University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 5

Due 12:30pm Monday, May 6, 2019

1. **Brain-warmers on spin coherent states.**

   (a) Show that
   $$\vec{n} = z^\dagger \vec{\boldsymbol{\sigma}} z$$
   where $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is a unit vector, and z_α are the components of the (normalized) spin coherent state $|\vec{n}\rangle$ ($\boldsymbol{\sigma}\cdot\vec{n}\,|\vec{n}\rangle = |\vec{n}\rangle$) in the Z-eigenbasis.

   (b) In the same notation, check that
   $$\langle \vec{n}_1 | \vec{n}_2 \rangle = z_1^\dagger z_2 .$$

   (c) Check that
   $$\mathbb{1}_{2\times 2} = \int \frac{d^2 n}{2\pi} |\vec{n}\rangle\langle\vec{n}| .$$

   (d) Check that
   $$\int dt \mathbf{i} z^\dagger \dot{z} = \int dt \mathbf{i} \frac{1}{2}\left(\cos\theta\dot{\phi} + \dot{\psi}\right) = 2\pi W_0[\hat{n}] .$$

   (e) Show that
   $$\langle \check{n}| \vec{h}\cdot\vec{\mathbf{S}} |\check{n}\rangle = s\vec{h}\cdot\check{n}$$
   where $|\check{n}\rangle = \mathcal{R}\,|s,s\rangle$ is a coherent state of spin s (where $|s,s\rangle$ is the eigenvector of $\mathbf{S}^z$ with maximal eigenvalue, and $\mathcal{R}$ is the rotation operator which takes $\check{z}$ to $\check{n}$).

   (f) Show that for several spins and $i \neq j$
   $$\langle \check{n}| \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j |\check{n}\rangle = s^2 \check{n}_i \cdot \check{n}_j ,$$
   where now $|\check{n}\rangle \equiv \otimes_j \left(\mathcal{R}_i\,|s_i\rangle\right)$ is a product of coherent states of each of the spins individually.

2. **Brain-warmer on Schwinger bosons.**

   Recall the Schwinger-boson representation of the $\mathsf{SU}(2)$ algebra:

$$\mathbf{S}^+ = a^\dagger b, \ \mathbf{S}^- = b^\dagger a, \ \mathbf{S}^z = a^\dagger a - b^\dagger b,$$

   where the modes a, b satisfy $[a, a^\dagger] = 1 = [b, b^\dagger], [a, b] = [a, b^\dagger] = 0$. This is the algebra of a simple harmonic oscillator in two dimensions,

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right).$$

   Is the $\mathsf{SU}(2)$ a symmetry of this Hamiltonian? How does it act on the oscillator coordinates? Check that the oscillator algebra does indeed imply that $\vec{\mathbf{S}}$ defined this way satisfy the $\mathsf{SU}(2)$ algebra.

3. **Geometric Quantization of the 2-torus.**

   Redo the analysis that we did in lecture for the two-sphere for the case of the two-torus, $S^1 \times S^1$. The coordinates on the torus are $(x, y) \simeq (x + 2\pi, y + 2\pi)$; use $\frac{N}{2\pi} \mathrm{d}x \wedge \mathrm{d}y$ as the symplectic form. Show that the resulting Hilbert space represents the Heisenberg algebra

$$e^{\mathbf{i}\mathbf{x}} e^{\mathbf{i}\mathbf{y}} = e^{\mathbf{i}\mathbf{y}} e^{\mathbf{i}\mathbf{x}} e^{\frac{2\pi \mathbf{i}}{N}}.$$

   (I am using boldface letters for operators.) The irreducible representation of this algebra is the same Hilbert space as a particle on a periodic one-dimensional lattice with N sites.

4. **Particle on a sphere with a monopole inside.**

   Consider a particle of mass m and electric charge e with action

$$S[\vec{x}] = \int \mathrm{d}t \left(\frac{1}{2} m \dot{\vec{x}}^2 + e \dot{\vec{x}} \cdot \vec{A}(\vec{x})\right)$$

   constrained to move on a two sphere of radius r in three-space, $\vec{x}^2 = r^2$. Suppose further that there is a *magnetic monopole* inside this sphere: this means that $4\pi g = \int_{S^2} \vec{B} \cdot \mathrm{d}\vec{a} = \int_{S^2} F$, where $F = \mathrm{d}A$. (Since the particle lives only at $\vec{x}^2 = r^2$, the form of the field in the core of the monopole is not relevant here.)

   (a) Find an expression for $A = A_i \mathrm{d}x^i = A_\theta \mathrm{d}\theta + A_\varphi \mathrm{d}\varphi$ such that $F = \mathrm{d}A$ has flux $4\pi g$ through the sphere.

   (b) Show that the Witten argument gives the Dirac quantization condition $2eg \in \mathbb{Z}$.

   (c) Take the limit $m \to \infty$. Count the states in the lowest Landau level. Compare with the calculation in lecture for coherent state quantization of a spin-s.