University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 4 – Solutions

Due 12:30pm Monday, April 29, 2019

1. **Diagrammatic understanding of BCS instability of Fermi liquid theory.**

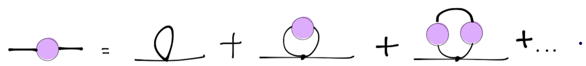
   (a) Recall that only the four-fermion interactions with special kinematics are marginal. Keeping only these interactions, show that cactus diagrams (like this: ) dominate.

   The diagrams which dominate are made of the marginal 4-fermion vertices, which have the momenta equal and opposite in pairs, *i.e.* $V(k_1, k_2, k_3, k_4) = V(k, -k, k', -k')$. This is automatic in cactus diagrams. The model which keeps only these terms is called the *Reduced BCS model.*

   (b) To sum the cacti, we can make bubbles with a corrected propagator. Argue that this correction to the propagator is innocuous and can be ignored.

   These diagrams do not depend on the external momenta. Therefore, they are merely a renormalization of the chemical potential. Fixing the propagator according to the correct particle density therefore removes all effects of these diagrams.

   To resum their effects we use the self-energy with the pink blob which satisfies

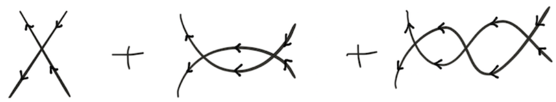
   (c) Armed with these results, compute diagrammatically the Cooper-channel susceptibility (two-particle Green's function),

$$\chi(\omega_0) \equiv \left\langle \mathcal{T} \psi^\dagger_{\vec{k},\omega_3,\downarrow} \psi^\dagger_{-\vec{k},\omega_4,\uparrow} \psi_{\vec{p},\omega_1,\downarrow} \psi_{-\vec{p},\omega_2,\uparrow} \right\rangle$$

   as a function of $\omega_0 \equiv \omega_1 + \omega_2$, the frequencies of the incoming particles. Think of χ as a two point function of the Cooper pair field $\Phi = \epsilon_{\alpha\beta} \psi^\dagger_\alpha \psi_\alpha$ at zero momentum.

   Sum the geometric series in terms of a (one-loop) integral kernel.

$$\chi(\omega_0) = $$



$$+\cdots \tag{1}$$

$$= -\mathbf{i}V + (-\mathbf{i}V)^2\frac{1}{2}\int đ^d k d\epsilon G(\epsilon+\omega_0,\vec{k})G(-\epsilon,-\vec{k}) + (-\mathbf{i}V)^3\left(\frac{1}{2}\right)^2\int GG\int GG + \cdots \tag{2}$$

$$\equiv -\mathbf{i}V\left(1 - \frac{\mathbf{i}}{2}V\int GG + (-\frac{\mathbf{i}}{}V\int GG)^2 + \cdots\right) \tag{3}$$

$$= -\mathbf{i}V\left(1 - \mathcal{I} + \mathcal{I}^2 + \cdots\right) = \frac{-\mathbf{i}V}{1+\mathcal{I}}. \tag{4}$$

The $\frac{1}{2}$ is a symmetry factor.

(d) Do the integrals. In the loops, restrict the range of energies to $|\omega| < E_D$ (or $|\epsilon(k)| < E_D$), the Debye energy, since it is electrons with these energies which experience attractive interactions.

Consider for simplicity a round Fermi surface. For doing integrals of functions singular near a round Fermi surface, make the approximation $\epsilon(k) \simeq v_F(|k| - k_F)$, so that $d^d k \simeq k_F^{d-1}\frac{d\xi}{v_F}d\Omega_{d-1}$.

Now we have to do the integral.

$$\mathcal{I} = \frac{\mathbf{i}}{2}V\int đ^d k d\epsilon G(\epsilon+\omega_0,\vec{k})G(-\epsilon,-\vec{k}) \tag{5}$$

$$= \frac{\mathbf{i}}{2}V\int đ^d k d\epsilon \frac{1}{(\epsilon+\omega_0)(1+\mathbf{i}\eta)-\xi(\vec{k})}\frac{1}{(-\epsilon)(1+\mathbf{i}\eta)-\xi(-\vec{k})} \tag{6}$$

$$= \frac{\mathbf{i}}{2}V\int đ^d k \frac{2\pi\mathbf{i}}{2\pi}(-1)^{\text{sign}(\xi(k))}\frac{1}{\omega_0 - 2\xi(k)} \tag{7}$$

$$= -\frac{V}{2}\int đ^d k (-1)^{\text{sign}(\xi(k))}\frac{1}{\omega_0 - 2\xi(k)} \tag{8}$$

In the third line we assumed parity $\xi(k) = \xi(-k)$, and did the frequency integral by residues, as recommended. The orientation of the contour depends on the sign of $\xi(k)$. Now we use the approximation $d^d k \simeq k_F^{d-1}\frac{d\xi}{v_F}d\Omega_{d-1}$ to write

$$\mathcal{I} = -V \underbrace{\frac{\int đ^{d-1}k}{2v_F}}_{\equiv N} \left(\int_0^{E_D} \frac{d\xi}{\omega_0 - 2\xi} - \int_{-E_D}^0 \frac{d\xi}{\omega_0 - 2\xi} \right) \quad (9)$$

$$= -NV \left(\int_0^{E_D} \frac{d\xi}{\omega_0 - 2\xi} - \int_0^{E_D} \frac{d\xi}{\omega_0 + 2\xi} \right) \quad (10)$$

$$= -NV \left(-\frac{1}{2} \log \frac{\omega_0 - 2E_D}{\omega_0} - \frac{1}{2} \log \frac{\omega_0 + 2E_D}{\omega_0} \right) \quad (11)$$

$$\stackrel{\omega_0 \ll E_D}{\simeq} NV \left(\frac{1}{2} \log \frac{-2E_D}{\omega_0} + \frac{1}{2} \log \frac{+2E_D}{\omega_0} \right) \quad (12)$$

$$= NV \left(\log \frac{2E_D}{\omega_0} + \frac{\mathbf{i}\pi}{2} \right). \quad (13)$$

(e) Show that when $V < 0$ is attractive, $\chi(\omega_0)$ has a pole. Does it represent a bound-state? Interpret this pole in the two-particle Green's function as indicating an instability of the Fermi liquid to superconductivity. Compare the location of the pole to the energy E_{BCS} where the Cooper-channel interaction becomes strong.

The pole occurs at

$$0 = 1 + \mathcal{I} = 1 + NV \left(\log \frac{2E_D}{\omega_0} + \frac{\mathbf{i}\pi}{2} \right)$$

which says

$$\omega_0 = 2\mathbf{i} E_D e^{-\frac{1}{NV}}.$$

Note the crucial factor of $\mathbf{i}$. This says that the pole is in the UHP of the ω_0 plane. The fact that the pole occurs in the UHP of the ω_0 plane means that the Fourier transform of this quantity grows exponentially in time (for short times at least).

(f) **Cooper problem.** [optional] We can compare this result to Cooper's influential analysis of the problem of two electrons interacting with each other in the presence of an inert Fermi sea. Consider a state with two electrons with antipodal momenta and opposite spin

$$|\psi\rangle = \sum_k a_k \psi^\dagger_{k,\uparrow} \psi^\dagger_{-k,\downarrow} |F\rangle$$

where $|F\rangle = \prod_{k<k_F} \psi^\dagger_{k,\uparrow} \psi^\dagger_{k,\downarrow} |0\rangle$ is a filled Fermi sea. Consider the Hamiltonian

$$H = \sum_k \epsilon_k \psi^\dagger_{k,\sigma} \psi_{k,\sigma} + \sum_{k,k'} V_{k,k'} \psi^\dagger_{k,\sigma} \psi_{k,\sigma} \psi^\dagger_{k',\sigma'} \psi_{k',\sigma'}.$$

Write the Schrödinger equation as

$$(\omega - 2\epsilon_k)a_k = \sum_{k'} V_{k,k'} a_{k'}.$$

Now assume (following Cooper) that the potential has the following form:

$$V_{k,k'} = V w^\star_{k'} w_k, \qquad w_k = \begin{cases} 1, & 0 < \epsilon_k < E_D \\ 0, & \text{else} \end{cases}.$$

Defining $C \equiv \sum_k \omega^\star_k a_k$, show that the Schrödinger equation requires

$$1 = V \sum_k \frac{|w_k|^2}{\omega - 2\epsilon_k}. \tag{14}$$

Assuming V is attractive, find a bound state. Compare (1) to the condition for a pole found from the bubble chains above.

This leads to a boundstate at ω such that

$$1 = VN \int_0^{E_D} \frac{d\xi}{\omega - 2\xi} = -\frac{VN}{2} \log\left(\frac{-2E_D}{\omega}\right)$$

which says

$$\omega = -2E_D e^{-\frac{2}{|V|N}}.$$

The Cooper bound-state equation (1) is just what we would get if we left out the contribution of the virtual electrons with $\xi < 0$ – the ones below the Fermi energy (which in fact I did when I was first writing this problem). This results in a factor of two in the exponent (so the Cooper pair binding energy is exponentially larger than the magnitude frequency found above). More importantly it results in a minus sign rather than a factor of $\mathbf{i}$ (a boundstate energy should be negative). Including (correctly) the effects of fluctuations below Fermi sea level changes the boundstate to an instability. I recommend the book by Schrieffer (called *Superconductivity*) for this subject.

2. **Topological terms in QM.** [from Abanov]

The purpose of this problem is to demonstrate that total derivative terms in the action (like the θ term in QCD) do affect the physics.

The euclidean path integral for a particle on a ring with magnetic flux $\theta = \int \vec{B} \cdot d\vec{a}$ through the ring is given by

$$Z = \int [D\phi] e^{-\int_0^\beta d\tau \left(\frac{m}{2}\dot{\phi}^2 - \mathbf{i}\frac{\theta}{2\pi}\dot{\phi}\right)}.$$

Here

$$\phi \equiv \phi + 2\pi \tag{15}$$

is a coordinate on the ring. Because of the identification (2), ϕ need not be a single-valued function of τ – it can wind around the ring. On the other hand, $\dot{\phi}$ is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

$$\phi(\tau) = \frac{2\pi}{\beta} Q\tau + \sum_{\ell \in \mathbb{Z} \setminus 0} \phi_\ell e^{\mathbf{i}\frac{2\pi}{\beta}\ell\tau}. \tag{16}$$

(a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.

(b) Using the decomposition (3), write the partition function as a sum over topological sectors labelled by the *winding number* $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula

$$\sum_{n \in \mathbb{Z}} e^{-\frac{1}{2}tn^2 + \mathbf{i}zn} = \sqrt{\frac{2\pi}{t}} \sum_{\ell \in \mathbb{Z}} e^{-\frac{1}{2t}(z - 2\pi\ell)^2}. \;]$$

I should have mentioned that more generally the Poisson resummation formula says

$$\sum_n f(n) = \sum_l \hat{f}(2\pi l)$$

where $\hat{f}(p) = \int dx e^{-\mathbf{i}px} f(x)$ is the fourier transform of f.

Using the given mode expansion and $\int_0^\beta dt e^{\frac{2\pi \mathbf{i}(l-l')\tau}{\beta}} = \beta\delta_{l,l'}$ the action is

$$S[\phi] = \mathbf{i}\theta Q + \frac{m(2\pi Q)^2}{2\beta} + \sum_{\ell \neq 0} \frac{(2\pi\ell)^2 m}{2\beta} \phi_\ell \phi_{-\ell}$$

where $\phi_\ell = \phi^\star_{-\ell}$. Thus

$$Z = \sum_{Q \in \mathbb{Z}} e^{-\mathbf{i}\theta Q + \frac{m(2\pi Q)^2}{2\beta}} \prod_{\ell \neq 0} \int d^2\phi_\ell e^{\frac{(2\pi\ell)^2 m}{2\beta}\phi_\ell \phi^\star_\ell} \tag{17}$$

$$= \sum_{Q \in \mathbb{Z}} e^{-\mathbf{i}\theta Q + \frac{m(2\pi Q)^2}{2\beta}} \prod_{\ell \neq 0} \left(\frac{\beta}{2\pi\ell^2 m} \right) \tag{18}$$

$$\propto \sum_{n \in \mathbb{Z}} e^{-\beta \frac{1}{2m(2\pi)^2}(\theta - 2\pi n)^2} = \sum_{n \in \mathbb{Z}} e^{-\beta \frac{1}{2m}\left(n - \frac{\theta}{2\pi}\right)^2} \tag{19}$$

where in the last step we used the above Poisson summation formula with $z = \theta$ and $t = \frac{m(2\pi)^2}{\beta}$.

(c) Use the result from the previous part to determine the energy spectrum as a function of θ.

After the Poisson resummation, this is manifestly the partition function of a system with energies $E_n = \frac{1}{2m}(n - \frac{\theta}{2\pi})^2$.

(d) Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

Note that the action given above is the *Euclidean* action. The real time action (from which we should derive the hamiltonian) is

$$S = \int dt \left(\frac{1}{2} m\dot{\phi}^2 + \dot{\phi}\frac{\theta}{2\pi} \right).$$

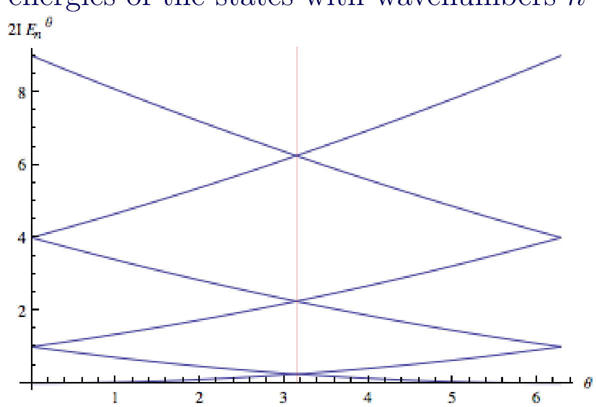
This gives $p = \frac{\partial L}{\partial \dot{\phi}} = m\dot{\phi} + \frac{\theta}{2\pi}$, and hence

$$H = \frac{\left(p - \frac{\theta}{2\pi}\right)^2}{2m}.$$

Now, since $\phi \equiv \phi + 2\pi$, its canonical momentum is quantized, $p \in \mathbb{Z}$, so

$$E_n = \frac{1}{2m}\left(n - \frac{\theta}{2\pi}\right)^2$$

as above. We find the following spectrum for various θ (I am plotting the energies of the states with wavenumbers $n \in [-3, 2]$):



(In the axis label, I is the moment of inertia of the rotor.) Notice that when $\theta = \pi$, the groundstate becomes doubly degenerate.

(e) Consider what happens in the limit $m \to 0, \theta \to \pi$ with $X \equiv \frac{\theta - \pi}{m} \sim \beta^{-1}$ fixed. Interpret the result as the partition function for a spin 1/2 particle. What is the meaning of the ratio X in this interpretation?

In this limit, the higher bands of energies go off to ∞, and we are left with a two-state system. X is a Zeeman field splitting the two states.

3. **Grassmann brain-warmers.**

(a) A useful device is the integral representation of the grassmann delta function. Show that

$$-\int d\bar{\psi}_1 e^{-\bar{\psi}_1(\psi_1-\psi_2)} = \delta(\psi_1 - \psi_2)$$

in the sense that $\int d\psi_1 \delta(\psi_1 - \psi_2) f(\psi_1) = f(\psi_2)$ for any grassmann function f. (Notice that since the grassmann delta function is not even, it matters on which side of the δ we put the function: $\int d\psi_1 f(\psi_1)\delta(\psi_1 - \psi_2) = f(-\psi_2) \neq f(\psi_2)$.)

(b) Recall the resolution of the identity on a single qbit in terms of fermion coherent states

$$\mathbb{1} = \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} \left|\psi\right\rangle \left\langle\bar{\psi}\right| . \tag{20}$$

Show that $\mathbb{1}^2 = \mathbb{1}$. (The previous part may be useful.)

(c) In lecture I claimed that a representation of the trace of a bosonic operator was

$$\text{tr}\mathbf{A} = \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} \left\langle -\bar{\psi}\right| \mathbf{A} \left|\psi\right\rangle \;,$$

and the minus sign in the bra had important consequences.

(Here $\left\langle -\bar{\psi}\right| \mathbf{c}^\dagger = \left\langle -\bar{\psi}\right| (-\bar{\psi})$).

Check that using this expression you get the correct answer for

$$\text{tr}(a + b\mathbf{c}^\dagger\mathbf{c})$$

where a, b are ordinary numbers.

(d) Prove the identity (4) by expanding the coherent states in the number basis.

Using $\left|\psi\right\rangle = \left|0\right\rangle + \psi\left|1\right\rangle, \left\langle -\bar{\psi}\right| = \left\langle 0\right| - \bar{\psi}\left\langle 1\right|$, we have

$$\begin{aligned}\int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} \left|\psi\right\rangle \left\langle\bar{\psi}\right| &= \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} \left(\left|0\right\rangle + \psi\left|1\right\rangle\right) \left(\left\langle 0\right| - \bar{\psi}\left\langle 1\right|\right) \\ &= \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} \left(\left|0\right\rangle\left\langle 0\right| - \psi\bar{\psi}\left|1\right\rangle\left\langle 1\right|\right) \\ &= \left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right| = \mathbb{1}.\end{aligned} \tag{21}$$

4. **Fermionic coherent state exercise.**

Consider a collection of fermionic modes c_i with quadratic hamiltonian $H = \sum_{ij} h_{ij} c_i^\dagger c_j$, with $h = h^\dagger$.

(a) Compute $\mathrm{tr} e^{-\beta H}$ by changing basis to the eigenstates of h_{ij} (the single-particle hamiltonian) and performing the trace in that basis: $\mathrm{tr}\ldots = \prod_\epsilon \sum_{n_\epsilon = c_\epsilon^\dagger c_\epsilon = 0,1} \ldots$.

In the eigenbasis of h_{ij},

$$H = \sum_{ij} h_{ij} c_i^\dagger c_j = \sum_\alpha \epsilon_\alpha c_\alpha^\dagger c_\alpha,$$

the trace factorizes:

$$\mathrm{tr} e^{-\beta H} = \prod_\alpha \sum_{n_\alpha = c_\alpha^\dagger c_\alpha = 0,1} e^{-\beta \epsilon_\alpha n_\alpha} = \prod_\alpha \left(1 + e^{-\beta \epsilon_\alpha}\right) = \det\left(1 + e^{-\beta h}\right).$$

(b) Compute $\mathrm{tr} e^{-\beta H}$ by coherent state path integral. Compare!

In lecture we showed for a single fermionic mode how to write the thermal partition function as a grassmann path integral

$$\mathrm{tr} e^{-\beta H(c^\dagger, c)} = \int [D\psi D\bar{\psi}] e^{-\int_0^\beta d\tau \left(\bar{\psi}\partial_\tau \psi - H(\bar{\psi}, \psi)\right)}$$

as long as H is normal-ordered. Here we just have many copies of that problem:

$$\mathrm{tr} e^{-\beta H(c_i^\dagger, c_j)} = \int \prod_i [D\psi_i D\bar{\psi}_i] e^{-\int_0^\beta d\tau \left(\bar{\psi}_i \partial_\tau \psi_i - h_{ij}\bar{\psi}_i \psi_j\right)}.$$

To do this integral, let's go to frequency space:

$$\psi_i(\tau) = \sum_n e^{-\omega_n \tau} \psi_{ni}, \quad \omega_n = \pi T(2n+1).$$

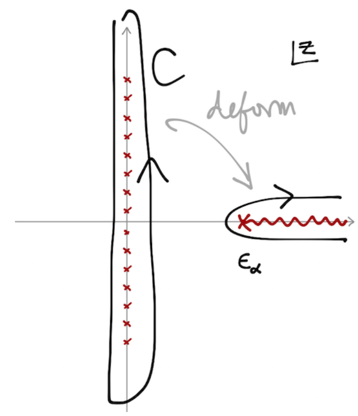
Further, let's change coordinates to diagonalize h, so we have

$$Z = \int \prod_{\alpha,n} d\psi_{\alpha,n} d\bar{\psi}_{\alpha,n} \prod_{\alpha,n} e^{-\bar{\psi}_{\alpha,n}(\mathbf{i}\omega_n - \epsilon_\alpha)\psi_{\alpha,n}} \tag{22}$$

$$= \prod_{\alpha,n} (\mathbf{i}\omega_n - \epsilon_\alpha) = e^{\sum_{\alpha,n} \log(\mathbf{i}\omega_n - \epsilon_\alpha)} \tag{23}$$

So

$$
\begin{aligned}
\log Z &= \sum_{\alpha,n} \log\left(\mathbf{i}\omega_n - \epsilon_\alpha\right) \qquad (24)\\
&= \sum_\alpha \frac{1}{2\pi\mathbf{i}} \oint_C dz \frac{\beta}{e^{\beta z}+1} \log\left(\mathbf{i}\omega_n - \epsilon_\alpha\right)
\end{aligned}
$$



$$
\begin{aligned}
&= \frac{1}{2\pi\mathbf{i}} \sum_\alpha \int_{\epsilon_\alpha}^{\infty} dz \mathrm{disc}\left(\frac{\beta}{e^{\beta z}+1} \log\left(\mathbf{i}\omega_n - \epsilon_\alpha\right)\right)\\
&= \frac{1}{2\pi\mathbf{i}} \sum_\alpha \int_{\epsilon_\alpha}^{\infty} dz \frac{\beta}{e^{\beta z}+1} 2\pi\mathbf{i}\\
&= \sum_\alpha \int_{\epsilon_\alpha}^{\infty} dz \frac{\beta}{e^{\beta z}+1} = \sum_\alpha \log\left(1 + e^{-\beta\epsilon_\alpha}\right),
\end{aligned}
$$

which gives the same answer as above.

(c) [super bonus problem] Consider the case where h_{ij} is a random matrix. What can you say about the thermodynamics?