University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 3

Due 12:30pm Monday, April 22, 2019

1. **Anomaly cancellation in the Standard Model.** If we try to gauge a chiral symmetry (such as hypercharge in the Standard Model (SM)), it is important that it is actually a symmetry, *i.e.* is not anomalous. In $D = 3 + 1$, a possible anomaly is associated with a choice of three currents, out of which to make a triangle diagram. We'll call a "$\mathsf{G}_1\mathsf{G}_2\mathsf{G}_3$ anomaly" the diagram with insertions of currents for $\mathsf{G}_1, \mathsf{G}_2$ and G_3. Generalizing a little, we showed that the divergence of the current for G_1 is

$$\partial_\mu j_1^{A\mu} = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F^{2B}_{\mu\nu}F^{3C}_{\rho\sigma}\sum_f (-1)^f \mathrm{tr}_{R(f)}\{T_1^A, T_2^B\}T_3^C .$$

The sum is over each Weyl fermion, $R(f)$ is its representation under the combined group $\mathsf{G}_1 \times \mathsf{G}_2 \times \mathsf{G}_3$, and T_1^A are a basis of generators of the Lie algebra of G_1 etc. in the representation of the field f. By $(-1)^f$ I mean $\pm$ for left- and right-handed fermions respectively.

We consider the possibilities in turn.

   (a) Convince yourself that the divergence of the $\mathsf{U}(1)_Y$ hypercharge current gets a contribution of the form

$$\partial_\mu J^\mu_Y = \left(\sum_{\text{left}} Y_l^3 - \sum_{\text{right}} Y_r^3\right)\frac{g'^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}B_{\rho\sigma}$$

   from the triangle with three insertions of the current itself (here B is the hypercharge gauge field strength). The sum on the RHS is over all left- and right-handed Weyl spinors weighted by the cube of their hypercharge. Check that this sum evaluates to zero in the SM.

   (b) Show that any anomaly of the form $\mathsf{SU}(N)\mathsf{U}(1)^2$ or $\mathsf{SU}(N)G_1G_2$ is zero.

   (c) (Easy) Convince yourself that there is no $\mathsf{SU}(3)^3$ anomaly for QCD.

   (d) Check that there is never an $\mathsf{SU}(2)^3$ anomaly. (Hint: the generators satisfy $\{\tau^a, \tau^b\} = 2\delta^{ab}$.)

   (e) Show that the $\mathsf{SU}(3)^2\mathsf{U}(1)_Y$ anomaly demands that $2Y_Q - Y_u - Y_d = 0$. Check that this is true in the SM.

(f) Show that a necessary condition for hypercharge to not have an anomaly with the Electroweak gauge bosons on the RHS is $Y_L + 3Y_Q = 0$, where Y_L and Y_Q are the hypercharges of the left-handed leptons and quarks. Check that this works out in the SM.

(g) There is another kind of anomaly called a *gravitational anomaly.* This is a violation of current conservation in response to coupling to curved space. An example is of the form

$$\partial_\mu j_Y^\mu = a \mathrm{tr} \mathcal{R} \wedge \mathcal{R}$$

where $\mathcal{R}$ is a two-form related to the curvature of spacetime (analogous to the field strength F). The coefficient a is proportional to $\sum_{\text{left}} \mathrm{tr} Y_l - \sum_{\text{right}} \mathrm{tr} Y_r$. Check that this too vanishes for hypercharge in the Standard Model.

These conditions, plus the assumption that the right-handed neutrino is neutral, actually determine all the hypercharge assignments.

2. **Right-handed neutrinos.**

Consider adding a right-handed singlet (under all gauge groups) neutrino N_R to the Standard Model. It may have a majorana mass M; and it may have a coupling g_ν to leptons, so that all the dimension ≤ 4 operators are

$$\mathcal{L}_N = \bar{N}_R \mathbf{i} \partial\!\!\!/ N_R - \frac{M}{2} \bar{N}_R^c N_R - \frac{M}{2} \bar{N}_R N_R^c + \left(g_\nu \bar{N}_R H_i^T L_j \epsilon^{ij} + h.c. \right)$$

where $N_R^c = C \left(\bar{N}_R \right)^T$ is the the charge conjugate field, $C = \mathbf{i} \gamma_2 \gamma_0$ (in the Dirac representation), H is the Higgs doublet, L is the left-handed lepton doublet, containing ν_L and e_L. Take the mass M to be large compared to the electroweak scale. Integrate out the right-handed neutrinos at tree level. [Hint: you may find it useful to work in terms of the Majorana field

$$N \equiv N_R + N_R^c$$

which satisfies $N = N^c$.]

Show that the leading term in the expansion in $1/M$ is a dimension-5 operator made of Standard Model fields. Explain the consequences of this operator for neutrino physics, assuming a vacuum expectation value for the Higgs field.

Place a bound on M assuming that the observed neutrinos have masses $m_\nu < 0.5$ eV.

3. **Gross-Neveu model.**

   Here's an example which illustrates the manipulations we did in describing the BCS phenomenon. Now that we've learned about fermionic path integrals, consider the partition function for an N-vector of fermionic spinor fields in D dimensions:

$$Z = \int [d\psi d\bar{\psi}] e^{\mathbf{i}S[\psi]}, \quad S[\vec{\psi}] = \int \mathrm{d}^D x \left(\bar{\psi}^a \mathbf{i} \partial\!\!\!/ \psi^a - \frac{g}{N} \left(\bar{\psi}^a \psi^a \right)^2 \right).$$

   (a) At the free fixed point, what is the dimension of the coupling g as a function of the number of spacetime dimensions D? Show that it is classically marginal in $D = 2$, so that this action is (classically) scale invariant.

   (b) We will show that this model in $D = 2$ exhibits dimensional transmutation in the form of a dynamically generated mass gap. Here are the steps: first use the Hubbard-Stratonovich trick to replace ψ^4 by $\sigma\psi^2 + \sigma^2$ in the action, where σ is a scalar field. Then integrate out the ψ fields. Find the saddle point equation for σ; argue that the saddle point dominates the integral for large N. Find a translation invariant saddle point. Plug the saddle point configuration of σ back into the action for ψ and describe the resulting dynamics.

4. **Polyacetylene returns.**

   On HW01 problem 4, you may have wondered what is the connection between the field theory we were studying (a scalar coupled to fermions in $D = 2$) and polyacetylene. I'd like to explain that connection a bit.

   Consider an extension of the model above to include also *phonon* modes, *i.e.* degrees of freedom encoding the positions of the ions in the solid. (Again we ignore the spins of the electrons for simplicity.)

$$H = -t \sum_n (1 + u_n) c_n^\dagger c_{n+1} + h.c. + \sum_n K (u_n - u_{n+1})^2 \equiv H_F + H_E.$$

   Here u_n is the deviation of the nth ion from its equilibrium position (in the $+x$ direction), so the second term represents an elastic energy.

   (a) (The part with the free massless Dirac field you had time to do on HW2.)

   (b) Consider a configuration

$$u_n = \phi(-1)^n \tag{1}$$

   where the ions move closer in pairs. Compute the electronic spectrum. (Hint: this enlarges the unit cell. Define $c_{2n} \equiv a_n, c_{2n+1} \equiv b_n$, and solve in

Fourier space, $a_n \equiv \oint đk e^{2\mathbf{i}kn} a_k$ *etc.*) You should find that when $\phi \neq 0$ there is a gap in the electron spectrum (unlike $\phi = 0$). Expand the spectrum near the minimum gap and include the effects of the field ϕ in the continuum theory.

(c) **Peierls' instability.** Compute the groundstate energy of the electrons H_F in the configuration (7), at half-filling (*i.e.* the number of electrons is half the number of available states). Check that you recover the previous answer when $\phi = 0$. Interpret the answer when $\phi = 1$.

Compute H_E in this configuration, and minimize the sum of the two as a function of ϕ.

(d) You should find that the energy is independent of the *sign* of ϕ. This means that there are two groundstates. We can consider a domain wall between a region of $+$ and a region of $-$. Show that this domain wall carries a fermion mode whose energy lies in the bandgap and has fermion number $\pm\frac{1}{2}$.

(e) Verify the result of the previous part by diagonalizing the relevant tight-binding matrix.

(f) Time-reversal played an important role here. If we allow complex hopping amplitudes, we can make a domain wall without midgap modes. Explain this from field theory. Bonus: explain this from the lattice hamiltonian.