Brain-warmer on spin coherent states.

Show that

\[ \langle \hat{n} | \vec{ h} \cdot \vec{ S} | \hat{n} \rangle = s \vec{ h} \cdot \vec{ n} \]

where \( | n \rangle = R | s, s \rangle \) is a coherent state of spin \( s \) (where \( | s, s \rangle \) is the eigenvector of \( S_z \) with maximal eigenvalue, and \( R \) is the rotation operator which takes \( \hat{z} \) to \( \hat{n} \)).

Show that for several spins and \( i \neq j \)

\[ \langle \hat{n} | \vec{ S}_i \cdot \vec{ S}_j | \hat{n} \rangle = s^2 \vec{n}_i \cdot \vec{n}_j, \]

where now \( | n \rangle \equiv \otimes_j (R | s_i \rangle) \) is a product of coherent states of each of the spins individually.

Brain-warmer on Schwinger bosons.

Recall the Schwinger-boson representation of the \( SU(2) \) algebra:

\[ S^+ = a^\dagger b, \quad S^- = b^\dagger a, \quad S_z = a^\dagger a - b^\dagger b, \]

where the modes \( a, b \) satisfy \([a, a^\dagger] = 1 = [b, b^\dagger], [a, b] = [a, b^\dagger] = 0 \). This is the algebra of a simple harmonic oscillator in two dimensions,

\[ H = \frac{1}{2} (p_x^2 + p_y^2 + x^2 + y^2). \]

Is the \( SU(2) \) a symmetry of this Hamiltonian? How does it act on the oscillator coordinates? Check that the oscillator algebra does indeed imply that \( \vec{S} \) defined this way satisfy the \( SU(2) \) algebra.

Simplicial homology and the toric code.

In lecture we discussed the (de Rham) cohomology of the exterior derivative \( d \) acting on vector spaces (over \( \mathbb{R} \)) of differential forms on some smooth manifold \( X \). The dimensions \( b^p(X, \mathbb{R}) \) of the cohomology groups are topological properties of \( X \). This same data is manifested in many other ways; in this problem we study another one, along with an important and familiar physical realization of it.
(a) The toric code we’ve discussed so far has qbits on the links \( \ell \in \Delta_1(\Delta) \) of a graph \( \Delta \). But the definition of the Hamiltonian involves more information than just the links of the graph: we have to know which vertices \( v \) lie at the boundaries of each link \( \ell \), and we have to know which links are boundaries of which faces. The Hamiltonian has two kinds of terms: a ‘plaquette’ operator \( B_p = \prod_{\ell \in \partial p} X_\ell \) associated with each 2-cell (plaquette) \( p \in \Delta_2(\Delta) \), and ‘star’ operators, \( A_s = \prod_{\ell \in \partial^{-1}(s)} Z_\ell \), associated with each 0-cell (site) \( s \in \Delta_0(\Delta) \). Here I’ve introduced some notation that will be useful, please be patient: \( \Delta_k \) denotes a collection of \( k \)-dimensional polyhedra which I’ll call \( k \)-simplices or more accurately \( k \)-cells – \( k \)-dimensional objects making up the space. (It is important that each of these objects is topologically a \( k \)-ball.) This information constitutes (part of) a simplicial complex, which says how these parts are glued together:

\[
\Delta_d \xrightarrow{\partial} \Delta_{d-1} \xrightarrow{\partial} \cdots \Delta_1 \xrightarrow{\partial} \Delta_0
\]

where \( \partial \) is the (signed) boundary operator. For example, the boundary of a link is \( \partial \ell = s_1 - s_0 \), the difference of the vertices at its ends. The boundary of a face \( \partial p = \sum_{\ell \in \partial p} \ell \) is the (oriented) sum of the edges bounding it. By \( \partial^{-1}(s) \) I mean the set of links which contain the site \( s \) in their boundary (with sign).

Think of this collection of objects as a triangulation (or more generally some chopping-up) of a smooth manifold \( X \). Convince yourself that the sequence of maps (1) is a complex in the sense that \( \partial^2 = 0 \).

(b) [not actually a question] This means that the simplicial complex defines a set of homology groups, which are topological invariants of \( X \), in the following way. (It is homology and not cohomology because \( \partial \) decreases the degree \( k \)). To define these groups, we should introduce one more gadget, which is a collection of vector spaces over some ring \( R \) (for the ordinary toric code, \( R = \mathbb{Z}_2 \))

\[
\Omega_p(\Delta, R), \ p = 0...d \equiv \dim(X)
\]

basis vectors for which are \( p \)-simplices:

\[
\Omega_p(\Delta, R) = \text{span}_R \{ \sigma \in \Delta_p \}
\]

- that is, we associate a(n orthonormal) basis vector to each \( p \)-simplex (which I’ve just called \( \sigma \)), and these vector spaces are made by taking linear combinations of these spaces, with coefficients in \( R \). Such a linear combination of \( p \)-simplices is called a \( p \)-chain. It’s important that we can add (and subtract) \( p \)-chains, \( C + C' \in \Omega_p \). A \( p \)-chain with a negative coefficient can
be regarded as having the opposite orientation. We’ll see below how better to interpret the coefficients.

The boundary operation on $\Delta_p$ induces one on $\Omega_p$. A chain $C$ satisfying $\partial C = 0$ is called a cycle, and is said to be closed.

So the $p$th homology is the group of equivalence classes of $p$-cycles, modulo boundaries of $p+1$ cycles:

$$H_p(X, R) \equiv \frac{\ker (\partial : \Omega_p \to \Omega_{p-1}) \subset \Omega_p}{\text{Im} (\partial : \Omega_{p+1} \to \Omega_p) \subset \Omega_p}$$

This makes sense because $\partial^2 = 0$ – the image of $\partial : \Omega_{p+1} \to \Omega_p$ is a subset of $\ker (\partial : \Omega_p \to \Omega_{p-1})$. It’s a theorem that the dimensions of these groups are the same for different (faithful-enough) discretizations $\Delta$ of $X$. Furthermore, their dimensions (as vector spaces over $R$) contain (much of) the same information as the Betti numbers defined by de Rham cohomology. For more information and proofs, see the great book by Bott and Tu, Differential forms in algebraic topology.

(c) A state of the toric code on a cell-complex $\Delta$ can be written (for the hamiltonian described above, this is in the basis where $Z_\ell$ is diagonal) as an element of $\Omega_1(X, \mathbb{Z}_2)$,

$$|\Psi\rangle = \sum_C \Psi(C) |C\rangle$$

where $C$ is an assignment of an element of $\mathbb{Z}_2$ in $X$ (the eigenvalue of $Z_\ell$). For the case of $\mathbb{Z}_2$ coefficients, $1 = -1 \mod 2$ and we don’t care about the orientations of the cells. Show that the conditions for a state $\Psi(C)$ to be a groundstate of the toric code ($A_s |\Psi\rangle = |\Psi\rangle \forall s$ and $B_p |\Psi\rangle = |\Psi\rangle \forall p$) are exactly those defining an element of $H_1(X, \mathbb{Z}_2)$.

(d) Consider putting a spin variable on the $p$-simplices of $\Delta$. More generally, let’s put an $N$-dimensional hilbert space $\mathcal{H}_N \equiv \text{span}\{|n\rangle, n = 1..N\}$ on each $p$-simplex, on which act the operators

$$X \equiv \sum_{n=1}^N |n\rangle \langle n| \omega^n = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & \omega & 0 & \ldots \\ 0 & 0 & \omega^2 & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix}, \quad Z \equiv \sum_{n=1}^N |n\rangle \langle n+1| = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \\ 1 & 0 & 0 & \ldots \end{pmatrix}$$

where $\omega^N = 1$ is an $n$th root of unity. If you haven’t already, check that satisfy the clock-shift algebra: $XZ = \omega ZX$. For $N = 2$ these are Pauli matrices and $\omega = -1$.

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1I don’t want to talk about torsion homology.
Consider the Hamiltonian

\[ H = -J_{p-1} \sum_{s \in \Delta_{p-1}} A_s - J_{p+1} \sum_{\mu \in \Delta_{p+1}} B_\mu - g_p \sum_{\sigma \in \Delta_p} Z_\sigma \]

with

\[ A_s \equiv \prod_{\sigma \in \partial^{-1}(s) \subset \Delta_p} Z_\sigma \]
\[ B_\mu \equiv \prod_{\sigma \in \partial_\mu} X_\sigma . \]

This is a lattice version of \( p \)-form \( \mathbb{Z}_N \) gauge theory, at a particular, special point in its phase diagram.

Show that

\[ 0 = [A_s, A_{s'}] = [B_\mu, B_{\mu'}] = [A_s, B_\mu], \quad \forall s, s', \mu, \mu' \]

so that for \( g_p = 0 \) this is solvable.

(e) Show that the groundstates of \( H_p \) (with \( g_p = 0 \)) are in one-to-one correspondence with elements of \( H_p(\Delta, \mathbb{Z}_N) \).

4. **Non-linear sigma models on more general spaces.** [Warning: some knowledge of general relativity is helpful here.]

In lecture we considered the 2d non-linear sigma model whose target space was a round 2-sphere, motivated by the low-energy physics of antiferromagnets. At weak coupling (large radius of sphere, which means large spin), we saw that the sphere wants to shrink in the IR.

Consider now a 2d non-linear sigma model (NLSM) whose target space is a more general manifold \( X \) with Riemannian metric \( ds^2 = L^2 g_{ij}(x)dx^i dx^j \). Assume that the space is big, in the sense that we will treat the parameter \( L^{-1} \) as a small parameter, and smooth in the sense that we can Taylor expand around any point.

The NLSM is a field theory whose fields \( x^i(\sigma) \) are maps from spacetime (here 2d flat space) to the target space \( X \). The simplest action is

\[ S[x(\sigma)] = \int d^2 \sigma L^2 g_{ij}(x) \partial_\sigma x^i \partial_\sigma x^j \eta^{\mu\nu} \]

where \( \eta^{\mu\nu} \) is the flat metric on the 2d spacetime ‘worldsheet’.

\( D = 2 \) is special because the free scalar field \( x(\sigma) \) is dimensionless. As long as \( g_{ij} \) is nonsingular, in the limit \( L \to \infty \), the local coordinate field becomes free.
Regard $g_{ij}(x)$ as a coupling function. What is the leading beta function (actually beta functional) for this set of couplings?

Hint: use the fact that the answer must be covariant under changes of coordinates on $X$ plus dimensional analysis.

5. **Haldane phase.** [bonus problem]

Consider the $D = 1 + 1$ nonlinear sigma model with target space $S^2$ at $\theta = 2\pi$. The $\theta$ term is a total derivative in the action, so it can manifest itself when we study the path integral on a spacetime with boundary.

(a) Put this field theory on the half-line $x > 0$. Suppose that the boundary conditions respect the $SO(3)$ symmetry, so that the boundary values $\vec{n}(\tau, x = 0)$ are free to fluctuate. By remembering that the $\theta$-term is a total derivative, and considering the strong-coupling (IR) limit, $g \to \infty$, show that there is a spin-$\frac{1}{2}$ at the boundary. (Hint: Recall the coherent state path integral for a spin-$\frac{1}{2}$.)

(b) Now cut the path integral open at some fixed euclidean time $\tau = 0$. (Consider periodic boundary conditions in space.) Such a path integral computes the groundstate wavefunction, as a function of the boundary values of the fields, $\vec{S}(x, \tau = 0)$. Find the groundstate wavefunctional is $\Psi[\vec{n}(x, \tau = 0)]$ in the strong coupling limit $g \to \infty$ (where the gap is big).