University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 6 – Solutions

Due 12:30pm Monday, May 13, 2019

1. **Brain-warmer.**

   Compute the expectation values of $\mathbf{X}$ and $\mathbf{Z}$ in the spin-coherent state $|\check{n}\rangle$.

2. **Mean field theory is product states.**

   Consider the transverse field Ising model on an arbitrary lattice:

$$\mathbf{H} = -J\left(\sum_{\langle x,y\rangle} Z_x Z_y + g\sum_x X_x\right).$$

   We will study the mean field state:

$$|\psi_{\rm MF}\rangle \equiv \otimes_x \left(\sum_{s_x \pm} \psi_{s_x} |s_x\rangle\right). \tag{1}$$

   Restrict to the case where the state of each spin is the same.

   (a) Write the variational energy for the mean field state, $E(\hat{n}) \equiv \langle \psi_{\rm MF}|\mathbf{H}|\psi_{\rm MF}\rangle$.

   (b) Assuming s_x is independent of x, minimize it for each value of the dimensionless parameter g. Find the groundstate magnetization $\langle\psi|Z_x|\psi\rangle$ in this approximation, as a function of g. Draw the mean-field phase diagram.

Mean field theory means that we completely ignore entanglement between different sites, and suppose that the state is a product state

$$|MFT\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \cdots |\psi_j\rangle \cdots ...$$

If we further assume translational invariance then the state at every site is the same and we have one bloch sphere to minimize over for each g:

$$|\check{n}\rangle = \otimes_j |{\uparrow_{\check{n}}}\rangle_j = \otimes_j \left(\cos\frac{\theta}{2}e^{\mathbf{i}\varphi/2}|{\rightarrow}\rangle + \sin\frac{\theta}{2}e^{-\mathbf{i}\varphi/2}|{\leftarrow}\rangle\right)_j.$$

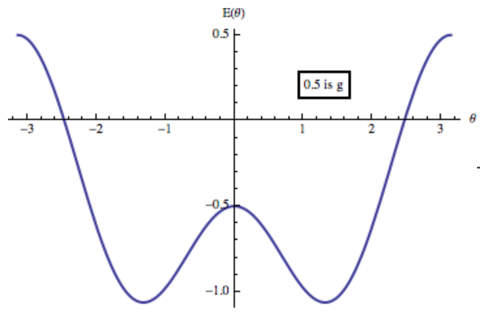
(Here θ is the angle $\check{n}$ makes with the x axis, and φ is the azimuthal angle in the yz plane, from the z-axis.) To evaluate the energy expectation in this state, we only need to know single-qbit expectations:

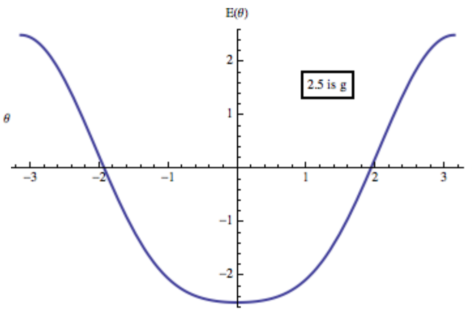
$$\langle{\uparrow_{\check{n}}}|\,\mathbf{X}\,|{\uparrow_{\check{n}}}\rangle = \cos\theta, \quad \langle{\uparrow_{\check{n}}}|\,\mathbf{Z}\,|{\uparrow_{\check{n}}}\rangle = \sin\theta\cos\varphi.$$

So the energy expectation is

$$E(\theta, \varphi) = -NJ\left(\frac{z}{2}\sin^2\theta\cos^2\varphi + g\cos\theta\right)$$

where z is the coordination number of the lattice; I set $z = 2$ for $d = 1$.





This is extremized when $\varphi = 0, \pi$ and when

$$0 = \partial_\theta E = NJ\sin\theta\,(2\cos\theta - g)\,.$$

Notice that when $\theta = 0$, the two solutions of φ are the same, since the φ coordinate degenerates at the pole. The solutions at $\cos\theta = g/2$ only exist when $g/2 < 1$. In that case they are minima (see the figure) since $\partial_\theta^2 E|_{\cos\theta = g/2} > 0$, while $\partial_\theta^2 E|_{\theta=0} = NJ(g-2)$ is negative for $g < 2$. (Notice that $\varphi = \pi$ can be included by allowing $\theta \in (-\pi, \pi]$, as in the figure.)

So mean field theory predicts a phase transition at $g = 2$, from a state where $\langle \mathbf{Z}_j \rangle = \sin\theta$ to one where $\langle \mathbf{Z} \rangle = 0$. It overestimates the range of the ordered phase because it leaves out fluctuations which tend to destroy the order.

Let's study the behavior near the transition, where θ is small. Then the energy can be approximated by its Taylor expansion

$$E(\theta) \simeq NJ\left(-2 + \frac{g-2}{2}\theta^2 + \frac{1}{4}\theta^4\right)$$

(where I have set $g = g_c = 2$ except in the crucial quadratic term). This has minima at

$$\langle \mathbf{Z}_j \rangle = \sin\theta \simeq \theta = \pm\sqrt{g_c - g}\;. \tag{2}$$

The energy behaves like

$$E_{MFT}(g) = \begin{cases} \frac{3}{4}(g_c - g)^2\,, & g < g_c \\ \\ 0\,, & g \geq g_c \end{cases}$$

Notice that $\partial_g E$ is continuous at the transition. (Recall that the groundstate energy of the quantum system is equal to the free energy of the corresponding stat mech system, so $\partial_g E \propto \partial_T F$ continuous is the same criterion for a continuous transition.) So mean field theory (correctly) predicts a continuous quantum phase transition between the ordered phase and the disordered phase. The location of the transition is wrong (mean field theory overestimates the size of the ordered region because it leaves out lots of order-destroying fluctuations), and so are other properties, such as the exponent in (2), which should be 1/8 instead of 1/2.

3. **Potentials for matrix-valued fields.**

   (a) Convince yourself that by a symmetry transformation $\Sigma \to g_L \Sigma g_R^\dagger$ we can put the complex matrix Σ in the form $\Sigma = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$.

   The transformation is a general similarity transformation, by which we may diagonalize Σ. Actually a completely general complex matrix may have Jordan blocks, where upper triangular bits can't be removed.

   (b) Consider the $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R$-symmetric potential

$$V(\Sigma) = -m^2 \mathrm{tr}\Sigma\Sigma^\dagger + \frac{\lambda}{4}\left(\mathrm{tr}\Sigma\Sigma^\dagger\right)^2 + g\mathrm{tr}\Sigma\Sigma^\dagger\Sigma\Sigma^\dagger. \tag{3}$$

   Show that for any $g > 0$ this potential has a minimum at $\langle\Sigma\rangle = \frac{v}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find v. Show that if $g = 0$ there are other minima which are not related by rotations $\Sigma \to g_L \Sigma g_R^\dagger$.

   Diagonalizing Σ as in the previous part, the potential is

$$V = -m^2\left(|v_1|^2 + |v_2|^2\right) + \frac{\lambda}{4}\left(|v_1|^2 + |v_2|^2\right)^2 + g(|v_1|^4 + |v_2|^4).$$

   The equations $\partial_{v_1} V = 0, \partial_{v_2} V = 0$ are solved by $v_1^2 = v_2^2 = \frac{m^2}{\lambda + 2g}$.

   When $g = 0$, the two equations $\partial_{v_a} V = 0$ are the same and we may have $\langle\Sigma\rangle \propto \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$ with $v_1^2 + v_2^2 = \frac{2m^2}{\lambda}$. (Indeed, when $g = 0$, the potential depends only on the combination $v_1^2 + v_2^2$.) Since the eigenvalues are different, no similarity transformation can turn this into something proportional to the identity.

   (c) [bonus problem] Now consider a hermitian-matrix-valued field $\Phi = \Phi^a T^a$. Suppose T^a are generators of the adjoint of $\mathsf{SU}(5)$, so there are 24 components of Φ^a. In order for grand unification to work, there must be a potential for such a Higgs field Φ which has a minimum of the form

$$\langle\Phi\rangle = v\mathrm{diag}(2,2,2,-3,-3) \equiv \Phi_{3,2}$$

which breaks $\mathsf{SU}(5)$ down to $\mathsf{SU}(3)_{\mathrm{color}} \times \mathsf{SU}(2)_{\mathrm{weak}}$. Consider the most general quartic potential for Φ which is invariant under $\mathsf{SU}(5)$:

$$V = -m^2\mathrm{tr}\Phi^2 + a\mathrm{tr}\Phi^4 + b\left(\mathrm{tr}\Phi^2\right)^2.$$

Choose a basis where $\Phi = v\mathrm{diag}(a_1, a_2, a_3, a_4, a_5)$, with $\sum_{i=1}^5 a_i = 0$. (Impose this last condition with a Lagrange multiplier.)

For what values of m, a, b is $\Phi_{3,2}$ an extremum?

Show that $\Phi_{3,2}$ is a minimum.

Find all possible minima of this potential.

Here is a nice argument from Brian Vermilyea: the equation for one of the eigenvalues

$$0 = \partial_{a_j}\left(V + \lambda\sum_j a_j\right) = a_j\left(-2m^2 + 2b\sum_j a_j^2\right) + aa_j^3$$

has at most 3 different solutions. Therefore at most 3 of the a_j can be different. So the possible solutions with $\sum_j a_j = 0$ are

$$\Phi_{3,2}, \Phi_{4,1}, \Phi_{2,2,1} \equiv v(1,1,a,a,-2a-2), \Phi_{3,1,1} \equiv v(1,1,1,b,-b-3).$$

We can determine $a = -1$ and $b = 0$ by varying with respect to these parameters.

For the minimum of the form $\langle\Phi\rangle \equiv \Phi_{4,1} = v\mathrm{diag}(1,1,1,1,-4)$, what are the masses of the massive gauge bosons, and what is the unbroken gauge group?

The unbroken group is $\mathsf{SU}(4) \times \mathsf{U}(1)$, and the masses of the gauge boson A^A in $A = A^A T^A$ is $\mathrm{tr}\left([T^A, \Phi_{4,1}]^2\right)$.