University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 2

Due 12:30pm Monday, April 15, 2019

1. **Emergence of the Dirac equation.** Consider a chain of free fermions with

$$H = -t\sum_n c_n^\dagger c_{n+1} + h.c.$$

(a) Show that the low-energy excitations at a generic value of the filling are described the the massless Dirac lagrangian in 1+1 dimensions. Find an explicit choice of 1+1-d gamma matrices which matches the answer from the lattice model. Show that the right-movers are right-handed $\gamma^5 \equiv \gamma^0\gamma^1 = 1$ and the left-movers are left-handed.

[I am adding the rest of this problem on Friday April 12, so it is all bonus material. It will carry over to HW3.]

On the previous problem set problem 4, you may have wondered what is the connection between the field theory we were studying (a scalar coupled to fermions in $D = 2$) and polyacetylene. I'd like to explain that connection a bit.

Consider an extension of the model above to include also *phonon* modes, *i.e.* degrees of freedom encoding the positions of the ions in the solid. (Again we ignore the spins of the electrons for simplicity.)

$$H = -t\sum_n (1+u_n)c_n^\dagger c_{n+1} + h.c. + \sum_n K(u_n - u_{n+1})^2 \equiv H_F + H_E.$$

Here u_n is the deviation of the nth ion from its equilibrium position (in the $+x$ direction), so the second term represents an elastic energy.

(b) Consider a configuration

$$u_n = \phi(-1)^n \tag{1}$$

where the ions move closer in pairs. Compute the electronic spectrum. (Hint: this enlarges the unit cell. Define $c_{2n} \equiv a_n, c_{2n+1} \equiv b_n$, and solve in Fourier space, $a_n \equiv \oint đk e^{2\mathbf{i}kn} a_k$ *etc.*) You should find that when $\phi \neq 0$ there is a gap in the electron spectrum (unlike $\phi = 0$). Expand the spectrum near the minimum gap and include the effects of the field ϕ in the continuum theory.

(c) **Peierls' instability.** Compute the groundstate energy of the electrons H_F in the configuration (1), at half-filling (*i.e.* the number of electrons is half the number of available states). Check that you recover the previous answer when $\phi = 0$. Interpret the answer when $\phi = 1$.

Compute H_E in this configuration, and minimize the sum of the two as a function of ϕ.

(d) You should find that the energy is independent of the *sign* of ϕ. This means that there are two groundstates. We can consider a domain wall between a region of $+$ and a region of $-$. Show that this domain wall carries a fermion mode whose energy lies in the bandgap and which carries fermion number $\frac{1}{2}$.

(e) Verify the result of the previous part by diagonalizing the relevant tight-binding matrix.

(f) Time-reversal played an important role here. If we allow complex hopping amplitudes, we can make a domain wall without midgap modes. Explain this from field theory. Bonus: explain this from the lattice hamiltonian.

2. **An application of effective field theory in quantum mechanics.**

Consider a model of two canonical quantum variables ($[\mathbf{x}, \mathbf{p}_x] = \mathbf{i} = [\mathbf{y}, \mathbf{p}_y], 0 = [\mathbf{x}, \mathbf{p}_y] = [\mathbf{x}, \mathbf{y}]$, etc) with Hamiltonian

$$\mathbf{H} = \mathbf{p}_x^2 + \mathbf{p}_y^2 + \lambda \mathbf{x}^2 \mathbf{y}^2.$$

(This is similar to the degenerate limit of the model studied in lecture with two QM variables where both natural frequencies are taken to zero.)

(a) Based on a semiclassical analysis, would you think that the spectrum is discrete?

(b) Study large, fixed x near $y = 0$. We will treat x as the slow (= low-energy) variable, while y gets a large restoring force from the background x value. Solve the y dynamics, and find the groundstate energy as a function of x:

$$V_{\text{eff}}(x) = E_{\text{g.s. of y}}(x).$$

(c) The result is not analytic in x at $x = 0$. Why?

(d) Is the spectrum of the resulting 1d model with

$$\mathbf{H}_{\text{eff}} = \mathbf{p}_x^2 + V_{\text{eff}}(\mathbf{x})$$

discrete? Is this description valid in the regime which matters for the semi-classical analysis?

[Bonus: determine the spectrum of $\mathbf{H}_{\text{eff}}$.]

3. **Matching with massive electrons.**

Consider QED in the regime where the photon momenta q^μ are much smaller than the electron mass m_e. In this regime, we can integrate out the electron and write an effective field theory involving only the photon.

(a) Calculate the QED one-loop vacuum polarization diagram using dimensional regularization, in the $\overline{\text{MS}}$ scheme. (You've done this before. Here's the new ingredient:) Expand $\Pi(q^2)$ *through* first order in $\frac{q^2}{m_e^2}$.

(b) Write down a Lagrangian involving only the photon field operator that reproduces the first two terms in the expansion. (Hint: it should be gauge invariant and Lorentz invariant. There is essentially (up to integration by parts) only one addition to the Maxwell term). Use the calculation above to match between the photon-only EFT and QED at $\mu = m_e$, at this order in the fine structure constant α.

(c) What symmetry of QED forbids dimension-6 operators involving three field strengths?

(d) At dimension 8, there are operators in the photons-only EFT which describe light-by-light scattering. Write them down (there are two). Draw the QED Feynman diagram which matches to these terms, and determine the number of factors of α in their coefficients. (Don't do the integrals unless you find it enjoyable.)

(e) [bonus] Use dimensional analysis in the low-energy EFT to estimate the size of the $\gamma\gamma \to \gamma\gamma$ cross section.