_{c,a}\left(T^A T^B \tau^a\right).$$

5.1.2 Zeromodes of the Dirac operator

Do you see now why I said that the step involving the fermion Green's function was full of danger? The danger arises because the Dirac operator (whose inverse is the Green's function) can have *zeromodes*, eigenspinors with eigenvalue $\epsilon_n = 0$. In that case, $\mathbf{i}\not{D}$ is *not* invertible, and the expression (5.5) for G is ambiguous. This factor of ϵ_n is about to be cancelled when we compute the divergence of the current and arrive at (5.2). Usually this kind of thing is not a problem because we can lift the zeromodes a little and put them back at the end. But here it is actually hiding something important. The zeromodes cannot just be lifted. This is true because nonzero modes of $\mathbf{i}\not{D}$ must come in left-right pairs: this is because $\{\gamma^5, \mathbf{i}\not{D}\} = 0$, so $\mathbf{i}\not{D}$ and γ^5 cannot be simultaneously diagonalized in general. That is: if $\mathbf{i}\not{D}\xi = \epsilon\xi$ then $(\gamma^5\xi)$ is also an eigenvector of $\mathbf{i}\not{D}\xi$, with eigenvalue $-\epsilon$. Only for $\epsilon = 0$ does this fail, so zeromodes can come by themselves. So you can't just smoothly change the eigenvalue of some ξ_0 from zero unless it has a partner with whom to pair up. By taking linear combinations

$$\chi^{L/R}_n = \frac{1}{2}\left(1 \pm \gamma^5\right)\xi_n$$

these two partners can be arranged into a pair of simultaneous eigenvectors of $(\mathbf{i}\not{D})^2$ (with eigenvalue ϵ_n^2) and of γ^5 with $\gamma^5 = \pm$ respectively.

This leads us to a deep fact, called the (Atiyah-Singer) *index theorem*: *only* zeromodes can contribute to the anomaly. Any mode ξ_n with nonzero eigenvalue has a partner with the opposite sign of γ^5; hence they cancel exactly in

$$\sum_n \bar{\xi}_n \gamma^5 \xi_n e^{-s\epsilon_n^2} \quad !$$

So the anomaly equation tells us that the number of zeromodes of the Dirac operator, weighted by handedness (*i.e.* with a + for L and - for R) is equal to

$$N_L - N_R = \int d^D x \mathcal{A}(x) = \int \frac{1}{16\pi^2} F \wedge F.$$

A practical consequence for us is that it makes manifest that the result is independent of the regulator s.

5.1.3 The physics of the anomaly

[Polyakov, page 102; Kaplan 0912.2560 §2.1; Alvarez-Gaumé] Consider non-relativistic free (*i.e.* no 4-fermion interactions) fermions in 1+1 dimensions, *e.g.* with 1-particle dispersion $\omega_k = \frac{1}{2m}\vec{k}^2$. The groundstate of N such fermions is described by filling the N lowest-energy single particle levels, up the Fermi momentum: $|k| \leq k_F$ are filled. We must introduce an infrared regulator so that the levels are discrete – put them in a box of length L, so that $k_n = \frac{2\pi n}{L}$. (In Figure 16, the red circles are possible 1-particle states, and the green ones are the occupied ones.) The lowest-energy excitations of this groundstate come from taking a fermion just below the Fermi level $|k_1| \lesssim k_F$ and putting it just above $|k_2| \gtrsim k_F$; the energy cost is

$$E_{k_1 - k_2} = \frac{1}{2m}(k_F + k_1)^2 - \frac{1}{2m}(k_F - k_2)^2 \simeq \frac{k_F}{m}(k_1 - k_2)$$

– we get relativistic dispersion with velocity $v_F = \frac{k_F}{m}$. The fields near these Fermi points in k-space satisfy the Dirac equation[49]:

$$(\omega - \delta k)\,\psi_L = 0, \quad (\omega + \delta k)\,\psi_R = 0.$$

[49] This example is worthwhile for us also because we see the relativistic Dirac equation is emerging from a non-relativistic model; in fact we could have started from an even more distant starting point – *e.g.* from a lattice model, like

$$H = -t \sum_n c_n^\dagger c_{n+1} + h.c.$$

where the dispersion would be $\omega_k = -2t(\cos ka - 1) \sim \frac{1}{2m}k^2 + \mathcal{O}(k^4)$ with $\frac{1}{2m} = ta^2$.

It would therefore seem to imply a conserved axial current – the number of left moving fermions minus the number of right moving fermions. But the fields ψ_L and ψ_R are not independent; with high-enough energy excitations, you reach the bottom of the band (near $k = 0$ here) and you can't tell the difference. This means that the numbers are *not* separately conserved.

We can do better in this 1+1d example and show that the amount by which the axial current is violated is given by the anomaly formula. Consider subjecting our poor 1+1d free fermions to an electric field $E_x(t)$ which is constant in space and slowly varies in time. Suppose we gradually turn it on and then turn it off; here gradually means slowly enough that the process is adiabatic. Then each particle experiences a force $\partial_t p = eE_x$ and its net change in momentum is

$$\Delta p = e \int \mathrm{d}t E_x(t).$$

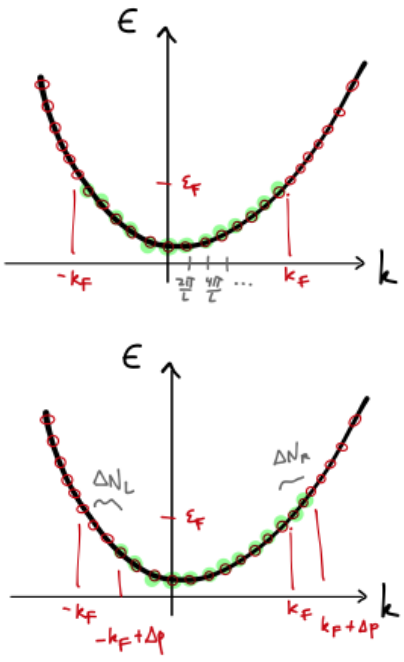


Figure 16: Green dots represent occupied 1-particle states. Top: In the groundstate. Bottom: After applying $E_x(t)$.

This means that the electric field puts the fermions in a state where the Fermi surface $k = k_F$ has shifted to the right by Δp, as in the figure. Notice that the total number of fermions is of course the same – charge is conserved.

Now consider the point of view of the low-energy theory at the Fermi points. This theory has the action

$$S[\psi] = \int \mathrm{d}x\mathrm{d}t\bar{\psi}\left(\mathbf{i}\gamma^\mu\partial_\mu\right)\psi \ ,$$

where γ^μ are 2×2 and the upper/lower component of ψ creates fermions near the left/right Fermi point. In the process above, we have added N_R right-moving particles and taken away N_L left-moving particles, that is *added* N_L left-moving holes (aka anti-particles). The axial charge of the state has changed by

$$\Delta Q_A = \Delta(N_L - N_R) = 2\frac{\Delta p}{2\pi/L} = \frac{L}{\pi}\Delta p = \frac{L}{\pi}e\int \mathrm{d}t E_x(t) = \frac{e}{\pi}\int \mathrm{d}t\mathrm{d}x E_x = \frac{e}{2\pi}\int \epsilon_{\mu\nu}F^{\mu\nu}$$

On the other hand, the LHS is $\Delta Q_A = \int \partial^\mu J^A_\mu$. We can infer a local version of this equation by letting E vary slowly in space as well, and we conclude that

$$\partial_\mu J^\mu_A = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}.$$

This agrees exactly with the anomaly equation in $D = 1+1$ produced by the calculation above in (5.6) (see Problem Set 7).

5.2 Topological terms in QM and QFT

5.2.1 Differential forms and some simple topological invariants of manifolds

[Zee section IV.4] This is nothing fancy, mostly just some book-keeping. It's some notation that we'll find useful.

Suppose we are given a smooth manifold X on which we can do calculus. For now, we don't even need a metric on X.

A p-form on X is a completely antisymmetric p-index tensor,

$$A \equiv \frac{1}{p!} A_{m_1 \ldots m_p} \mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_p}.$$

The point in life of a p-form is that it can be integrated over a p-dimensional space. The order of its indices keeps track of the orientation (and it saves us the trouble of writing them). It is a geometric object, in the sense that it is something that can be (wants to be) integrated over a p-dimensional subspace of X, and its integral will only depend on the subspace, not on the coordinates we use to describe it.

Familiar examples include the gauge potential $A = A_\mu \mathrm{d}x^\mu$, and its field strength $F = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. Given a curve C in X parameterized as $x^\mu(s)$, we have

$$\int_C A \equiv \int_C dx^\mu A_\mu(x) = \int ds \frac{dx^\mu}{ds} A_\mu(x(s))$$

and this would be the same if we chose some other parameterization or some other local coordinates.

The *wedge product* of a p-form A and a q-form B is a $p+q$ form

$$A \wedge B = A_{m_1 .. m_p} B_{m_{p+1} \ldots m_{p+q}} \mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_{p+1}}$$

[50] The space of p-forms on a manifold X is sometimes denoted $\Omega^p(X)$.

The *exterior derivative* d acts on forms as

$$\begin{array}{rcl} \mathrm{d} : \Omega^p(X) & \to & \Omega^{p+1} \\ A & \mapsto & \mathrm{d}A \end{array}$$

[50] The components of $A \wedge B$ are then

$$(A \wedge B)_{m_1 \ldots m_{p+q}} = \frac{(p+q)!}{p!q!} A_{[m_1 \ldots m_p} B_{m_{p+1} \ldots m_{p+q}]}$$

where [..] means sum over permutations with a -1 for odd permutations. Try not to get caught up in the numerical prefactors.

by

$$\mathrm{d}A = \partial_{m_1}(A)_{m_2 \ldots m_{p+1}} \mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_{p+1}}.$$

You can check that

$$\mathrm{d}^2 = 0$$

basically because derivatives commute. Notice that $F = \mathrm{d}A$ in the example above. Denoting the boundary of a region D by ∂D, Stokes' theorem is

$$\int_D d\alpha = \int_{\partial D} \alpha.$$

And notice that $\Omega^{p>\dim(X)}(X) = 0$ – there are no forms of rank larger than the dimension of the space.

A form ω_p is *closed* if it is killed by d: $\mathrm{d}\omega_p = 0$.

A form ω_p is *exact* if it is d of something: $\omega_p = \mathrm{d}\alpha_{p-1}$. That something must be a $(p-1)$-form.

Because of the property $\mathrm{d}^2 = 0$, it is possible to define *cohomology* – the image of one $\mathrm{d} : \Omega^p \to \Omega^{p+1}$ is in the kernel of the next $\mathrm{d} : \Omega^{p+1} \to \Omega^{p+2}$ (*i.e.* the Ω^ps form a chain complex). The pth de Rham cohomology group of the space X is defined to be

$$H^p(X) \equiv \frac{\text{closed } p\text{-forms on } X}{\text{exact } p\text{-forms on } X} = \frac{\ker(\mathrm{d}) \in \Omega^p}{\mathrm{Im}(\mathrm{d}) \in \Omega^p}.$$

That is, two closed p-forms are equivalent in cohomology if they differ by an exact form:

$$[\omega_p] - [\omega_p + \mathrm{d}\alpha_{p-1}] = 0 \in H^p(X),$$

where $[\omega_p]$ denotes the equivalence class. The dimension of this group is $b^p \equiv \dim H^p(X)$ called the pth betti number and is a topological invariant of X. The euler characteristic of X, which you can get by triangulating X and counting edges and faces and stuff is

$$\chi(X) = \sum_{p=0}^{d=\dim(X)} (-1)^p b^p(X).$$

Now suppose we have a volume element on X, *i.e.* a way of integrating d-forms. This is guaranteed if we have a metric, since then we can integrate $\int \sqrt{\det g} \ldots$. Then we can define the Hodge star operation $\star$ which maps a p-form into a $(d-p)$-form:

$$\star : \Omega^p \to \Omega^{d-p}$$

by

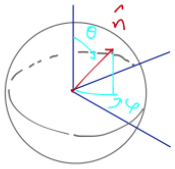
$$\left(\star A^{(p)}\right)_{\mu_1\ldots\mu_{d-p}} \equiv \epsilon_{\mu_1\ldots\mu_d} A^{(p)\ \mu_{d-p+1}\ldots\mu_d}$$

An application: consider the Maxwell action. $S[A] = \int F \wedge \star F$. Show that this is the same as $\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$. (Don't trust my numerical prefactor.)

Derive the Maxwell EOM by $0 = \frac{\delta S}{\delta A}$.

5.2.2 Geometric quantization and coherent state quantization of spin systems

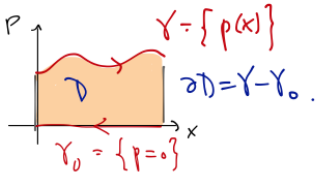
[Zinn-Justin, Appendix A3; XGW §2.3] We're going to spend some time talking about QFT in $D = 0+1$, then we'll work our way up to $D = 1+1$. Consider the nice, round two-sphere. It has an area element which can be written



$$\omega = s\mathrm{d}\cos\theta \wedge \mathrm{d}\varphi \qquad \text{and satisfies} \qquad \int_{S^2} \omega = 4\pi s.$$

Suppose we think of this sphere as the *phase space* of some dynamical system. We can use ω as the symplectic form. What is the associated quantum mechanics system?

Let me remind you what I mean by 'the symplectic form'. Recall the phase space formulation of classical dynamics. The action associated to a trajectory is



$$\mathcal{A}[x(t), p(t)] = \int_{t_1}^{t_2} \mathrm{d}t\,(p\dot{x} - H(x,p)) = \int_\gamma p(x)\mathrm{d}x - \int H\mathrm{d}t$$

where γ is the trajectory through the phase space.
The first term is the area 'under the graph' in the classical phase space – the area between (p, x) and $(p = 0, x)$. We can rewrite it as

$$\int p(t)\dot{x}(t)\mathrm{d}t = \int_{\partial D} p\mathrm{d}x = \int_D \mathrm{d}p \wedge \mathrm{d}x$$

using Stokes' theorem; here ∂D is the closed curve made by the classical trajectory and some reference trajectory ($p = 0$) and it bounds some region D. Here $\omega = \mathrm{d}p\wedge\mathrm{d}x$ is the symplectic form. More generally, we can consider an $2n$-dimensional phase space with coordinates u_α and symplectic form

$$\omega = \omega_{\alpha\beta}\mathrm{d}u^\alpha \wedge \mathrm{d}u^\beta$$

and action

$$\mathcal{A}[u] = \int_D \omega - \int_{\partial D} \mathrm{d}t H(u,t).$$

It's important that $\mathrm{d}\omega = 0$ so that the equations of motion resulting from $\mathcal{A}$ depend only on ∂D and not on the interior. The equations of motion from varying u are

$$\omega_{\alpha\beta}\dot{u}^\beta = \frac{\partial H}{\partial u^\alpha}.$$

Locally, we can find coordinates p, x so that $\omega = \mathrm{d}(p\mathrm{d}x)$. Globally on the phase space this is not guaranteed – the symplectic form needs to be closed, but need not be exact.

So the example above of the two-sphere is one where the symplectic form is closed (there are no three-forms on the two sphere, so $\mathrm{d}\omega = 0$ automatically), but is not exact. One way to see that it isn't exact is that if we integrate it over the whole two-sphere, we get the area:

$$\int_{S^2} \omega = 4\pi s \ .$$

On the other hand, the integral of an exact form over a closed manifold (meaning a manifold without boundary, like our sphere) is zero:

$$\int_C \mathrm{d}\alpha = \int_{\partial C} \alpha = 0.$$

So there can't be a *globally defined* one form α such that $\mathrm{d}\alpha = \omega$. Locally, we can find one; for example:

$$\alpha = s\cos\theta \mathrm{d}\varphi \ ,$$

but this is singular at the poles, where φ is not a good coordinate.

So: what I mean by "what is the associated quantum system..." is the following: let's construct a system whose path integral is

$$Z = \int [\mathrm{d}\theta \mathrm{d}\varphi] e^{\frac{\mathrm{i}}{\hbar}\mathcal{A}[\theta,\varphi]} \tag{5.9}$$

with the action above, and where $[\mathrm{d}x]$ denotes the path integral measure:

$$[\mathrm{d}x] \equiv \aleph \prod_{i=1}^{N} \mathrm{d}x(t_i)$$

where $\aleph$ involves lots of awful constants that drop out of ratios. It is important that the measure does not depend on our choice of coordinates on the sphere.

- Hint 1: the model has an action of O(3), by rotations of the sphere.
- Hint 2: We actually didn't specify the model yet, since we didn't choose the Hamiltonian. For definiteness, let's pick the hamiltonian to be

  $$H = -s\vec{h}\cdot\vec{n}$$

  where $\vec{n} \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. WLOG, we can take the polar axis to be along the 'magnetic field': $\vec{h} = \hat{z}h$. The equations of motion are

  $$0 = \frac{\delta\mathcal{A}}{\delta\theta(t)} = -s\sin\theta\left(\dot{\varphi} - h\right), 0 = \frac{\delta\mathcal{A}}{\delta\varphi(t)} = -\partial_t\left(s\cos\theta\right)$$

which by rotation invariance can be written better as

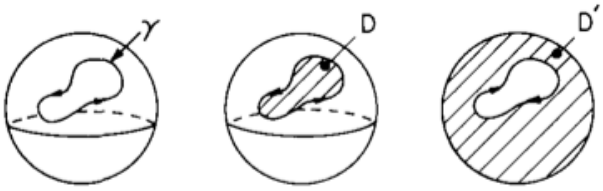
$$\partial_t \vec{n} = \vec{h} \times \vec{n}.$$

This is a big hint about the answer to the question.

- Hint 3: Semiclassical expectations. semiclassically, each patch of phase space of area $\hbar$ contributes one quantum state. Therefore we expect that if our whole phase space has area $4\pi s$, we should get approximately $\frac{4\pi s}{2\pi\hbar} = \frac{2s}{\hbar}$ states, at least at large $s/\hbar$. (Notice that s appears out front of the action.) This will turn out to be very close – the right answer is $2s+1$ (when the spin is measured in units with $\hbar = 1$)!

Notice that we can add a total derivative without changing the path integral on a closed manifold.

In QM we care that the action produces a well-defined phase – the action must be defined modulo additions of 2π times an integer. We should get the same answer whether we fill in one side D of the trajectory γ or the other D'. The difference between them is



[from Witten]

$$s\left(\int_D - \int_{D'}\right) \text{area} = s \int_{S^2} \text{area}\ .$$

So in this difference s multiplies $\int_{S^2} \text{area} = 4\pi$ (actually, this can be multiplied by an integer which is the number of times the area is covered). Our path integral will be well-defined (*i.e.* independent of our arbitrary choice of 'inside' and 'outside') only if $4\pi s \in 2\pi\mathbb{Z}$, that is if $\boxed{2s \in \mathbb{Z} \text{ is an integer}}$.

The conclusion of this discussion is that the coefficient of the area term must be an integer. We will interpret this integer below.

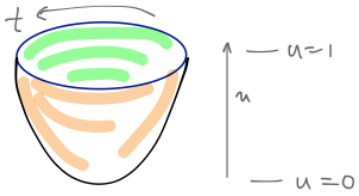
WZW term. We have a nice geometric interpretation of the 'area' term in our action $\mathcal{A}$ – it's the solid angle swept out by the particle's trajectory. But how do we write it in a manifestly SU(2) invariant way? We'd like to be able to write not in terms of the annoying coordinates θ, ϕ, but directly in terms of

$$n^a \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^a.$$

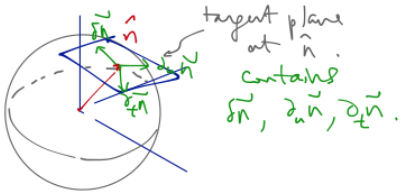
One answer is to add an extra dimension:

$$\frac{1}{4\pi}\int \mathrm{d}t\,(1-\cos\theta)\,\partial_t\phi = \frac{1}{8\pi}\int_0^1 \mathrm{d}u \int \mathrm{d}t \epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c \epsilon^{abc} \equiv W_0[\vec{n}]$$

where $x^\mu = (t, u)$, and the ϵ tensors are completely antisymmetric in their indices with all nonzero entries 1 and -1.



In order to write this formula we have to extend the $\vec{n}$-field into the extra dimension whose coordinate is u. We do this in such a way that the real spin lives at $u = 1$: $\vec{n}(t, u = 1) = \vec{n}(t)$, and $\vec{n}(t, u = 0) = (0, 0, 1)$ – it goes to the north pole at the other end of the extra dimension for all t. If we consider periodic boundary conditions in time $n(\beta) = n(0)$, then this means that the space is really a disk with the origin at $u = 0$, and the boundary at $u = 1$. Call this disk B, its boundary ∂B is the real spacetime.



This WZW term has the property that its variation with respect to $\vec{n}$ depends only on the values at the boundary (that is: δW_0 is a total derivative). The crucial reason is that allowed variations $\delta\vec{n}$ lie on the 2-sphere, as do derivatives $\partial_\mu\vec{n}$; this means $\epsilon^{abc}\delta n^a\partial_\mu n^b\partial_\nu n^c = 0$, since they all lie in a two-dimensional tangent plane to the 2-sphere at $\vec{n}(t)$. Therefore:

$$\begin{aligned}\delta W_0 &= \int_0^1 \mathrm{d}u \int \mathrm{d}t \frac{1}{4\pi}\epsilon^{\mu\nu} n^a \partial_\mu \delta n^b \partial_\nu n^c \epsilon^{abc} = \int_B \frac{1}{4\pi} n^a \mathrm{d}\delta n^b \wedge \mathrm{d}n^c \epsilon^{abc} \\ &= \int_0^1 \mathrm{d}u \int \mathrm{d}t\, \partial_\mu \left(\frac{1}{4\pi}\epsilon^{\mu\nu} n^a \delta n^b \partial_\nu n^c \epsilon^{abc}\right) = \int_B \mathrm{d}\left(\frac{1}{4\pi} n^a \delta n^b \mathrm{d}n^c \epsilon^{abc}\right) \\ &\overset{\text{Stokes}}{=} \frac{1}{4\pi}\int \mathrm{d}t \delta\vec{n}\cdot\left(\dot{\vec{n}}\times\vec{n}\right). \end{aligned} \tag{5.10}$$

(Note that $\epsilon^{abc} n^a m^b \ell^c = \vec{n}\cdot\left(\vec{m}\times\vec{\ell}\right)$. The right expressions in red in each line are a rewriting in terms of differential forms; notice how much prettier they are.) So the equations of motion coming from this term do not depend on how we extend it into the auxiliary dimension.

And in fact they are the same as the ones we found earlier:

$$0 = \frac{\delta}{\delta\vec{n}(t)}\left(4\pi s W_0[n] + s\vec{h}\cdot\vec{n} + \lambda\left(\vec{n}^2 - 1\right)\right) = s\partial_t\vec{n}\times\vec{n} + s\vec{h} + 2\lambda\vec{n}$$

(λ is a Lagrange multiplier to enforce unit length.) The cross product of this equation with $\vec{n}$ is $\partial_t\vec{n} = \vec{h}\times\vec{n}$.

In QM we also care that the action produces a well-defined phase – the action must be defined modulo additions of 2π times an integer. There may be many ways to extend $\hat{n}$ into an extra dimension; another obvious way is shown in the figure at right. The demand that the action is the same modulo $2\pi\mathbb{Z}$ gives the same quantization law as above for the coefficient of the WZW term.

So the WZW term is topological in the sense that because of topology its coefficient must

be quantized.

Coherent quantization of spin systems. [Wen §2.3.1, Fradkin, Sachdev, *QPT*, chapter 13 and §2.2 of cond-mat/0109419] To understand more about the path integral we've just constructed, we now go in the opposite direction. Start with a spin one-half system, with

$$\mathcal{H}_{\frac{1}{2}} \equiv \text{span}\{|\uparrow\rangle, |\downarrow\rangle\}.$$

Define spin coherent states $|\vec{n}\rangle$ by[51]:

$$\vec{\boldsymbol{\sigma}} \cdot \vec{n}|\vec{n}\rangle = |\vec{n}\rangle \ .$$

These states form another basis for $\mathcal{H}_{\frac{1}{2}}$; they are related to the basis where $\boldsymbol{\sigma}^z$ is diagonal by:

$$|\vec{n}\rangle = z_1|\uparrow\rangle + z_2|\downarrow\rangle, \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{\mathbf{i}\varphi/2}\cos\theta/2 \\ e^{-\mathbf{i}\varphi/2}\sin\theta/2 \end{pmatrix} \tag{5.11}$$

as you can see by diagonalizing $\vec{n}\cdot\vec{\boldsymbol{\sigma}}$ in the $\boldsymbol{\sigma}^z$ basis. Notice that

$$\vec{n} = z^\dagger \vec{\boldsymbol{\sigma}} z, \quad |z_1|^2 + |z_2|^2 = 1$$

and the phase of z_α does not affect $\vec{n}$ (this is the Hopf fibration $S^3 \to S^2$). In (5.11) I chose a representative of the phase. The space of independent states is a two-sphere:

$$S^2 = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 = 1\}/(z_\alpha \simeq e^{\mathbf{i}\varphi} z_\alpha).$$

It is just the ordinary Bloch sphere of pure states of a qbit.

The completeness relation in this basis is:

$$\int \frac{\mathrm{d}^2\vec{n}}{2\pi} |\vec{n}\rangle\langle\vec{n}| = \mathbb{1}_{2\times 2}. \tag{5.12}$$

As always, we can construct a path integral representation of any amplitude by inserting many copies of $\mathbb{1}$ in between successive time steps. For example, we can construct such a representation for the propagator using (5.12) many times:

$$\begin{aligned} \mathbf{i}G(\vec{n}_f, \vec{n}_1, t) &\equiv \langle \vec{n}_f | e^{-\mathbf{i}\mathbf{H}t} | \vec{n}_1 \rangle \\ &= \int \prod_{i=1}^{N \equiv \frac{t}{\mathrm{d}t}} \frac{\mathrm{d}^2\vec{n}(t_i)}{2\pi} \lim_{\mathrm{d}t\to 0} \langle \vec{n}(t)|\vec{n}(t_N)\rangle ... \langle \vec{n}(t_2)|\vec{n}(t_1)\rangle\langle \vec{n}(t_1)|\vec{n}(0)\rangle. \end{aligned} \tag{5.13}$$

[51] For more general spin representation with spin s, and spin operator $\vec{\mathbf{S}}$, we would generalize this equation to

$$\vec{\mathbf{S}} \cdot \vec{n}|\vec{n}\rangle = s|\vec{n}\rangle.$$

(Notice that $\mathbf{H} = 0$ here, so $\mathbf{U} \equiv e^{-\mathbf{i}\mathbf{H}t}$ is actually the identity.) The crucial ingredient is

$$\langle \vec{n}(t+\epsilon)|\vec{n}(t)\rangle = z^\dagger(\mathrm{d}t)z(0) = 1 - z^\dagger(\mathrm{d}t)\left(z(\mathrm{d}t) - z(0)\right) \approx e^{-z^\dagger \partial_t z \mathrm{d}t}.$$

$$\mathbf{i}G(\vec{n}_2, \vec{n}_1, t) = \int \left[\frac{D\vec{n}}{2\pi}\right] e^{\mathbf{i}S_B[\vec{n}(t)]}, \qquad S_B[\vec{n}(t)] = \int_0^t \mathrm{d}t \mathbf{i} z^\dagger \dot{z} \ . \tag{5.14}$$

Notice how weird this is: even though the Hamiltonian of the spins was zero – whatever their state, they have no potential energy and no kinetic energy – the action in the path integral is not zero. This phase $e^{\mathbf{i}S_B}$ is a quantum phenomenon called a Berry phase.

Starting from the action S_B and doing the Legendre transform to find the Hamiltonian you will get zero. The first-derivative action says that $z^\dagger$ is the canonical momentum conjugate to z: the space with coordinates $(z, z^\dagger)$ becomes the phase space (just like position and momentum)! But this phase space is curved. In fact it is the two-sphere

$$S^2 = \{(z_1, z_2)||z_1|^2 + |z_2|^2 = 1\}/(z_\alpha \simeq e^{\mathbf{i}\psi} z_\alpha).$$

In terms of the coordinates θ, φ above, we have

$$S_B[z] = S_B[\theta, \varphi] = \int dt \left(-\frac{1}{2}\cos\theta\dot{\phi} - \frac{1}{2}\dot{\phi}\right) = -4\pi s W_0[\hat{n}]|_{s=\frac{1}{2}}.$$

BIG CONCLUSION: This is the 'area' term that we studied above, with $s = \frac{1}{2}$! So the expression in terms of z in (5.14) gives another way to write the area term which is manifestly SU(2) invariant; this time the price is introducing these auxiliary z variables.

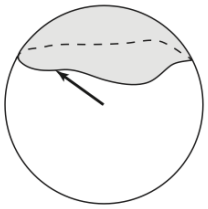
If we redo the above coherent-state quantization for a spin-s system we'll get the expression with general s. Notice that this only makes sense when $2s \in \mathbb{Z}$.

Different choices of gauge fixing for ψ can shift the constant in front of the second term; as we observed earlier, this term is a total derivative. Different choices of ψ affect the phase of the wavefunction, which doesn't change physics (recall that this is why the space of normalized states of a qbit is a two-sphere and not a three-sphere). Notice that $\mathcal{A}_t = z^\dagger \partial_t z$ is like the time component of a gauge field.

We can add a nonzero Hamiltonian for our spin; for example, we can put it in an external Zeeman field $\vec{h}$, which adds $\mathbf{H} = -\vec{h} \cdot \vec{\mathbf{S}}$. This will pass innocently through the construction of the path integral, adding a term to the action $S = S_B + S_h$,

$$S_h = \int \mathrm{d}t \left(s\vec{h} \cdot \vec{n}\right)$$

where S is the spin.



We are back at the system (5.9). We see that the system we get by 'geometric quantization' of the sphere is a quantum spin. The quantized coefficient of the area is $2s$: it determines the dimension of the spin space to be $2s+1$. Here the quantization of the WZW term is just quantization of angular momentum. (In higher-dimensional field theories, it is something else.)

Deep statement: the purpose in life of the WZW term is to enforce the commutation relation of the SU(2) generators, $[\mathbf{S}^i, \mathbf{S}^j] = \mathbf{i}\epsilon^{ijk}\mathbf{S}^k$. It says that the different components of the spin don't commute, and it says precisely what they don't commute to.

Incidentally, another way to realize this system whose action is proportional to the area of the sphere is to take a particle on the sphere, put a magnetic monopole in the center, and take the limit that the mass of the particle goes to zero. In that context, the quantization of $2s$ is Dirac quantization of magnetic charge. And the degeneracy of $2s+1$ states is the degeneracy of states in the lowest Landau level for a charged particle in a magnetic field; the $m \to 0$ limit gets rid of the higher Landau levels (which are separated from the lowest by the cylotron frequency, $\frac{eB}{mc}$).

5.2.3 Ferromagnets and antiferromagnets.

[Zee §6.5] Now we'll try $D = 1+1$. Consider a chain of spins, each of spin $s \in \mathbb{Z}/2$, with

$$\mathbf{H} = \sum_j J\vec{\mathbf{S}}_j \cdot \vec{\mathbf{S}}_{j+1}.$$

For $J < 0$, the classical ground state is ferromagnetic, with $\langle\vec{\mathbf{S}}_j\rangle = s\hat{z}$. For $J > 0$, the neighboring spins want to anti-align; this is an antiferromagnet: $\langle\vec{\mathbf{S}}_j\rangle = (-1)^j s\hat{z}$.

(Note that I am lying about there being spontaneous breaking of a continuous symmetry in 1+1 dimensions. Really there is only short-range order because of the Coleman-Mermin-Wagner theorem. But that is enough for this calculation.)

We can write down the action that we get by coherent-state quantization – it's just many copies of the above, where each spin plays the role of the external magnetic field for its neighbors:

$$L = \mathbf{i}s\sum_j z_j^\dagger \partial_t z_j - Js^2 \sum_j \vec{n}_j \cdot \vec{n}_{j+1}.$$

Spin waves in ferromagnets. Let's use this to find the equation of motion for small

fluctuations $\delta\vec{n}_i = \vec{S}_i - s\hat{z}$ about the ferromagnetic state. Once we recognize the existence of the Berry phase term, this is the easy case. In fact the discussion is not restricted to $D = 1 + 1$. The system is translation invariant, so we should fourier transform. The condition that $\vec{n}_j^2 = 1$ means that $\delta n_z(k) = 0$. Linearizing in $\delta\vec{n}$ (using (5.10)) and fourier transforming, we find

$$0 = \begin{pmatrix} h(k) & -\frac{\mathbf{i}}{2}\omega \\ \frac{\mathbf{i}}{2}\omega & h(k) \end{pmatrix} \begin{pmatrix} \delta n_x(k) \\ \delta n_y(k) \end{pmatrix}$$

with $h(k)$ determined by the exchange (J) term. For example for the square lattice, it is $h(k) = 4s|J|\,(2 - \cos k_x a - \cos k_y a) \overset{k\to 0}{\simeq} 2s|J|a^2k^2$, with a the lattice spacing. For small k, the eigenvectors have $\omega \sim k^2$, a $z = 2$ dispersion.

Antiferromagnets. [Fradkin, 2d ed, p. 203] Now, let's study instead the equation of motion for small fluctuations about the antiferromagnetic state. The conclusion will be that there is a linear dispersion relation. This would be the conclusion we came to if we simply erased the WZW/Berry phase term and replaced it with an ordinary kinetic term

$$\frac{1}{2g^2}\sum_j \partial_t\vec{n}_j \cdot \partial_t\vec{n}_j \ .$$

How this comes about is actually a bit more involved! It's very similar to the way the second order kinetic term for the Goldstone mode in the superfluid arose: the role of ρ will be played by the *ferro*magnetic fluctuation $\vec{\ell}_j$ in

$$\vec{n}_j = (-1)^j\vec{m}_j + a\vec{\ell}_j \ .$$

$\vec{m}_j$ is the AF fluctuation; a is the lattice spacing; $s \in \mathbb{Z}/2$ is the spin. The constraint $\vec{n}^2 = 1$ tells us that $\vec{m}^2 = 1$ and $\vec{m}\cdot\vec{\ell} = 0$. Why do we have to include both variables? Because $\vec{m}$ are the AF order-parameter fluctuations, but the total spin is still conserved, and therefore its local fluctuations $\vec{\ell}$ still constitute a slow mode.

The hopping term in the action is (using $\vec{n}_{2r} - \vec{n}_{2r-1} \approx a\,(\partial_x\vec{m}_{2r} + 2\ell_{2r}) + \mathcal{O}(a^2)$)

$$S_J[\vec{n}_j = (-1)^j\vec{m}_j + a\vec{\ell}_j] = -aJs^2\int \mathrm{d}x\mathrm{d}t\left(\frac{1}{2}\left(\partial_x\vec{m}\right)^2 + 2\ell^2\right) \ .$$

The WZW terms evaluate to[52]

$$S_W = s\sum_{j=1}^{N} W_0[(-1)^j m_j + \ell_j] \overset{N\to\infty, a\to 0, Na \text{ fixed}}{\simeq} \int \mathrm{d}x\mathrm{d}t\ \left(\frac{s}{2}\vec{m}\cdot(\partial_t\vec{m}\times\partial_x\vec{m}) + s\vec{\ell}\cdot(\vec{m}\times\partial_t\vec{m})\right).$$

[52] The essential ingredient is

$$\delta W_0[n] = \int \mathrm{d}t\delta\vec{n}\cdot(\vec{n}\times\partial_t\vec{n}) \ .$$

So

$$W_0[n_{2r}] - W[n_{2r-1}] = -\frac{1}{2}\frac{\mathrm{d}x}{a}\frac{\delta W_0}{\delta n^i}\partial_x\hat{n}^i a = -\frac{1}{2}\mathrm{d}x\hat{n}\times\partial_t\hat{n}\cdot\partial_x\hat{n}.$$

Altogether, we find that ℓ is an auxiliary field with no time derivative:

$$L[m,\ell] = -2aJs^2\vec{\ell}^2 + s\vec{\ell}\cdot(\vec{m}\times\partial_t\vec{m}) + L[m]$$

so we can integrate out ℓ (like we did ρ in the EFT of SF in §4.5.4) to find

$$S[\vec{m}] = \int \mathrm{d}x\mathrm{d}t \left(\frac{1}{2g}\left(\frac{1}{v_s}\left(\partial_t\vec{m}\right)^2 - v_s\left(\partial_x\vec{m}\right)^2\right) + \frac{\theta}{8\pi}\epsilon_{\mu\nu}\vec{m}\cdot(\partial_\mu\vec{m}\times\partial_\nu\vec{m})\right), \tag{5.15}$$

with $g = \frac{2}{s}$ and $v_s = 2aJs$, and $\theta = 2\pi s$. The equation of motion for small fluctuations of $\vec{m}$ therefore gives linear dispersion with velocity v_s. Notice that these fluctuations have wavenumber k close to π, since they are fluctuations of the AF order, that is, $\omega \sim |k-\pi|$.

The last ('theta') term in (5.15) is a total derivative. This means it doesn't affect the EOM, and it doesn't affect the Feynman rules. It is even more topological than the WZW term – its value only depends on the topology of the field configuration, and not on local variations. You might think then that it doesn't matter. Although it doesn't affect small fluctuations of the fields, it does affect the path integral; in particular, recall that the object multiplying theta counts the winding number of the field configuration $\vec{m}$, the number of times Q the map $\vec{m}:\mathbb{R}^2\to S^2$ covers its image (we can assume that the map $\vec{m}(|x|\to\infty)$ approaches a constant, say the north pole). We can break up the path integral into sectors, labelled by this number $Q \equiv \frac{1}{8\pi}\int \mathrm{d}x\mathrm{d}t\ \epsilon_{\mu\nu}\vec{m}\cdot(\partial_\mu\vec{m}\times\partial_\nu\vec{m})$:

$$Z = \int [D\vec{m}]e^{\mathbf{i}S} = \sum_{Q\in\mathbb{Z}}\int [D\vec{m}]_Q e^{\mathbf{i}S_{\theta=0}}e^{\mathbf{i}\theta Q}\ .$$

θ determines the relative phase of different topological sectors (for $\theta=\pi$, this a minus sign for odd Q).

Actually, the theta term makes a huge difference. (Perhaps it is not so surprising given Problem Set 7, Problem 1 with the particle on the ring with magnetic flux through it?) The model with even s flows to a trivial theory in the IR, while the model with odd s flows to a nontrivial fixed point, called the $\mathrm{SU}(2)_1$ WZW model. It can be described in terms of one free relativistic boson. Sadly, I have not figured out a way to arrive at this last conclusion in the time remaining. The 2nd edition of the book by Fradkin continues this discussion.

[End of Lecture 20]

Nonlinear sigma models in perturbation theory. Let us discuss what happens in perturbation theory in small g. A momentum-shell calculation integrating out fast modes (see the next subsection, §5.2.4) shows that

$$\frac{dg}{d\ell} = (D-2)g + (n-2)K_D g^2 + \mathcal{O}(g^3) \tag{5.16}$$

where ℓ is the logarithmic RG time, and $\ell \to \infty$ is the IR. n is the number of components of $\hat{n}$, here $n = 3$, and $K_D = \frac{\Omega_{D-1}}{(2\pi)^D}$ as usual. Cultural remark: the second term is proportional to the *curvature* of the target space, here S^{n-1}, which has positive curvature for $n > 1$. For $n = 2$, we get S^1 which is one-dimensional and hence flat and there is no perturbative beta function. In fact, for $n = 2$, it's just a free boson.

The fact that the RHS of (5.16) is positive in $D = 2$ says that this model is *asymptotically free* – the coupling is weak in the UV (though this isn't so important if we are starting from a lattice model) and becomes strong in the IR. This is opposite what happens in QED; the screening of the charge in QED makes sense in terms of polarization of the vacuum by virtual charges. Why does this antiscreening happen here? There's a nice answer: the effect of the short-wavelength fluctuations is to make the spin-ordering vector $\vec{n}$ effectively *smaller*. It is like what happens when you do the block spin procedure, only this time don't use majority rule, but just average the spins. But rescaling the variable $\vec{n} \to a\vec{n}$ with $a \lesssim 1$ is the same as rescaling the coupling $g \to g/a$ – the coupling gets bigger. (Beware Peskin's comments about the connection between this result and the Coleman-Mermin-Wagner theorem: it's true that the logs in 2d enhance this effect, but in fact the model can reach a fixed point at finite coupling; in fact, this is what happens when $\theta = \pi$.)

Beyond perturbation theory. Like in QCD, this *infrared slavery* (the dark side of asymptotic freedom) means that we don't really know what happens in the IR from this calculation. From other viewpoints (Bethe ansatz solutions, many other methods), we know that (for integer s) there is an energy gap above the groundstate (named after Haldane) of order

$$\Lambda_H \sim \Lambda_0 e^{-\frac{c}{g_0}},$$

analogous to the QCD scale. Here g_0 is the value of g at the scale Λ_0; so Λ_H is roughly the energy scale where g becomes large. This is dimensional transmutation again.

For $s \in \mathbb{Z}$, for studying bulk properties like the energy gap, we can ignore the theta term since it only appears as $e^{2\pi i n}$, with $n \in \mathbb{Z}$ in the path integral. For half-integer s, there is destructive interference between the topological sectors. The Bethe ansatz solution shows that this destroys the gap. This last sentence was a bit unsatisfying; more satisfying would be to understand the origin of the gap in the $\theta = 2\pi n$ case, and show that this interference removes that mechanism. This strategy is taken in this paper by Affleck.

5.2.4 The beta function for non-linear sigma models

[Polyakov §3.2; Peskin §13.3; Auerbach chapter 13] I can't resist explaining the result (5.16). Consider this action for a $D = 2$ non-linear sigma model with target space S^{n+1}, of radius R:

$$S = \int \mathrm{d}^2 x R^2 \partial_\mu \hat{n} \cdot \partial^\mu \hat{n} \equiv \int \mathrm{d}^2 x R^2 \mathrm{d}n^2.$$

Notice that R is a coupling constant (it's what I called $1/g$ earlier). In the second step I made some compact notation.

Since not all of the components of $\hat{n}$ are independent (recall that $\hat{n} \cdot \hat{n} = 1$!), the expansion into slow and fast modes here is a little trickier than in our previous examples. Following Polyakov, let

$$n^i(x) \equiv n^i_<(x)\sqrt{1-\phi_>^2} + \sum_{a=1}^{n-1} \phi_a^>(x) e_a^i(x). \tag{5.17}$$

Here the slow modes are represented by the unit vector $n^i_<(x)$, $\hat{n}_< \cdot \hat{n}_< = 1$; the variables e^i_a are a basis of unit vectors spanning the $n-1$ directions perpendicular to $\vec{n}_<(x)$

$$n_< \cdot \hat{e}_a = 0, \hat{e}_a \cdot \hat{e}_a = 1; \tag{5.18}$$

they are not dynamical variables and how we choose them does not matter.

The fast modes are encoded in $\phi_a^>(x) \equiv \int_{\Lambda/s}^{\Lambda}$, and $\phi_>^2 \equiv \sum_{a=1}^{n-1} \phi_a^> \phi_a^>$. Notice that differentiating the relations in (5.18) gives

$$\hat{n}_< \cdot \mathrm{d}\hat{n}_< = 0, \quad \hat{n}_< \cdot \mathrm{d}\hat{e}_a + \mathrm{d}\hat{n}_< \cdot \hat{e}_a = 0. \tag{5.19}$$

Below when I write ϕs, the $>$ symbol is implicit.

We need to plug the expansion (5.17) into the action, whose basic ingredient is

$$\mathrm{d}n^i = \mathrm{d}n^i_< \left(1-\phi^2\right)^{\frac{1}{2}} - n^i_< \frac{\phi \cdot \mathrm{d}\phi}{\sqrt{1-\phi^2}} + \mathrm{d}\phi \cdot e^i + \phi \cdot \mathrm{d}e^i.$$

So

$$\begin{aligned} L &= \frac{1}{2g^2} (\mathrm{d}\vec{n})^2 \\ &= \frac{1}{2g^2} \Bigg((\mathrm{d}n_<)^2 \left(1-\phi^2\right) + \underbrace{\mathrm{d}\phi^2}_{\text{kinetic term for } \phi} + 2\phi_a \mathrm{d}\phi_b \vec{e}_a \cdot \mathrm{d}\vec{e}_b \\ &\qquad + \underbrace{\mathrm{d}\phi_a \mathrm{d}\vec{n}_< \cdot \vec{e}_a}_{\text{source for } \phi} + \phi_a \phi_b \mathrm{d}\vec{e}_a \cdot \mathrm{d}\vec{e}_b + \mathcal{O}(\phi^3) \Bigg) \end{aligned} \tag{5.20}$$

So let's do the integral over ϕ, by treating the $\mathrm{d}\phi^2$ term as the kinetic term in a gaussian integral, and the rest as perturbations:

$$e^{-S_{\text{eff}}[n_<]} = \int [D\phi_>]_{\Lambda/s}^{\Lambda} e^{-\int L} = \int [D\phi_>]_{\Lambda/s}^{\Lambda} e^{-\frac{1}{2g^2}\int (\mathrm{d}\phi)^2} \text{ (all the rest)} \equiv \langle \text{all the rest} \rangle_{>,0} Z_{>,0} \quad .$$

The $\langle ... \rangle_{>,0}$s that follow are with respect to this measure.

$$\implies L_{\text{eff}}[n_<] = \frac{1}{2g^2} (\mathrm{d}n_<)^2 \left(1 - \langle \phi^2 \rangle_{>,0}\right) + \langle \phi_a \phi_b \rangle_{>,0} \mathrm{d}\vec{e}_a \cdot \mathrm{d}\vec{e}_b + \text{terms with more derivatives}$$

$$\langle \phi_a \phi_b \rangle_{>,0} = \delta_{ab} g^2 \int_{\Lambda/s}^{\Lambda} \frac{\mathrm{d}^2 k}{k^2} = g^2 K_2 \log(s) \delta_{ab}, \qquad K_2 = \frac{1}{2\pi}.$$

What to do with this $\mathrm{d}\vec{e}_a \cdot \mathrm{d}\vec{e}_b$ nonsense? Remember, $\vec{e}_a$ are just some arbitrary basis of the space perpendicular to $\hat{n}_<$; its variation can be expanded as

$$\mathrm{d}\vec{e}_a = (\mathrm{d}_a \cdot \hat{n}_<)\, \hat{n}_< + \sum_{c=1}^{n-1} \underbrace{(\mathrm{d}\vec{e}_a \cdot \vec{e}_c)}_{\overset{(5.19)}{=} -\mathrm{d}\hat{n}_< \cdot \vec{e}_a} \vec{e}_c$$

Therefore

$$\mathrm{d}\vec{e}_a \cdot \mathrm{d}\vec{e}_a = +(\mathrm{d}n_<)^2 + \sum_{c,a} (\vec{e}_c \cdot \mathrm{d}\vec{e}_a)^2$$

where the second term is a higher-derivative operator that we can ignore for our present purposes. Therefore

$$\begin{aligned} L_{\text{eff}}[n] &= \frac{1}{2g^2} (\mathrm{d}\hat{n}_<)^2 \left(1 - ((N-1) - 1)\, g^2 K_2 \log s\right) + ... \\ &\simeq \frac{1}{2} \left(g^2 + \frac{g^4}{4\pi} (N-2) \log s + ... \right) (\mathrm{d}\hat{n}_<)^2 + ... \end{aligned} \tag{5.21}$$

Differentiating this running coupling with respect to s gives the one-loop term in the beta function quoted above. The tree-level (order g) term comes from engineering dimensions.

5.2.5 Coherent state quantization of bosons

[Wen §3.3] Consider a system of free bosons described by the Hamiltonian

$$\mathbf{H}_0 = \sum_{\vec{k}} \left(\epsilon_{\vec{k}} - \mu\right) \mathbf{a}^\dagger_{\vec{k}} \mathbf{a}_{\vec{k}} \ .$$

Here the $\mathbf{a}$s are harmonic oscillators

$$[\mathbf{a}_{\vec{k}}, \mathbf{a}^\dagger_{\vec{k}'}] = \delta^d(\vec{k} - \vec{k}')$$

labelled by a d-dimensional spatial momentum. The Hilbert space is $\otimes_{\vec{k}} \mathcal{H}_{\vec{k}}$ where $\mathcal{H}_{\vec{k}} = \text{span}\{|n\rangle_{\vec{k}}, n = 0, 1, 2...\}$. The object $\epsilon_{\vec{k}} - \mu$ determines the energy of the state with one boson of momentum $\vec{k}$: $\mathbf{a}^\dagger_{\vec{k}}|0\rangle$. The chemical potential μ shifts the energy of any state by an amount proportional to

$$\left\langle \sum_{\vec{k}} \mathbf{a}^\dagger_{\vec{k}} \mathbf{a}_{\vec{k}} \right\rangle = N$$

the number of bosons.

For each of these oscillators we can construct coherent states

$$\mathbf{a}_k|a_k\rangle = a_k|a_k\rangle, \qquad |a_k\rangle = \mathcal{N} e^{a_k \mathbf{a}^\dagger_k}|0\rangle, \quad \mathcal{N} = e^{-|a_k|^2/2}.$$

These SHO coherent states satisfy an (over)completeness relation

$$\mathbb{1}_k = \int \frac{da_k da^\star_k}{2\pi} e^{-|a_k|^2/2} |a_k\rangle\langle a_k|.$$

(Here $\mathbb{1}_{\vec{k}}$ is the identity on the Hilbert space of a single oscillator.)

And we may construct a coherent state path integral by inserting many copies of the identity $\mathbb{1} = \prod_{\vec{k}} \mathbb{1}_{\vec{k}}$,

$$Z = \int [Da] e^{\mathbf{i} \int dt \sum_{\vec{k}} \left(\frac{\mathbf{i}}{2}\left(a^\star_{\vec{k}} \dot{a}_{\vec{k}} - a_{\vec{k}} \dot{a}^\star_{\vec{k}}\right) - \left(\epsilon_{\vec{k}} - \mu\right) a^\star_{\vec{k}} a_{\vec{k}} \right)}.$$

In real space $a_{\vec{k}} = \int \mathrm{d}^{D-1} x e^{\mathbf{i}\vec{k}\cdot\vec{x}} \psi(\vec{x})$, Taylor expanding $\epsilon_{\vec{k}} - \mu = -\mu + \frac{\vec{k}^2}{2m} + \mathcal{O}(k^4)$, this is

$$Z = \int [D\psi] e^{\mathbf{i} \int \mathrm{d}^d \vec{x} \mathrm{d}t \left(\frac{\mathbf{i}}{2}(\psi^\star \partial_t \psi - \psi \partial_t \psi^\star) - \frac{1}{2m} \vec{\nabla}\psi^\star \cdot \vec{\nabla}\psi - \mu \psi^\star \psi \right)}.$$

This the non-relativistic boson path integral we wrote earlier. The field ψ is actually the coherent state eigenvalue!

An interaction between the bosons can be written as

$$S_i = \int \mathrm{d}t \int d^d x \int d^d y \frac{1}{2} \psi^\star(x,t)\psi(x,t) V(x-y) \psi^\star(y,t)\psi(y,t) \ .$$

In the special case $V(x-y) = V(x)\delta^d(x-y)$, this is the local quartic interaction we considered earlier.

5.2.6 Where do topological terms come from?

[Abanov ch 7] Consider a 0+1 dimensional model of fermions ψ coupled to an order parameter field $\vec{n}$,

$$Z = \int [D\psi D\bar{\psi} D\vec{n}] e^{-\mathbf{i}\int_0^T \mathrm{d}t \bar{\psi}(\partial_t - M\vec{n}\cdot\vec{\sigma})\psi}$$

where $\psi = (\psi_1, \psi_2)$ is a two-component Grassmann spinor, and $\vec{\sigma}$ are Pauli matrices acting on its spinor indices. $\vec{n}^2 = 1$. It is coupled to the spin of the fermion $\bar{\psi}\vec{\sigma}\psi$.

We can do the (gaussian) integral over the fermion:

$$Z = \int [D\vec{n}] e^{-S_{\text{eff}}[\vec{n}]}$$

with

$$S_{\text{eff}}[\vec{n}] = -\log\det\left(\mathbf{i}\partial_t - \mathbf{i}M\vec{n}\cdot\vec{\sigma}\right) \equiv -\log\det D.$$

The variation of the effective action under a variation of $\vec{n}$ is:

$$\delta S_{\text{eff}} = -\mathrm{tr}\left(\delta D D^{-1}\right) = -\mathrm{tr}\left(\delta D D^{\dagger}\left(DD^{\dagger}\right)^{-1}\right)$$

where $D^{\dagger} = \mathbf{i}\partial_t + \mathbf{i}M\vec{n}\cdot\vec{\sigma}$. This is

$$\delta S_{\text{eff}} = \mathbf{i}M\mathrm{tr}\left(\delta\vec{n}\cdot\vec{\sigma}\left(\mathbf{i}\partial_t + \mathbf{i}M\vec{n}\cdot\vec{\sigma}\right)\left(-\partial_t^2 + M^2 - M\dot{\vec{n}}\cdot\vec{\sigma}\right)^{-1}\right).$$

We can expand the denominator in $\dot{\vec{n}}/M$ to get

$$\delta S_{\text{eff}} = \int \mathrm{d}t \left(\frac{1}{4M}\delta\dot{\vec{n}}\dot{\vec{n}} - \frac{\mathbf{i}}{2}\delta\vec{n}\cdot\left(\vec{n}\times\dot{\vec{n}}\right)\right) + ...$$

where ... is higher order in the expansion and we ignore it. But we know this is the variation of

$$S_{\text{eff}} = \int_0^T \mathrm{d}t \left(\frac{1}{8M}\dot{\vec{n}}^2\right) - 2\pi\mathbf{i}W_0$$

where W_0 is the WZW term.

Topological terms are one way in which some (topological) information from short distances can persist in the low energy effective action. Here the information is that the system is made of fermions.