University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239/139 Spring 2018
Assignment 6

Due 12:30pm Monday, May 14, 2018

1. **Brainwarmers on Kraus operators.**

   (a) Check that the Kraus operators

   $$\mathcal{K}_i = \langle i|U|0\rangle$$

   (where U is a unitary on $A \otimes \bar{A}$, $\{|i\rangle\}$ is an ON basis of $\bar{A}$, and $|0\rangle$ is a reference state in $\bar{A}$) satisfy the condition

   $$\sum_i \mathcal{K}_i^\dagger \mathcal{K}_i = \mathbb{1}_A \tag{1}$$

   (b) Check that the condition (1) implies that the Kraus evolution $\boldsymbol{\rho} \to \sum_i \mathcal{K}_i \boldsymbol{\rho} \mathcal{K}_i^\dagger$ preserves the trace.

   (c) Find a set of Kraus operators for the erasure (or reset) channel that takes $\boldsymbol{\rho} \mapsto |0\rangle\langle 0|$ for every $\boldsymbol{\rho}$. Check that they satisfy (1).

2. **Stationary states of unital channels.**

   Check that the unital condition $\sum_i \mathcal{K}_i^\dagger \mathcal{K}_i = \mathbb{1}$ implies that the uniform density matrix $\mathbf{u} \equiv \mathbb{1}\frac{1}{|\mathcal{H}|}$ is a fixed point of the associated quantum channel, $\mathcal{E} : \mathcal{H} \to \mathcal{H}$,

   $$\mathcal{E}(\mathbf{u}) = \mathbf{u}.$$

3. **Shannon entropy is concave.**

   Consider a collection of probability distributions π^α on a random variable X, so $\sum_x \pi_x^\alpha = 1, \pi_x^\alpha \geq 0, \forall x$. Then a convex combination of these $\pi_{\text{av}} \equiv \sum_\alpha p_\alpha \pi^\alpha$ is also a probability distribution on X. Show that the entropy of the average distribution is larger than the average of the entropies:

   $$H(\pi_{\text{av}}) \geq \sum_\alpha p_\alpha H(\pi^\alpha).$$

   My earlier claim that the upper bound is always saturated classically is not true. Thanks to Ahmed Akhtar for pointing this out.

4. **Random quantum expanders.** [optional. somewhat open-ended and numerical]

   Consider the family of quantum channels of the form

$$\boldsymbol{\rho} \mapsto \mathcal{E}_\chi(\boldsymbol{\rho}) = \sum_{i=1}^{\chi} p_i \mathbf{U}_i \boldsymbol{\rho} \mathbf{U}_i^\dagger$$

   with $\{\mathbf{U}_i\}$ a collection of unitaries. Such a channel is called a *quantum expander*. Show that such a channel is unital.

   Sample χ random unitaries from the Haar measure on $\mathsf{U}(d)$ *e.g.* in Mathematica[1]. (You can take $p_i = 1/\chi$ for definiteness if you wish.)

   Sample a random initial density matrix[2].

   Consider the rate at which repeated action of the channel $\mathcal{E}_\chi$, $\boldsymbol{\rho}_n = \mathcal{E}^n(\boldsymbol{\rho})$ mixes the initial state $\boldsymbol{\rho}$ as a function of χ (and d). We can use the von Neumann entropy as a measure of this mixing. Make some plots and some estimates.

   If n is very large, how many terms do I actually need to include in the sum in

$$\mathcal{E}^n(\boldsymbol{\rho}) = \sum_{i_1..i_n} p_{i_n} \cdots p_{i_1} \mathbf{U}_{i_1} \cdots \mathbf{U}_{i_n} \boldsymbol{\rho} \mathbf{U}_{i_n}^\dagger \cdots \mathbf{U}_{i_1}^\dagger \quad ?$$

   Consider the eigenstates (eigenoperators) of the (super)operator $\mathcal{E}_\chi$. Can you show that any state orthogonal (in the Hilbert-Schmidt norm) to $\mathbb{1}$ has a an eigenvalue less than 1?

5. **Scramble.**

   For this problem $\mathcal{H}_A$ has dimension d.

   (a) **Warmup.** The set of linear operators $\mathrm{End}(\mathcal{H}_A)$ is itself a Hilbert space with the Hilbert-Schmidt inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{tr}\mathbf{A}^\dagger \mathbf{B}$. Find an orthogonal basis $\{\mathbf{U}_a\}$ for this space (over $\mathbb{C}$) whose elements are themselves unitary operators, $\mathrm{tr}\mathbf{U}_a^\dagger \mathbf{U}_b = d\delta_{ab}$.

   [Hint: consider the algebra generated by the unitaries $\mathbf{X}, \mathbf{Z}$ on the qdit teleportation problem on the previous problem set.]

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[1] Haar measure means the measure which is invariant under the group action. I did this by choosing a $d \times d$ complex matrix X with entries chosen from the gaussian distribution (which is indeed invariant under $\mathsf{U}(d)$) and then taking $Y = X + X^\dagger$ to make it hermitian, and then using the matrix U which diagonalizes Y.

[2] I did this by choosing a complex matrix X with entries chosen from the gaussian distribution, and then taking $Y = X + X^\dagger$ to make it hermitian and then taking $Z = Y^2$ to make it positive and then taking $\boldsymbol{\rho} = Z/\mathrm{tr}Z$ to make it a density matrix. What distribution did I use?

Bonus: For the case of $|A| = 2^k$ find such a basis whose elements square to one.

(b) Consider a maximally entangled state $|\Phi\rangle \equiv \frac{1}{\sqrt{d}} \sum_i |ii\rangle \in \mathcal{H}_A \otimes \mathcal{H}_A$. Show that the d^2 maximally entangled states

$$|\Phi_a\rangle \equiv \mathbf{U}_a \otimes \mathbb{1} |\Phi\rangle$$

form an orthonormal basis of $\mathcal{H}_A \otimes \mathcal{H}_A$.

(c) Check your answers to the previous two parts for the case of qbits $d = 2$. Make a basis of product states from linear combinations of maximally entangled states.

(d) For an arbitrary operator $\mathbf{A} \in \mathrm{End}(A)$, find $\{p_a, \mathbf{U}_a\}$ with p_a probabilities and $\mathbf{U}_a$ unitary such that the associated channel *scrambles* $\mathbf{A}$ in the sense that

$$\sum_a p_a \mathbf{U}_a \mathbf{A} \mathbf{U}_a^\dagger = \frac{\mathrm{tr}\mathbf{A}}{d} \mathbb{1}.$$

(e) Use the previous result and the concavity of the entropy to show that the uniform state $\mathbf{u} = \mathbb{1}/d$ has the maximum von Neumann entropy on A.

(f) Bonus problem: for the case where $\mathbf{A}$ is Hermitian, find a set of only d unitaries which scramble $\mathbf{A}$.

The following is a collection of a few more examples of quantum channels, wherein it is fun and profitable to determine the long-term behavior on repeated action of the channel. Do as many of them as you find instructive.

6. **Decoherence by phase damping with non-orthogonal states** [from Preskill]

Suppose that a heavy particle A begins its life in outer space in a superposition of two positions

$$|\psi_0\rangle_A = a\,|x_0\rangle + b\,|x_1\rangle\,.$$

These positions are not too far apart. The particle interacts with the electromagnetic field, and in time dt, the whole system evolves according to

$$\mathbf{U}_{AE}\,|x_0\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\,|x_0\rangle_A \otimes |0\rangle_E + \sqrt{p}\,|x_0\rangle_A \otimes |\gamma_0\rangle_E$$

$$\mathbf{U}_{AE}\,|x_1\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\,|x_1\rangle_A \otimes |0\rangle_E + \sqrt{p}\,|x_1\rangle_A \otimes |\gamma_1\rangle_E$$

But because x_0 and x_1 are close, the (normalized) photon states $|\gamma_0\rangle\,, |\gamma_1\rangle$ have a large overlap:

$$\langle\gamma_0|\gamma_1\rangle_E = 1 - \epsilon, \quad \text{with } 0 < \epsilon \ll 1.$$

(a) Find the Kraus operators describing the time evolution of the reduced density matrix $\boldsymbol{\rho}_A$.

(b) How long does it take the superposition to decohere? More precisely, at what time t is $(\rho_A)_{01}(t) = \frac{1}{e}(\rho_A)_{01}(t=0)$?

7. **Decoherence on the Bloch sphere** [from Preskill]

Parametrize the density matrix of a single qubit as

$$\boldsymbol{\rho}_A = \frac{1}{2}\left(\mathbb{1} + \vec{P}\cdot\vec{\boldsymbol{\sigma}}\right).$$

(a) **Polarization-damping channel.**

Consider the (unitary) evolution of a qbit A coupled to a 4-state environment via

$$\mathbf{U}_{AE}\left|\phi\right\rangle_A \otimes \left|0\right\rangle_E = \sqrt{1-p}\left|\phi\right\rangle_A \otimes \left|0\right\rangle_E + \sqrt{p/3}\sum_{i=1}^{3}\boldsymbol{\sigma}^i_A \otimes \mathbb{1}_E\left|\phi\right\rangle_A \otimes \left|i\right\rangle_E$$

Show that this evolution can be accomplished with the Kraus operators

$$\mathbf{M}_0 = \sqrt{1-p}\mathbb{1}, \quad \mathbf{M}_i = \sqrt{p/3}\boldsymbol{\sigma}^i,$$

and show that they obey the completeness relation requred by unitarity of $\mathbf{U}_{AE}$.

Show that the polarization P_i of the qbit evolves according to

$$\vec{P} \to \left(1 - \frac{4p}{3}\right)\vec{P}.$$

Describe this evolution in terms of what happens to the Bloch ball.

What happens if $p > 3/4$?

(b) **Two-Pauli channel.**

Consider the (unitary) evolution of a qbit A coupled to a *three*-state environment via

$$\mathbf{U}_{AE}\left|\phi\right\rangle_A \otimes \left|0\right\rangle_E = \sqrt{1-p}\left|\phi\right\rangle_A \otimes \left|0\right\rangle_E + \sqrt{p/2}\sum_{i=1}^{2}\boldsymbol{\sigma}^i_A \otimes \mathbb{1}_E\left|\phi\right\rangle_A \otimes \left|i\right\rangle_E$$

Show that this evolution can be accomplished with the Kraus operators

$$\mathbf{M}_0 = \sqrt{1-p}\mathbb{1}, \quad \mathbf{M}_i = \sqrt{p/2}\boldsymbol{\sigma}^i, i = 1, 2$$

and show that they obey the completeness relation requred by unitarity of $\mathbf{U}_{AE}$.

Describe this evolution in terms of what happens on the Bloch ball, and evaluate the purity.

(c) **Phase-damping channel.**

For the evolution of problem 6,

$$\mathbf{U}_{AE} |0\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |0\rangle_A \otimes |0\rangle_E + \sqrt{p}\, |0\rangle_A \otimes |\gamma_0\rangle_E$$

$$\mathbf{U}_{AE} |1\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |1\rangle_A \otimes |0\rangle_E + \sqrt{p}\, |1\rangle_A \otimes |\gamma_1\rangle_E$$

now thinking of A as a qbit, describe the evolution of its polarization vector on the Bloch ball.

8. **Turtles all the way down.** [optional, open-ended]

A question you may have about our discussion of polarization-damping as a model of decoherence is: why does the environment reset to the reference state $|0\rangle_E$?

We can postpone the question a bit by coupling the environment to its own environment, according to an amplitude damping channel. On the previous problem set, you saw that the result of the repeated action of such a channel can set $\boldsymbol{\rho}_E = |0\rangle\langle 0|$. This statement in turn assumes a forgetful meta-environment. A thermodynamic limit is required to postpone the question indefinitely. Construct such a thermodynamic limit.