University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 4

Due 12:30pm Monday, April 29, 2019

1. **Diagrammatic understanding of BCS instability of Fermi liquid theory.**

   (a) Recall that only the four-fermion interactions with special kinematics are marginal. Keeping only these interactions, show that cactus diagrams (like this: ) dominate.

   (b) To sum the cacti, we can make bubbles with a corrected propagator. Argue that this correction to the propagator is innocuous and can be ignored.

   (c) Armed with these results, compute diagrammatically the Cooper-channel susceptibility (two-particle Green's function),

   $$\chi(\omega_0) \equiv \left\langle \mathcal{T} \psi^\dagger_{\vec{k},\omega_3,\downarrow} \psi^\dagger_{-\vec{k},\omega_4,\uparrow} \psi_{\vec{p},\omega_1,\downarrow} \psi_{-\vec{p},\omega_2,\uparrow} \right\rangle$$

   as a function of $\omega_0 \equiv \omega_1 + \omega_2$, the frequencies of the incoming particles. Think of χ as a two point function of the Cooper pair field $\Phi = \epsilon_{\alpha\beta}\psi^\dagger_\alpha \psi_\alpha$ at zero momentum.

   Sum the geometric series in terms of a (one-loop) integral kernel.

   (d) Do the integrals. In the loops, restrict the range of energies to $|\omega| < E_D$ (or $|\epsilon(k)| < E_D$), the Debye energy, since it is electrons with these energies which experience attractive interactions.

   Consider for simplicity a round Fermi surface. For doing integrals of functions singular near a round Fermi surface, make the approximation $\epsilon(k) \simeq v_F(|k| - k_F)$, so that $d^dk \simeq k_F^{d-1} \frac{d\xi}{v_F} d\Omega_{d-1}$.

   (e) Show that when $V < 0$ is attractive, $\chi(\omega_0)$ has a pole. Does it represent a bound-state? Interpret this pole in the two-particle Green's function as indicating an instability of the Fermi liquid to superconductivity. Compare the location of the pole to the energy E_{BCS} where the Cooper-channel interaction becomes strong.

   (f) **Cooper problem.** [optional] We can compare this result to Cooper's influential analysis of the problem of two electrons interacting with each other in the presence of an inert Fermi sea. Consider a state with two electrons with antipodal momenta and opposite spin

$$|\psi\rangle = \sum_k a_k \psi^\dagger_{k,\uparrow}\psi^\dagger_{-k,\downarrow}|F\rangle$$

where $|F\rangle = \prod_{k<k_F}\psi^\dagger_{k,\uparrow}\psi^\dagger_{k,\downarrow}|0\rangle$ is a filled Fermi sea. Consider the Hamiltonian

$$H = \sum_k \epsilon_k \psi^\dagger_{k,\sigma}\psi_{k,\sigma} + \sum_{k,k'} V_{k,k'}\psi^\dagger_{k,\sigma}\psi_{k,\sigma}\psi^\dagger_{k',\sigma'}\psi_{k',\sigma'}.$$

Write the Schrödinger equation as

$$(\omega - 2\epsilon_k)a_k = \sum_{k'} V_{k,k'}a_{k'}.$$

Now assume (following Cooper) that the potential has the following form:

$$V_{k,k'} = V w^\star_{k'} w_k, \qquad w_k = \begin{cases} 1, & 0 < \epsilon_k < E_D \\ 0, & \text{else} \end{cases}.$$

Defining $C \equiv \sum_k \omega^\star_k a_k$, show that the Schrödinger equation requires

$$1 = V\sum_k \frac{|w_k|^2}{\omega - 2\epsilon_k}. \tag{1}$$

Assuming V is attractive, find a bound state. Compare (14) to the condition for a pole found from the bubble chains above.

2. **Topological terms in QM.**

The purpose of this problem is to demonstrate that total derivative terms in the action (like the θ term in QCD) do affect the physics.

The euclidean path integral for a particle on a ring with magnetic flux $\theta = \int \vec{B}\cdot d\vec{a}$ through the ring is given by

$$Z = \int [D\phi] e^{-\int_0^\beta d\tau\left(\frac{m}{2}\dot\phi^2 - \mathbf{i}\frac{\theta}{2\pi}\dot\phi\right)}.$$

Here

$$\phi \equiv \phi + 2\pi \tag{2}$$

is a coordinate on the ring. Because of the identification (15), ϕ need not be a single-valued function of τ – it can wind around the ring. On the other hand, $\dot\phi$ is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

$$\phi(\tau) = \frac{2\pi}{\beta}Q\tau + \sum_{\ell\in\mathbb{Z}\setminus 0}\phi_\ell e^{\mathbf{i}\frac{2\pi}{\beta}\ell\tau}. \tag{3}$$

(a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.

(b) Using the decomposition (16), write the partition function as a sum over topological sectors labelled by the *winding number* $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}tn^2+\mathbf{i}zn} = \sqrt{\frac{2\pi}{t}} \sum_{\ell\in\mathbb{Z}} e^{-\frac{1}{2t}(z-2\pi\ell)^2}. \quad]$$

(c) Use the result from the previous part to determine the energy spectrum as a function of θ.

(d) Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

(e) Consider what happens in the limit $m \to 0, \theta \to \pi$ with $X \equiv \frac{\theta-\pi}{m} \sim \beta^{-1}$ fixed. Interpret the result as the partition function for a spin 1/2 particle. What is the meaning of the ratio X in this interpretation?

3. **Grassmann brain-warmers.**

(a) A useful device is the integral representation of the grassmann delta function. Show that

$$-\int d\bar{\psi}_1 e^{-\bar{\psi}_1(\psi_1-\psi_2)} = \delta(\psi_1 - \psi_2)$$

in the sense that $\int d\psi_1 \delta(\psi_1 - \psi_2) f(\psi_1) = f(\psi_2)$ for any grassmann function f. (Notice that since the grassmann delta function is not even, it matters on which side of the δ we put the function: $\int d\psi_1 f(\psi_1)\delta(\psi_1 - \psi_2) = f(-\psi_2) \neq f(\psi_2)$.)

(b) Recall the resolution of the identity on a single qbit in terms of fermion coherent states

$$\mathbb{1} = \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} |\psi\rangle \langle\bar{\psi}| \,. \tag{4}$$

Show that $\mathbb{1}^2 = \mathbb{1}$. (The previous part may be useful.)

(c) In lecture I claimed that a representation of the trace of a bosonic operator was

$$\mathrm{tr}\mathbf{A} = \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} \langle -\bar{\psi}| \mathbf{A} |\psi\rangle \;,$$

and the minus sign in the bra had important consequences.
(Here $\langle -\bar{\psi}| \mathbf{c}^\dagger = \langle -\bar{\psi}| (-\bar{\psi})$).

Check that using this expression you get the correct answer for

$$\mathrm{tr}(a + b\mathbf{c}^\dagger \mathbf{c})$$

where a, b are ordinary numbers.

(d) Prove the identity (20) by expanding the coherent states in the number basis.

4. **Fermionic coherent state exercise.**

   Consider a collection of fermionic modes c_i with quadratic hamiltonian $H = \sum_{ij} h_{ij} c_i^\dagger c_j$, with $h = h^\dagger$.

   (a) Compute $\mathrm{tr} e^{-\beta H}$ by changing basis to the eigenstates of h_{ij} (the single-particle hamiltonian) and performing the trace in that basis: $\mathrm{tr} ... = \prod_\epsilon \sum_{n_\epsilon = c_\epsilon^\dagger c_\epsilon = 0,1} ...$.

   (b) Compute $\mathrm{tr} e^{-\beta H}$ by coherent state path integral. Compare!

   (c) [super bonus problem] Consider the case where h_{ij} is a random matrix. What can you say about the thermodynamics?