University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 1 – Solutions

Due 12:30pm Monday, April 8, 2019

Some of you will have seen some of these problems in Winter 2018 215B. Please do the parts you didn't do then.

1. **Brain-warmer: chiral anomaly in two dimensions.**

   Consider a massive relativistic Dirac fermion in 1+1 dimensions, with

   $$S = \int \mathrm{d}x\mathrm{d}t\bar{\psi}\left(\mathbf{i}\gamma^\mu\left(\partial_\mu + eA_\mu\right) - m\right)\psi.$$

   By heat-kernel regularization of its expectation value, show that the divergence of the axial current $j^5_\mu \equiv \mathbf{i}\bar{\psi}\gamma_\mu\gamma^5\psi$ is

   $$\partial_\mu j^5_\mu = 2\mathbf{i}m\bar{\psi}\gamma^5\psi + \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}.$$

   The calculation follows very closely the one in the lecture notes. There are two new ingredients: the mass, and the change to $D = 2$. We know that classically the mass contributes to a violation of the axial current (in any even dimension). From the derivation in terms of the path integral measure we can see that the effects of this and the anomaly contribute additively to the divergence of j^5_μ.

   Again we expand the exponent using $(\mathbf{i}\not{D})^2 = D^2 + \frac{1}{2}i\Sigma^{\mu\nu}F_{\mu\nu}$. In $D = 2$, we use the fact that $\mathrm{tr}\gamma^5\Sigma^{\mu\nu} = 2\epsilon^{\mu\nu}$ (check by picking a basis or using the Clifford algebra directly) to find that the contribution from the anomaly is:

   $$\partial^\mu\left\langle j^5_\mu\right\rangle = s\mathrm{tr}\gamma^5\Sigma^{\mu\nu}F_{\mu\nu}\underbrace{\int đ^2k e^{-sk^2}}_{=\frac{1}{4\pi s}} + \mathcal{O}(s) \overset{s\to 0}{=} \frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}.$$

2. **Where to find a Chern-Simons term.**

   Consider a field theory in $D = 2+1$ of a massive Dirac fermion, coupled to a background $\mathsf{U}(1)$ gauge field A:

   $$S[\psi, A] = \int d^3x\bar{\psi}\left(\mathbf{i}\not{D} - m\right)\psi$$

   where $D_\mu = \partial_\mu - \mathbf{i}A_\mu$.

(a) Convince yourself that the mass term for the Dirac fermion in $D = 2 + 1$ breaks parity symmetry. That is, parity takes $m \to -m$. (Note that the definition of a parity transformation in d spatial dimensions is an element of $\mathsf{O}(d, 1)$ that's not in $\mathsf{SO}(d, 1)$, *i.e.* one with $\det(g) = -1$.)

First: the definition of parity is an element of $\mathsf{O}(d, 1)$ that's not in $\mathsf{SO}(d, 1)$, *i.e.* one with $\det(g) = -1$. In three spatial dimensions this is accomplished by $(t, \vec{x}) \to (t, -\vec{x})$. But in two spatial dimensions, the analogous transformation has only two minus signs and so has determinant one – it is just a π rotation. (Certainly $\bar{\psi}\psi$ is invariant under it. And in fact Peskin's argument for the transformation of the Dirac field goes through exactly – it picks up a γ^0.) Instead we must do something like $(t, x, y) \to (t, x, -y)$ (other transformations are related by composing with a rotation).

Now we must figure out what this does to the Dirac spinor. First recall that the clifford algebra in $D = 2 + 1$ can be represented by 2×2 matrices (*e.g.* the Paulis, times some factors of $\mathbf{i}$ to get the squares right) and there is no notion of chirality, since the product of the three Paulis is proportional to the identity. We want an operation on $\psi(t, x, -y)$ which gives back the (massless) Dirac equation:

$$0 = \left(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^2\partial_y\right)\psi(t, x, -y) = \left(\gamma^0\partial_t + \gamma^1\partial_x - \gamma^2\partial_{\tilde{y}}\right)\psi(t, x, \tilde{y})$$

with $\tilde{y} \equiv -y$. Inserting $1 = -\gamma_2^2$ before ψ we have

$$0 = \left(\gamma^0\partial_t + \gamma^1\partial_x - \gamma^2\partial_{\tilde{y}}\right)\left(-\gamma_2^2\right)\psi(t, x, \tilde{y}) = \gamma_2\left(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^2\partial_{\tilde{y}}\right)\gamma_2\psi(t, x, \tilde{y})$$

which is proportional to $\tilde{\partial\!\!\!/}\gamma^2\psi(\tilde{x}) = 0$. We conclude that $P\psi(t, x, y)P = \gamma^2\psi(t, x, -y)$ will work (up to a sign).

This gives $\bar{\psi}\psi \mapsto \left(\psi^\dagger\gamma^{2\dagger}\right)\gamma^0\gamma^2\psi = \bar{\psi}\left(\gamma^2\right)^2\psi = -\bar{\psi}\psi$, while $\bar{\psi}D\!\!\!\!/\,\psi \to \bar{\psi}D\!\!\!\!/\,\psi$.
Here we used $\left(\gamma^\mu\right)^\dagger\gamma^0 = \gamma^0\gamma^\mu$, and $A_\mu(t, x, y) \to (A_0(t, x, -y), A_x(t, x, -y), -A_y(t, x, -y))_\mu$.

(b) We would like to study the effective action for the gauge field that results from integrating out the fermion field

$$e^{-S_{eff}[A]} = \int [D\psi] e^{-S[\psi, A]}.$$

Focus on the term quadratic in A:

$$S_{eff}[A] = \int d^D q A_\mu(q)\Pi^{\mu\nu}(q)A_\nu(-q) + \ldots$$

We can compute $\Pi^{\mu\nu}$ by Feynman diagrams. Convince yourself that Π comes from a single loop of ψ with two A insertions.

(c) Evaluate this diagram using dim reg near $D = 3$. Show that, in the low-energy limit $q \ll m$ (where we can't make on-shell fermions),

$$\Pi^{\mu\nu} = a\frac{m}{|m|}\epsilon^{\mu\nu\rho}q_\rho + \ldots$$

for some constant a. Find a. Convince yourself that in position space this is a Chern-Simons term with level $k = \frac{1}{2}\frac{m}{|m|}$.

The key ingredient is that in $D = 3$ we have $\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho = -2\epsilon^{\mu\nu\rho}$. Note that this would have been zero in $D = 4$, as in Peskin's calculation on page 247-248.

Clearly this shows that the mass term is odd under parity, since the Chern-Simons term it generates is proportional to sign(m).

(d) [bonus] Redo this calculation by doing the Gaussian path integral over ψ.

Roughly:

$$\int [D\psi D\bar{\psi}] e^{S[\psi,\bar{\psi},A]} = \det\left(\mathbf{i}\not{D} - m\right) = e^{\mathrm{tr}\log\left(\mathbf{i}\not{D}-m\right)}.$$

Therefore

$$S_{\text{eff}}[A] = \mathrm{tr}\log\left(\mathbf{i}\not{\partial} - \not{A} - m\right) = \mathrm{tr}\log\left(\mathbf{i}\not{\partial} - m\right)\left(1 + \not{A}\left(\mathbf{i}\not{\partial} - m\right)^{-1}\right).$$

We need to expand this in A to second order to get Π, and the result is

$$S_{\text{eff}}[A] = \ldots + \int d^3x \, \langle x| \not{A}\left(\mathbf{i}\not{\partial} - m\right)^{-1} \not{A}\left(\mathbf{i}\not{\partial} - m\right)^{-1}|x\rangle \tag{1}$$

$$= \ldots + \int d^3x \int đ^d q e^{-\mathbf{i}qx} A_\mu(q) A_\nu(-q) \int đ^d p \, \mathrm{tr}\left(\gamma^\mu \frac{1}{\not{p} - m}\gamma^\nu \frac{1}{\not{p} - \not{q} - m}\right) \tag{2}$$

which is the same as the diagrammatic calculation above.

3. **A bit more about Chern-Simons theory.**

Consider again $\mathsf{U}(1)$ gauge theory in $D = 2+1$ dimensions with the Chern-Simons action

$$S[a] = \frac{k}{4\pi}\int_\Sigma a \wedge da.$$

(Here I've changed the name of the dynamical gauge field to a lowercase a to distinguish it from the electromagnetic field A which will appear anon.)

(a) Show that the Chern-Simons action is gauge invariant under $a \to a + d\lambda$, as long as there is no boundary of spacetime Σ. Compute the variation of the action in the presence of a boundary of Σ.

(b) [bonus] Actually, the situation is a bit more subtle than the previous part suggests. The actual gauge transformation is

$$a \to g^{-1} a g + \frac{1}{\mathbf{i}} g^{-1} dg$$

which reduces to the previous if we set $g = e^{\mathbf{i}\lambda}$. That expression, however, ignores the global structure of the gauge group (*e.g.* in the abelian case, the fact that g is a periodic function). Consider the case where spacetime is $\Sigma = S^1 \times S^2$, and consider a *large gauge transformation*:

$$g = e^{\mathbf{i}n\theta}$$

where θ is the coordinate on the circle. Show that the variation of the CS term is $\frac{k}{4\pi} \int g^{-1} \partial g \wedge f$ (where $f = da$). Since the action appears in the path integral in the form $e^{\mathbf{i}S}$, convince yourself that the path integrand is gauge invariant if

(1) $\int_\Gamma f \in 2\pi\mathbb{Z}$ for all closed 2-surfaces Γ in spacetime, and

(2) $k \in \mathbb{Z}$.

The first condition is called flux quantization, and is closely related to Dirac's condition.

(c) [bonus] In the case where G is a non-abelian lie group, the argument for quantization of the level (k) is more straightforward. Show that the variation of the CS Lagrangian

$$\mathcal{L}_{CS} = \frac{k}{4\pi} \mathrm{tr} \left(a \wedge da + \frac{2}{3} a \wedge a \wedge a \right)$$

under $a \to gag^{-1} - \partial g g^{-1}$ is

$$\mathcal{L}_{CS} \to \mathcal{L}_{CS} + \frac{k}{4\pi} d\mathrm{tr} dg g^{-1} \wedge a + \frac{k}{12\pi} \mathrm{tr} \left(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \right).$$

The integral of the second term over any closed surface is an integer. Conclude that $e^{\mathbf{i}S_{CS}}$ is gauge invariant if $k \in \mathbb{Z}$.

The first term integrates to zero on a closed manifold. The second term is the winding number of the map $g : \Sigma \to \mathsf{G}$

(d) If there is a boundary of spacetime, something must be done to fix up this problem. Consider the case where $\Sigma = \mathbb{R} \times \mathrm{UHP}$ where $\mathbb{R}$ is the time direction, and UHP is the upper half-plane $y > 0$. One way to fix the problem is simply to declare that the would-be gauge transformations which do not vanish at $y = 0$ are not redundancies. This means that they represent physical degrees of freedom.

The exterior derivative on this spacetime decomposes into $d = \partial_t dt + \tilde{d}$ where $\tilde{d}$ is just the spatial part, and similarly the gauge field is $a = a_0 dt + \tilde{a}$.

Let us choose the gauge $a_0 = 0$. We must still impose the equations of motion for a_0 (in the path integral it is a Lagrange multiplier). Solve this equation, and evaluate the action for the resulting solution.

We must still impose the equations of motion for a_0 (in the path integral it is a Lagrange multiplier) which says $\tilde{d}\tilde{a} = 0$ (just the spatial part). (If you took seriously the boundary terms in the variation of the action, you would also conclude that $\tilde{a}|_{\partial\Sigma} = 0$, but this conclusion can be modified by adding boundary terms to the action.) This equation is solved by $\tilde{a} = \tilde{d}\phi$ (or rather $\tilde{a} = g^{-1}dg$ where g is a $\mathsf{U}(1)$-valued function). This is pure gauge except at the boundary. Plugging this into the CS term gives

$$\begin{aligned} S &= \frac{k}{4\pi}\int_{\mathbb{R}\times D} \tilde{a} \wedge \left(dt\partial_t + \tilde{d}\right)\tilde{a} && (3) \\ &= \frac{k}{4\pi}\int_{\mathbb{R}\times D} \tilde{d}\phi \wedge dt\partial_t \tilde{d}\phi && (4) \\ &= \frac{k}{4\pi}\int_{\mathbb{R}\times D} \tilde{d}\left(\phi \wedge dt\partial_t \tilde{d}\phi\right) && (5) \\ &\overset{\text{Stokes}}{=} \frac{k}{4\pi}\int_{\mathbb{R}\times \partial D} \phi dt\partial_t \tilde{d}\phi && (6) \\ &= \frac{k}{4\pi}\int_{\mathbb{R}\times \partial D} dx dt \phi \partial_t \partial_x \phi && (7) \\ &\overset{\text{IBP}}{=} -\int_{\mathbb{R}\times \partial D} dx dt \partial_x \phi \partial_t \phi. && (8) \end{aligned}$$

We can also add local terms at the boundary to the action. Consider adding $\Delta S = g\int_{\partial\Sigma} \tilde{a}_x^2$ (for some coupling constant g). In the presence of such a boundary term, find the equations of motion for the boundary degrees of freedom.

This term evaluates to $\Delta S = \int_{\partial\Sigma} v\left(\partial_x\phi\right)^2$. Altogether we now have

$$S_{\text{edge}}[\phi] = \int_{y=0} dx dt \partial_x \phi \left(\frac{k}{4\pi}\partial_t\phi + g\partial_x\phi\right).$$

The EoM is then

$$\frac{\delta}{\delta\phi(x)} S_{\text{edge}}[\phi] \propto \partial_x\left(\frac{k}{4\pi}\partial_t\phi + g\partial_x\phi\right)$$

which is solved if $\frac{k}{4\pi}\partial_t\phi + g\partial_x\phi = 0$. This describes a dispersionless wave which moves only in the $\text{sign}(k)$ direction – a chiral bosonic edge mode.

I should mention that this physics is realized in integer quantum Hall states and incompressible fractional quantum Hall states. For more, I recommend the textbook by Xiao-Gang Wen.

Interpretation: the Chern-Simons theory on a space with boundary necessarily produces a chiral edge mode.

This was pointed out by Witten here (in §5).

(e) Suppose we had a system in $2+1$ dimensions with a gap to all excitations, which breaks parity symmetry and time-reversal invariance, and involves a conserved current J^μ, with

$$0 = \partial^\mu J_\mu. \tag{9}$$

Solve this equation by writing $J^\mu = \epsilon^{\mu\nu\rho}\partial_\nu a_\rho$ in terms of a one-form $a = a_\mu dx^\mu$. Guess the leading terms in the action for a_μ in a derivative expansion.

Well, the CS term has dimension 3 so is marginal. It has just the right symmetries. We can also add a Maxwell term, but that has dimension 4 so we can ignore it at low energies.

(f) Now suppose the current J^μ is coupled to an external electromagnetic field A_μ by $S \ni \int J^\mu A_\mu$. Ignoring the Maxwell term for a, compute the Hall conductivity, σ^{xy}, which is defined by Ohm's law $J^x = \sigma^{xy} E^y$.

Using the action

$$S[a, A] = \int \left(\frac{k}{4\pi} a \wedge da + J^\mu A_\mu \right) = \int d^3x \frac{k}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \epsilon^{\mu\nu\rho}\partial_\nu a_\rho A_\mu$$

we find the EoM

$$0 = \frac{\delta S}{\delta a} \propto \frac{k}{2\pi} f^{\mu\nu} + F^{\mu\nu}.$$

Using $J = \star da$ we can rewrite this as

$$F^{\mu\nu} = \frac{k}{2\pi} \epsilon^{\mu\nu\rho} J_\rho.$$

The components of this equation with $\mu, \nu = 0, i)$ say $E^i = \frac{k}{2\pi}\epsilon^{ij} J_j$ or

$$J_j = \frac{2\pi}{k} \epsilon_{ij} E^j$$

which says $\sigma^{xy} = \frac{4\pi}{k}$ (in natural units, which means $\sigma^{xy} = \frac{1}{k}\frac{e^2}{h}$).

4. **An application of the anomaly to a theory without gauge fields.**

Consider a 1+1d theory of Dirac fermions coupled to a background scalar field θ as follows:

$$\mathcal{L} = \bar{\Psi} \left(\mathbf{i} \partial\!\!\!/ + m e^{\mathbf{i}\theta\gamma^5} \right) \Psi.$$

We wish to ask: if we subject the fermion to various configurations of $\theta(x)$ (such as a domain wall where $\theta(x) = \pi + \theta(x)$) what does the fermion number do in the groundstate?

(a) Convince yourself that when θ is constant

$$\langle j^\mu \rangle = 0$$

where $j^\mu = \bar{\Psi}\gamma^\mu \Psi$ is the fermion number current.

Lorentz invariance forbids an expectation value for a vector quantity.

(b) Minimally couple the fermion to a *background* gauge field A_μ. Let $e^{\mathbf{i}\Gamma[A,\theta]} = \int [d\Psi] e^{\mathbf{i}S}$. Convince yourself that the term linear in A in $\Gamma[A,\theta] = \text{const} + \int A_\mu J^\mu + \mathcal{O}(A^2)$ is the vacuum expectation value of the current $\langle j^\mu \rangle = J^\mu$.

A is a source for j in the path integral:

$$\frac{1}{\mathbf{i}} \frac{\delta}{\delta A_\mu(x)} \log Z[A] = Z^{-1} \int D\psi j^\mu(x) e^{\mathbf{i}S} = \langle j^\mu(x) \rangle .$$

(c) Show that by a local chiral transformation $\Psi \to e^{\mathbf{i}\theta(x)\gamma^5/2}\Psi$ we can remove the dependence on θ from the mass term.

(d) Where does the theta-dependence go? Use the 2d chiral anomaly to relate $\langle j^\mu \rangle$ to $\partial\theta$. Notice that the result is independent of m. [This relation was found by Goldstone and Wilczek. The associated physics is realized in Polyacetylene.]

$$\mathbf{i}\Gamma[A,\theta] = \operatorname{tr} \log \left(\mathbf{i}\not{D} + m e^{\mathbf{i}\theta} \right) = \mathbf{i} \int A_\mu J^\mu + \mathcal{O}(A^2).$$

It looks challenging to evaluate this determinant. But we've already done the necessary work in studying the chiral anomaly. The variation of the effective action under a (can be local!) chiral rotation by angle $\theta(x)$ is

$$\delta\Gamma = \int d^2x \theta(x) \frac{F_{\mu\nu}\epsilon^{\mu\nu}}{2\pi}.$$

Since we showed $\Gamma[\theta = \text{constant}] = 0$, the anomaly is the whole thing:

$$\Gamma[A,\theta] = \int d^2x \theta(x) \frac{F_{\mu\nu}\epsilon^{\mu\nu}}{2\pi} \overset{\text{IBP}}{=} - \int d^2x \frac{\partial_\mu\theta\epsilon^{\mu\nu}}{2\pi} A_\nu$$

Therefore

$$\langle j^\mu \rangle = -\frac{\partial_\mu\theta\epsilon^{\mu\nu}}{2\pi}.$$

(e) Show that a domain wall where θ jumps from 0 to π localizes *fractional* fermion number.

The charge on the domain wall is

$$Q = \int_{-\epsilon}^{\epsilon} dx j^0 = \int_{-\epsilon}^{\epsilon} dx \frac{\partial_x \theta}{2\pi} = \frac{1}{2\pi}\left(\theta(+\epsilon) - \theta(-\epsilon)\right) = \frac{1}{2}.$$

(f) [bonus problem] Consider the Dirac hamiltonian in the presence of such a soliton. Show that there is a localized mode of zero energy.

In the basis for the gamma matrices where

$$\gamma^0 = \sigma^1, \gamma^1 = \mathbf{i}\sigma^2, \gamma^5 \equiv \gamma^0\gamma^1 = -\sigma^3,$$

the $D = 2$ Dirac Hamiltonian is

$$H = \gamma^0(\mathbf{i}\vec{\gamma}\cdot\vec{\nabla} + m(x)) = -\sigma^3 \mathbf{i}\partial_x + \sigma^1 m(x), \quad m(x) = me^{\mathbf{i}\theta(x)\gamma^5} = \cos\theta - \mathbf{i}\sigma^3 \sin\theta$$

so

$$H = -\sigma^3 \mathbf{i}\partial_x + m(\sigma^1 \cos\theta(x) - \sigma^2 \sin\theta(x)).$$

The condition for a zero-mode is

$$0 = H\psi = \begin{pmatrix} -\mathbf{i}\partial_x & e^{\mathbf{i}\theta} \\ e^{-\mathbf{i}\theta} & \mathbf{i}\partial_x \end{pmatrix} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix},$$

with ψ normalizable. If $\theta(x) = \begin{cases} 0, x < 0 \\ \pi, x > 0 \end{cases}$, then $m(x) = \begin{cases} m, x < 0 \\ -m, x > 0 \end{cases}$.

Let's expand ψ in eigenstates of σ^2 ($\psi_\uparrow = \pm\mathbf{i}\psi_\downarrow$); since $\{H, \sigma^2\} = 0$, the terms don't mix, and we find

$$0 = -\mathbf{i}\partial_x\left(\pm\mathbf{i}\psi_\downarrow\right) + m(x)\psi_\downarrow = \left(\pm\partial_x + m(x)\right)\psi_\downarrow \implies \psi_\downarrow(x) = \psi_\downarrow(0)e^{\mp\int_0^x dx' m(x')}.$$

Only one of the two choices (the lower sign) is normalizable far from the domain wall (on both sides), which gives the localized zero-energy mode

$$\psi(x) = \psi_\downarrow(0)e^{-m|x|}\begin{pmatrix} -\mathbf{i} \\ 1 \end{pmatrix}.$$