University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239/139 Spring 2018
Assignment 10 ("Final Exam")

Due 12:30pm Wednesday, June 13, 2018

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1. **Checking the operational interpretation of trace distance.**

   (a) **Warmup.** Show that for two pure states $|1\rangle, |2\rangle$, their trace distance T and their fidelity F satisfy
   $$F^2 + T^2 = 1.$$

   (b) In lecture we proved a result relating the probability of success at distinguishing two states by a single measurement to their trace distance. Might it be possible to evade this theorem by considering POVMs which are not projective measurements?

   Consider two non-orthogonal pure states $|1\rangle, |2\rangle$. with overlap $\delta = |\langle 1|2\rangle|^2$ and consider the POVM made of :
   $$E_1 = \chi |1\rangle \langle 1|, E_2 = \alpha |2\rangle \langle 2|, E_3 = 1 - E_1 - E_2.$$

   For which χ, α is this a POVM?

   Find the probability of success of the strategy: if outcome is 1 guess 1, if outcome is 2 guess 2, if outcome is 3 do a little dance then guess randomly. Show that the bound we proved is not violated.

   (c) Nevertheless, POVMs (which are not projective measurements) are indeed useful for state discrimination. Find a POVM with the property that distinguishes between two non-orthogonal pure states $|1, 2\rangle$ in such a way that for one outcome we are *certain* that the state is $|1\rangle$ and for another we are *certain* that the state is $|2\rangle$. (There is a third outcome where we learn nothing from the measurement.)

2. **Entanglement negativity for pure states.** Show that when $\rho_{AB} = |\psi\rangle\langle\psi|$ is pure, the logarithmic negativity
   $$E_N(\rho_{AB}) \equiv \log \| \rho^{T_A} \|_1$$
   is the Renyi entropy of index 1/2, $S_{1/2}(\rho_A)$, with $S_\alpha(\rho) \equiv \frac{1}{1-\alpha} \log \mathrm{tr} \rho^\alpha$. Use the Schmidt decomposition.

3. **Bekenstein bound.**

In this problem, $\hbar = c = k_B = 1$.

(a) A black hole has a temperature $T_{BH} = \frac{1}{8\pi G_N M}$ and (in Einstein gravity) an entropy $S_{BH} = \frac{A}{4G_N}$, where $A = 4\pi R^2$ is the area of the event horizon, and $R = 2G_N M$ is the Schwarzchild radius. Check that this is consistent with the first law of thermodynamics $dE = TdS$, where $E = M$.

(b) The generalized second law then says that $S_{\text{total}} = S_{BH} + S_{\text{stuff}}$ is non-decreasing. Suppose have an object of linear size R (say it fits in a sphere of radius R) whose energy E and entropy S satisfy $S \overset{?}{>} 2\pi ER$. Then we can cram some extra stuff in there until the object undergoes gravitational collapse and forms a black hole. Convince yourself that this would violate the generalized second law. Thus we arrive at the *Bekenstein bound*, $S \leq 2\pi ER$. Notice that G_N has dropped out of this relation. Indeed a version of it follows simply from positivity of the relative entropy (see below).

(c) [optional bonus part which requires some general relativity] To understand why a black hole has a temperature, notice that near the horizon at $r = 2G_N M$, the Schwarzchild metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \ f(r) = 1 - \frac{2GN_M}{r}$$

looks like

$$ds^2_{\text{Rindler}} = -\kappa^2\rho^2 dt^2 + d\rho^2 + R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{1}$$

where $\rho = \sqrt{r(r - 2G_N M)}$ for a constant κ. Determine κ.

Show that regularity of this geometry in euclidean time $\tau \equiv \mathbf{i}t$ requires periodic euclidean time $\tau \simeq \tau + \beta(\kappa)$. Find $\beta(\kappa)$ and interpret it as an inverse temperature.

Moreover, show that in the coordinates $T = \kappa\rho\sinh\eta, Z = \kappa\rho\cosh\eta$ (with $\eta \equiv \kappa t$), the near-horizon metric (1) is

$$ds^2_{\text{Rindler}} = -dT^2 + dZ^2 + R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

namely $\mathbb{R}^{1,1} \times S^2$. However, only the region $Z > 0$ describes the region outside the horizon. This means that the system outside the horizon must be described by a density matrix which traces out the region $Z < 0$.

Now recall from lecture the Bisognano-Wichmann theorem: in the ground state of a relativistic quantum field theory, the entanglement Hamiltonian for a half-space cut is the boost generator

$$K = 2\pi \int_{x>0} dx x T_{00}.$$

That is, the reduced density matrix $\rho_0 = e^{-K}/\text{tr}e^{-K}$ is a thermal state with Hamiltonian K. Moreover, the Rindler rapidity η is proportional to the asymptotic Minkowski time coordinate t. Check that the temperature obtained this way agrees with the euclidean periodicity argument.

(d) Show that a version of the Bekenstein bound can be obtained from positivity of the relative entropy. More precisely, consider some region of space, and write the reduced density matrix of the vacuum state as $\rho_0 = \frac{e^{-K}}{\text{tr}e^{-K}}$. Show that $0 \leq D(\rho||\rho_0$ can be written as

$$S(\rho) - S(\rho_0) \leq \text{tr}\rho K - \text{tr}\rho_0 K.$$

Interpret the left hand side as the entropy above the vacuum, and the RHS as $(E - E_0)R$ where E_0 is the vacuum energy.

4. **Additivity of squashed entanglement.** [from Preskill]

(a) Use the chain rule for mutual information and the non-negativity of the conditional mutual information to show that

$$I(AA' : BB'|C) \geq I(A : B|C) + I(A' : B'|AC). \tag{2}$$

Conclude that the squashed entanglement is *superadditive*, *i.e.*

$$E_{sq}(AA' : BB') \geq E_{sq}(A : B) + E_{sq}(A' : B').$$

(b) Show that for the special case of product states of the form $\rho_{ABA'B'} = \rho_{AB} \otimes \rho_{A'B'}$, the inequality (2) is saturated:

$$E_{sq}(AA' : BB') \overset{\text{product states}}{=} E_{sq}(A : B) + E_{sq}(A' : B').$$

5. **Literature quest.** [optional] In lecture I mentioned some sufficient conditions (something like additivity, convexity) for an entanglement monotone $E_X(\rho)$ to satisfy

$$E_D(\rho) \leq E_X(\rho) \leq E_F(\rho)$$

where E_D, E_F are the entanglement of distillation and formation respectively. Find the right conditions and a proof that they are sufficient in the literature (or, more ambitiously, find them yourself).

6. **Random singlets.**

Consider qbits arranged on a chain. Suppose that the groundstate is made of random singlets, in the following sense: for a given site i, with probability $f(|i-j|a)$ (a is the lattice spacing), the spins at i and j are in the state $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ (and otherwise are uncorrelated). Consider in turn the case of short-range singlets where $f(x) \propto e^{-x/\xi}$, and long-range singlets where $f(x) \propto \frac{1}{x^2+\delta^2}$.

(a) Consider a region A which is an interval $\left[-\frac{R-\epsilon}{2}, \frac{R-\epsilon}{2}\right]$ ($\epsilon \ll R$) and B is what we called $\bar{A}^-$ (nearly the complement), more precisely: $B \equiv [-\infty, -\frac{R}{2}] \cup [\frac{R}{2}, \infty]$. Let $I_\epsilon(R) \equiv I(A:B) = S(A) + S(B) - S(AB)$ be their mutual information.

Find $\overline{\left\langle \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j \right\rangle}$ (where $\vec{\mathbf{S}} = \frac{1}{2}(\sigma^x, \sigma^y, \sigma^z)$) and $\overline{I_\epsilon(R)}$. In both cases assume the regions are big enough that you can average over regions and use a continuum approximation ($\xi, \delta \gg$ lattice spacing).

Check that the answer is consistent with the mutual information bound on correlations.

(b) [This part is optional since I added it late.]

Consider instead the case where $B = [-\infty, -\frac{R}{2} - L] \cup [\frac{R}{2} + L, \infty]$, so that A and B are separated by a distance L. Show that: for short-range singlets, (i) all (averaged) correlation functions decay exponentially in L (ii) $I(A:B) \sim e^{-L/\xi}$ for large L (and hence the mutual information satisfies an area law). For long-range singlets (i) (averaged) correlation functions have power law decay (ii) $I(A:B) \sim \log(2R-L)$ for large L, and there is no area law.