University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239/139 Spring 2018
Assignment 3

Due 12:30pm Monday, April 23, 2018

1. **Mutual information bounds correlations.**

   Consider again the distribution on two binary variables from last homework: $p_{yx} = \begin{pmatrix} 0 & a \\ b & b \end{pmatrix}_{yx}$, where $y = 1, -1$ is the row index and $x = 1, -1$ is the column index (so yx are like the indices on a matrix). Normalization implies $\sum_{xy} p_{xy} = a + 2b = 1$, so we have a one-parameter family of distributions, labelled by b.

   (a) I've changed the labels on the variables from $\uparrow, \downarrow$ to $1, -1$ so that we can consider correlation functions, such as the connected two-point function

   $$C \equiv \langle xy \rangle_c \equiv \langle xy \rangle - \langle x \rangle \langle y \rangle$$

   where $\langle A \rangle \equiv \sum_{xy} p_{yx} A$. Compute C as a function of b.

   (b) Compute the mutual information between X and Y

   $$I(X : Y) = \sum_{xy} p_{yx} \log \frac{p_{yx}}{p_y p_x}.$$

   (c) Check that

   $$I(X : Y) \geq \frac{1}{2} C^2$$

   for every value of b (for example, plot both functions).

   (d) [Bonus] The inequality I quoted in lecture, and which we will prove in the more general quantum case later, is

   $$I(X : Y) \geq \frac{1}{2} \frac{\langle \mathcal{O}_X \mathcal{O}_Y \rangle_c^2}{\| \mathcal{O}_X \|^2 \| \mathcal{O}_Y \|}$$

   where the norms are defined (in the classical case) by

   $$\| \mathcal{O}_X \|^2 \equiv \sup_{p | \sum_x p_x = 1} \{ \sum_x \mathcal{O}_x^\star \mathcal{O}_x p_x \}.$$

   Show that in the above example, the 'operators' x, y are normalized, in the sense that $\| x \| = \| y \| = 1$.

2. **Strong subadditivity, the classical case.** [From Barnett] Prove *strong subaddivity* of the Shannon entropy: for any distribution on three random variables,

$$H(ABC) + H(B) \leq H(AB) + H(BC).$$

(The corresponding statement about the von Neumann entropy is not so easy to show.)

Hint: $q(a,b,c) \equiv \frac{p(a,b)p(b,c)}{p(b)}$ is a perfectly cromulent probability distribution on ABC.

What is the name for the situation when equality holds? Write the condition for equality in terms of the conditional mutual information $I(A:C|B)$.

3. **Symbol coding problem.**

You are a mad scientist, but a sloppy one. You have 127 identical-looking jars of liquid, and you have forgotten which one is the poison one. You have at your disposal 7 rats on whom your poor moral compass will allow you to test the liquids. However (the rats have a strong social network and excellent spies) you only get one shot: the rats must drink all at once (or they will catch on to what is happening and revolt). You may mix the liquids in separate containers; any rat that drinks any amount of poison will turn bright orange. Design a protocol to uniquely identify the poison jar.

4. **Huffman code.**

Make the Huffman code for the probability distribution $p(x) = .5, .2, .15, .1, .05$.

Compare the average word length to the Shannon entropy.

Bonus: what property of the distribution determines the deviation from optimality?

5. **Huffman code decryption problem.** [Optional, but fun.]

```
000110100 010101010010110 1111100001000110001001 1 0010111 00101000
0111111100001001011000011 01 1010101001001 10111011 1110100
00001000100000100101000011 0000
110001001011101110100101001001000 10011
0001011001010010100011010111 0111 11111000000111 00101000
0001001010010110100 0111 0000 100011110010010001101 0100 010010101
1010000010 1111000?  1111100001000110001001 1 010010101 0010111
11010010000100 11011 0000 1111001011000010001011 01001011
110010110001 1111.
```

```
0000110111111000100101100101001100 1111100011
11001011010101101011011100000011 0101101000011011 00101000 00000011
010000010111100101011100001101 10000101011011110100001001101
0101101000011011, 10001111001011000010111 010110100000110111000
(10111000001011000100110000010101 01011110000000001011110111001
10000010000100100) 00001001011 101000011111110111001 111110000000011
00000011010101 10000101011011110100001001101 0101101000011011
101000011010110000101101000011 1111100011010101 00001001011
1011101001001011 10101110101100010010000101010001101011.
1001011100001100001001010111100 00010110010100101000110101110111:
101010010001100011 00000011 00001101010100010010010010111110010111
1010101001001 1000010011010001000000100011010100
000100101110110000000111111011010111 1011100000011010101
101000010000011010101 10000101011000111011101111000
0010001110000001010010100101100011 101000010101
0000100100101111100101110111111001001011 010110100000110111000
0000001100011 1101011101110001010111101-0001001010100100001000
010110100000110111000.
```

Hint: I used the letter frequencies from *The Origin of Species*.

You might want to use `Mathematica` to do this problem!

6. **Analogy with strong-disorder RG.** [open ended, more optional question]

   Test or decide the following consequence suggested by the analogy between Huffman coding and strong-disorder RG: The optimality of the Huffman code is better when the distribution is broader. A special case is the claim that the Huffman code is worst when all the probabilities are the same. Note that the outcome of the Huffman algorithm in this case depends on the number of elements of the alphabet.

   Measure the optimality by $\langle \ell \rangle - H[p]$ (or maybe $\frac{\langle \ell \rangle - H[p]}{H[p]}$?).

7. **Binary symmetric channel.**

   For the binary symmetric channel ABE defined in lecture, with $a, b, e \in \{0, 1\}$, and

   $$p(a) = (p, 1-p)_a, \; p(e) = (q, 1-q)_e, \quad \text{and} \;\; b = (a+e)_2,$$

   find all the quantities $p(a,b), p(b), p(b|a), p(a|b)$ and $H(B), H(B|A), I(B:A), I(B:A|E)$.