

Physics 239/139 Spring 2018

Assignment 9

Due 12:30pm Wednesday, June 6, 2018

1. Work and the Holevo bound.

(a) Show that the Holevo quantity $\chi(p_a, \rho_a) \equiv S(\rho_{av}) - \sum_a p_a S(\rho_a)$ (with $\rho_{av} \equiv \sum_a p_a \rho_a$) can be written as $\chi(p_a, \rho_a) = \sum_a p_a D(\rho_a || \rho_{av})$.

(b) Show that

$$\sum_a p_a D(\rho_a || \sigma) = \chi(p_a, \rho_a) + D(\rho_{av} || \sigma).$$

(c) Suppose A labors in contact with a heat bath at temperature T , and is governed by hamiltonian H . Convince yourself that in order to create the signal state ρ_a , the required work A must do is

$$W_a \geq F_T[\rho_a] - F_T[\rho_T] = (k_B T \ln 2) D(\rho_a || \rho_T),$$

where $F_T[\rho] \equiv \text{tr} \rho H - T S_{vN}[\rho]$ is the free energy functional.

(d) Show that the average work $\bar{W} \equiv \sum_a p_a W_a$ satisfies

$$\bar{W} \geq (k_B T \ln 2) \chi(p_a, \rho_a).$$

(hint: $D(\rho || \sigma) \geq 0$).

(e) Apply the Holevo bound to conclude

$$\bar{W} \geq (k_B T \ln 2) I(A : B),$$

so that that every bit of information A can convey to B requires average work at least $k_B T \ln 2$. Yay, Landauer.

(f) [optional] Estimate the amount of work done per bit sent to your cellular telephone.

2. Holevo quantity and channel capacity.

Consider a collection of mutually-commuting density matrices $\{\rho_a\}$. Show that in this case, the Holevo quantity

$$\chi(p_a, \rho_a) \equiv S(\rho_{av}) - \sum_a p_a S(\rho_a) = \sum_a p_a D(\rho_a || \rho_{av}), \quad \rho_{av} \equiv \sum_a p_a \rho_a$$

is the mutual information $I(A : B)$, where the random variable B is the variable b labelling the mutual eigenvectors of the ρ_a : $\rho_a = \sum_b \lambda_a^b |b\rangle\langle b|$.

This suggests that a good definition of the capacity of a quantum channel for sending classical information (let's call it classical capacity) is determined by the Holevo quantity as

$$C = \chi(p_a, \rho_a) / \mathcal{T}$$

(where \mathcal{T} is how long the information takes to go down the channel). And indeed, recall the Holevo bound, which says that $I(A : B) \leq \chi(p_a, \rho_a)$ where B is the outcomes of *any* measurement done on $\sum_a p_a \rho_a$.

3. Channel capacity of the radiation field.

Suppose (crazy idea) we wanted to send signals using the electromagnetic field.

The radiation field is a collection of quantum harmonic oscillators labelled by frequency, ω . For simplicity, let's consider a one-dimensional field with only one polarization, so there is one oscillator for each value of ω . In the first part of the problem, we'll put the system in a box, so that the allowed frequencies are integer multiples of some fundamental frequency, and the energy of a state with n_j photons in mode j is $E(\{n\}) = \sum_j j n_j h \equiv N h$ for some constant h .

The signal information could be stored for example in the number of photons $\bar{n}(\omega)$ with a given frequency. As in other examples, to send message a , A puts the field in the state ρ_a . And the message can be extracted by measurements on the resulting radiation field, for example by counting photons.

For practical reasons, we will fix the power P of the signal. There are several ways to implement this constraint; we'll consider two below.

At first we ignore the presence of noise.

- (a) Show that the Holevo quantity χ (and hence the channel capacity, no matter what measurement we do) is bounded by the entropy of the average signal $\sum_a p_a \rho_a$.
- (b) What is the ρ_{av} which maximizes the entropy, subject to the constraint of fixed energy $E(\{n\}) = P\mathcal{T}$ (where \mathcal{T} is the duration of the signal)?
- (c) As a useful intermediate step, show that the entropy for a single harmonic oscillator in thermal equilibrium can be written in terms of the average occupation number \bar{n} as $S_B(\bar{n})$ where

$$S_B(n) \equiv (n + 1) \log(n + 1) - n \log n.$$

- (d) Using the definition of classical capacity in the previous problem, determine the classical capacity of this channel at large \mathcal{T} .

You may use the Hardy-Ramanujan formula, which counts partitions of N at large N :

$$\mathcal{N}(N) = \frac{1}{4\sqrt{3N}} e^{\pi\sqrt{\frac{2}{3}N}} + \mathcal{O}\left(e^{\frac{\pi}{2}\sqrt{\frac{2}{3}N}}\right).$$

- (e) Alternatively, we may impose the condition of fixed power as a condition on the *average* energy. The state which maximizes entropy at fixed average energy is a thermal state. The temperature is determined by the average energy, which is in turn related to the power carried by the signal. Find the relation between T and P . Find a bound on the channel capacity at fixed average energy. (In this part of the problem you may take the infinite-volume limit.)

Inevitably there will be noise, represented by an additional number of photons $\bar{n}(\omega)$ at each frequency which are out of our control. Assume the noise is thermal, in equilibrium at temperature T_N . Suppose the power of the *signal* P (which is some amount of extra photons on top of the noise) is still fixed.

- (f) Convince yourself that the upper bound on the channel capacity is now reduced by the entropy of the noise:

$$C\mathcal{T} \leq S(\rho_{T_{S+N}}) - S(\rho_{T_N})$$

where ρ_T is the thermal density matrix with temperature T , T_N is the noise temperature, and T_{S+N} is the temperature at an average energy which includes both the noise and the signal. Find T_{S+N} in terms of T_N and P .

- (g) Do the integral over frequency. Study the high- and low-temperature limits of your answer. Confirm Landauer's principle in the former case in the following sense: compute the minimum power required to send a single bit.