University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239/139 Spring 2018
Assignment 8

Due 12:30pm Wednesday, May 30, 2018

1. **Brainwarmers.**

   (a) Is it true that $0 \leq S(A|C) + S(B|C)$? Prove or give a counterexample.

   (b) Show that the von Neumann entropy is the special case[1] $S(\boldsymbol{\rho}) = \lim_{\alpha \to 1} S_\alpha(\boldsymbol{\rho})$ of the Renyi entropies:

   $$S_\alpha(\boldsymbol{\rho}) \equiv \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \log \mathrm{tr} \boldsymbol{\rho}^\alpha = \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \log \sum_a p_a^\alpha \ .$$

2. **Direct application of Lieb's theorem.**

   We only used a very special case of Lieb's theorem to prove monotonicity of the relative entropy. Surely there is more to learn from it.

   Consider an ensemble of states $\boldsymbol{\rho} = \sum_i p_i \boldsymbol{\rho}_i$, and a unitary operator $\mathbf{U}$ (for example, it may be closed-system time evolution).

   Show that the relative entropy between $\boldsymbol{\rho}(t) \equiv \mathbf{U}\boldsymbol{\rho}\mathbf{U}^\dagger$ and $\boldsymbol{\rho}$ is convex in $\boldsymbol{\rho}$:

   $$D(\boldsymbol{\rho}(t)||\boldsymbol{\rho}) \leq \sum_i p_i D(\boldsymbol{\rho}_i(t)||\boldsymbol{\rho}_i).$$

   Open ended bonus problem: see if you can find a better result by directly applying Lieb's joint concavity theorem to a problem in many body physics.

3. **Majorization questions.**

   (a) Show that if a doubly stochastic map is reversible (invertible and the inverse is also doubly stochastic) then it is a permutation.

   (b) Show that the set of doubly stochastic maps is convex (that is: a convex combination $\sum_a p_a D_a$ of doubly stochastic maps is doubly stochastic). What are the extreme points of this set? (This is the easier direction of the Birkhoff theorem.)

[1] Thanks to Kelson Kaj for pointing out the error in the previous version.

(c) Show that a pure state and uniform state satisfy $(1, 0, 0 \cdots) \succ p \succ (1/L, 1/L \cdots)$ for any p on an L-item space.

(d) A useful visualization of majorization relations is called the 'Lorenz curve': this is just a plot of the cumulative probability $P_p(K) = \sum_{k=1}^{K} p_k$ as a function of K. What does $p \succ q$ mean for the Lorenz curves of p and q? Draw the Lorenz curves for the uniform distribution and for a pure state.

(e) Show that the set of probability vectors majorized by a fixed vector x is convex. That is: if $x \succ y$ and $x \succ z$ then $x \succ ty + (1-t)z, t \in [0, 1]$. Hints: (1) the analogous relation is true if we replace x, y, z with real numbers and $\succ$ with $\geq$. (2) Show that $P_{p^\downarrow}(K) \geq P_{\pi p^\downarrow}(K)$ (where $\pi p^\downarrow$ indicates any other ordering of the distribution).

(f) For the case of a 3-item sample space we can draw some useful pictures of the whole space of distributions. The space of probability distributions on three elements is the triangle $x_1 + x_2 + x_3 = 1, x_i \geq 0$, which can be drawn in the plane. We can simplify the picture further by ordering the elements $x_1 \geq x_2 \geq x_3$, since majorization does not care about the order. Pick some distribution x with $x_1 \neq x_2 \neq x_3$ and draw the set of distributions which x majorizes, the set of distributions majorized by x, and the set of distributions with which x does not participate in a majorization relation ('not comparable to x').