University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2019
Assignment 9

Due 12:30pm Wednesday, June 5, 2019

1. **Brain-warmer.** Find the coefficient $\mathcal{N}_s$ in the coherent state representation of the spin operator for general spin s
$$\mathbf{S}^a = \mathcal{N}_s \int dn \, |\check{n}\rangle \langle \check{n}| \, \check{n}^a.$$

2. **Topological charge.** How does the theta term appear in the $\mathbb{CP}^1$ representation of the NLSM on S^2? Show that
$$\epsilon_{abc} n^a dn^b \wedge dn^c = \alpha dA$$
for some constant α, and find the number α.

3. **Large-N saddle points in the $\mathsf{O}(N)$ model.** [This problem is optional, since by now we've done a number of similar problems.]

   Consider the partition function for an N-vector of scalar fields in D dimensions
$$Z = \int [D\phi] e^{\mathbf{i}S[\phi]}, \quad S[\vec{\phi}] = \int \mathrm{d}^D x \left(\frac{1}{2} \partial\phi^a \partial\phi^a - NV\left(\frac{\vec{\phi}^2}{N} \right) \right)$$
   with a general 2-derivative $O(N)$-invariant action. We're going to do this path integral by saddle point, which is a good idea at large N. As usual, the constant prefactors in Z drop out of physical ratios so you should ignore them.

   (a) Change variables to the $O(N)$ singlet field $\zeta \equiv \vec{\phi}^2/N$ by inserting the identity
$$1 = \int [D\zeta] \delta\left[\zeta - \frac{\vec{\phi}^2}{N} \right]$$
   into the path integral representation for Z. Represent the functional delta function as
$$\delta\left[\zeta - \frac{\vec{\phi}^2}{N} \right] = \int [D\sigma] e^{\mathbf{i} \int \mathrm{d}^D x \sigma \left(\vec{\phi}^2 - \zeta N \right)}.$$
   Do the integral over ϕ^a to obtain
$$Z = \int [D\zeta D\sigma] e^{\mathbf{i} N S_{\text{eff}}[\zeta, \sigma]}.$$
   Determine $S_{\text{eff}}[\zeta, \sigma]$.

(b) The integrals over ζ, σ have a well-peaked saddle point at large N. Obtain the coupled large-N saddle point equations for the saddle point configurations ζ_0, σ_0, and in particular the equation

$$\zeta_0(x) = \left(\frac{\mathbf{i}}{-\Box - 2V'(\zeta_0)} \right)_{xx}$$

(the subscript denotes a matrix element of the position-space operator).

(c) [more optional] Show that

$$\frac{\delta}{\delta\sigma(x)} \operatorname{tr}\log\left(-\Box + \sigma\right) = \left(\frac{1}{-\Box + \sigma} \right)_{xx}$$

by Taylor expansion.

(d) At large N, we know that

$$\zeta_0(x) \overset{N\to\infty}{=} \left\langle \frac{\vec{\phi}^2(x)}{N} \right\rangle = \zeta_0, \text{ constant.}$$

Use this to show that the saddle point equation is the *gap equation*

$$\zeta_0 = \int đ^D k_E \frac{1}{k_E^2 + 2V'(\zeta_0)}$$

which determines ζ_0, the expectation value of the order parameter $\left\langle \vec{\phi}^2/N \right\rangle$.

(e) What class of diagrams did you just sum?

(f) Compare and contrast the saddle point condition for $D = 2$ and $D > 2$. For $D > 2$ you should find a critical value of the coupling.

Compare the behavior near the critical point with the large-n limit of the Wilson-Fisher fixed point in the ϵ expansion.

(g) Evaluate the two point function $\langle \phi^a(x)\phi^a(0)\rangle$ at the saddle point with $\zeta_0 \neq 0$.

4. **The Hohenberg-Mermin-Wagner-Coleman Fact.**

(a) Consider a massless scalar X in 2d, with (Euclidean) action

$$S[X] = \frac{1}{4\pi g} \int d^2\sigma \partial_a X \partial^a X. \tag{1}$$

Show that the euclidean propagator

$$G_2(z, z') \equiv \langle X(z) X(z') \rangle$$

satisfies

$$\nabla^2 G_2(z, z') = b\delta^2(z - z') \tag{2}$$

where $z = \sigma_1^E + \mathbf{i}\sigma_2^E$, for some constant b; find b. Show that the solution is given by

$$G_2(z, z') = a \ln |z - z'|,$$

for some constant a (for example by Fourier transform); find a.

(b) The long-distance behavior of G_2 has important implications for the spontaneous breaking of continuous symmetries in $D = 2$ – it can't happen. Argue that if a system with a continuous (say $\mathsf{U}(1)$, for definiteness) symmetry were to have an unsymmetric groundstate, the excitations about that state would include a field X with the action (1). Conclude from the form of G_2 that there is in fact no long-range order.

5. **Correlators of composite operators made of free bosons in 1+1 dimensions.**

Consider a collection of n two-dimensional free bosons X^μ governed by the (Euclidean) action

$$S = \frac{1}{4\pi g} \int d^2\sigma \partial_a X_\mu \partial^a X^\mu.$$

Until further notice, we will assume that X takes values on the real line.

[If $X \in \mathbb{R}$, the coupling g can be absorbed into the definition of X if we prefer, but it is useful to leave this coupling constant arbitrary for several reasons. First, different physicists use different conventions for the normalization and as you will see this affects the appearance of the final answer. But more importantly, in part 5d, g will become meaningful.]

(a) Compute the Euclidean generating functional

$$Z[J] = \left\langle e^{\int (d^2\sigma)_E J^\mu X_\mu} \right\rangle \equiv Z_0^{-1} \int [dX] e^{-S} e^{\int (d^2\sigma)_E J^\mu X_\mu}$$

(where $Z_0^{-1} \equiv Z[J = 0]$ but please don't worry too much about the normalization of the path integral).

[Hint: use the Green function from the previous problem, and Wick's theorem. Or use our general formula for Gaussian integrals with sources.]

[Warning: In the problem at hand, even the euclidean kinetic operator has a kernel, namely the zero-momentum mode. You will need to do this integral separately.]

[Cultural remark 1: this field theory describes the propagation of featureless strings in n-dimensional flat space $\mathbb{R}^n$ – think of $X^\mu(\sigma)$ as the parametrizing the position in $\mathbb{R}^n$ to which the point σ is mapped.

Cultural remark 2: this is an example of a conformal field theory. In particular recall that massless scalars in $D = 2$ have engineering dimension zero.]

(b) Show that

$$\left\langle \prod_{i=1}^{N} : e^{-i\sqrt{2\alpha'}k_i \cdot X(\sigma^{(i)})} : \right\rangle = \delta^n \left(\sum_i k_i^\mu \right) \prod_{i,j=1}^{N} |z_i - z_j|^{-\alpha' g k_i \cdot k_j} \tag{3}$$

where $\sigma^{(i)}$ label points in 2d Euclidean space, $z_i \equiv \sigma_1^{(i)} + i\sigma_2^{(i)}$, α' is a parameter with dimensions of $[X^2/g]$ (called the 'Regge slope'), and k_i^μ are a set of arbitrary n-vectors in the target space. The : ... : indicate the following prescription for *defining* composite operators. The prescription is simply to leave out Wick contractions of objects within a pair of : ... :. Give a symmetry explanation of the delta function in k.

[Cultural remark: this calculation is the central ingredient in the *Veneziano amplitude* for scattering of bosonic strings at tree level.]

(c) Conclude that the composite operator $\mathcal{O}_a \equiv: e^{\mathbf{i}aX}$: has *scaling dimension* $\Delta_a = \frac{ga^2}{2}$, in the sense that

$$\left\langle \mathcal{O}_a(z)\mathcal{O}_b^\dagger(0) \right\rangle = \delta(a-b)\frac{1}{|z|^{2\Delta_a}}.$$

Notice that the correlation functions of these operators do not describe the propagation of *particles* in any sense. The operator $\mathcal{O}$ produces some power-law excitation of the CFT soup.

(d) Suppose we have one field ($n = 1$) X which takes values on the circle, that is, we identify

$$X \simeq X + 2\pi R\ .$$

What values of a label single-valued operators : $e^{\mathbf{i}aX}$: ? How should we modify (3)?

6. **T-duality: not just for the free theory.** [bonus problem]

Here is a path integral derivation of T-duality which is more general than just a single free boson.

Consider the sigma model whose action is

$$S(\partial X, Y) = S(Y) + \frac{1}{4\pi\alpha'} \int d^2 z \left(\delta^{ab} G_{XX}(Y) \partial_a X \partial_b X + \left(\delta^{ab} G_{\mu X} + \epsilon^{ab} B_{\mu X} \right) \partial_a X \partial_b Y^\mu \right) \ .$$

Here Y^μ are a bunch of coordinates on which the background fields G, B may depend in arbitrarily complicated ways. X only appears through its derivatives.

(a) Show that by replacing $\partial_\mu X$ by $\partial_\mu X + A_\mu$ we arrive at a theory with an invariance under local shifts of $X \to X + \alpha(x)$.

(b) Add a 2d θ term $\mathbf{i}\phi F_{\mu\nu}$, with $F = dA$ and the angle ϕ a dynamical field. Show that the path integral over ϕ undoes the previous step and returns us to the original model. Hint: use the gauge $\partial_\mu A^\mu = 0$.

(c) Instead choose the gauge $X = 0$ and do the integral over A_μ. Identify ϕ as the T-dual variable. To get the period right, you need to think about non-perturbative parts of the gauge field path integral.

7. **T-duality as EM duality of 0-forms.**

In this problem we will contextualize the form of the T-duality map

$$\phi(z, \bar{z}) = \phi_L(z) + \phi_R(\bar{z}) \mapsto \tilde{\phi}(z, \bar{z}) \equiv \phi_L(z) - \phi_R(\bar{z})$$

in terms of more general duality maps on form fields.

Consider a massless p-form field a in D (euclidean) dimensions, more specifically, on $\mathbb{R}^D$. We will treat it classically. Suppose its eom are

$$\mathrm{d} \star \mathrm{d}a = 0 \ .$$

[1]

This equation says $\star \mathrm{d}a$ is closed, which on $\mathbb{R}^D$ which has no nontrivial topology, this means it is exact: we can define $\star \mathrm{d}a = \mathrm{d}\tilde{a}$.

For abelian gauge theory in $D = 4$ show that this map $a \to \tilde{a}$ takes $(E, B) \to (\tilde{E}, \tilde{B}) = (B, -E)$.

[1] By this notation, I mean the following. The exterior derivative of a p-form is a $p+1$ form:

$$(\mathrm{d}a)_{\mu_1 \cdots \mu_{p+1}} = \left(\partial_{\mu_1} a_{\mu_2 \cdots \mu_{p+1}} \pm \text{perms} \right) \frac{1}{(p+1)!}$$

The Hodge dual of a k-form is a $d - k$ form:

$$(\star \omega_k))_{\mu_1 \cdots \mu_{d-k}} \equiv \epsilon_{\mu_1 \cdots \mu_d} (\omega_k)^{\mu_{d-k+1} \cdots \mu_d} \ .$$

Show that the map between ϕ and $\tilde{\phi}$ is of this form, if we regard ϕ as a 0-form potential.

For help see this paper by Chris Beasley.

8. **$\mathsf{SU}(2)$ current algebra from free scalar.** [bonus problem]

Consider again a compact free boson $\phi \simeq \phi + 2\pi$ in $D = 1 + 1$ with action

$$S[\phi] = \frac{R^2}{8\pi} \int \mathrm{d}x\mathrm{d}t \partial_\mu \phi \partial^\mu \phi. \tag{4}$$

[Notice that if we redefine $\tilde{\phi} \equiv R\phi$ then we absorb the coupling R from the action $S[\tilde{\phi}] = \frac{1}{8\pi} \int \mathrm{d}x\mathrm{d}t \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi}$ but now $\tilde{\phi} \simeq \tilde{\phi} + 2\pi R$ has a different period – hence the name 'radius'.[2]]

So: there is a special radius (naturally called the $\mathsf{SU}(2)$ radius) where new operators of dimension $(1, 0)$ and $(0, 1)$ appear, and which are charged under the boson number current $\partial_\pm \phi$. Their dimensions tell us that they are (chiral) currents, and their charges indicate that they combine with the obvious currents $\partial_\pm \phi$ to form the (Kac-Moody-Bardakci-Halpern) algebra $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R$.

Here you will verify that the model (4) does in fact host an $SU(2)_L \times SU(2)_R$ algebra involving *winding modes* – configurations of ϕ where the field winds around its target space circle as we go around the spatial circle. We'll focus on the holomorphic (R) part, $\phi(z) \equiv \phi_R(z)$; the antiholomorphic part will be identical, with bars on everything.

Define

$$J^\pm(z) \equiv : e^{\pm i\phi(z)} :, \qquad J^3 \equiv i\partial\phi(z).$$

The dots indicate a normal ordering prescription for defining the composite operator: no wick contractions between operators within a set of dots.

(a) Show that $J^3, J^\pm$ are single-valued under $\phi \to \phi + 2\pi$.

(b) Compute the scaling dimensions of $J^3, J^\pm$. Recall that the scaling dimension Δ of a holomorphic operator in 2d CFT can be extracted from its two-point correlation function:

$$\left\langle \mathcal{O}^\dagger(z)\mathcal{O}(0) \right\rangle \sim \frac{1}{z^{2\Delta}} .$$

For free bosons, all correlation functions of composite operators may be computed using Wick's theorem and

$$\langle \phi(z)\phi(0) \rangle = -\frac{1}{R^2} \log z.$$

[2]Relative to the notation I used in lecture, I have set $\pi T \equiv R^2$. A note for the string theorists: I am using units where $\alpha' = 2$.

Find the value of R where the vertex operators $J^{\pm}$ have dimension 1.

(c) Defining $J^{\pm} \equiv \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)$ show that the operator product algebra of these currents is

$$J^a(z)J^b(0) \sim \frac{k\delta^{ab}}{z^2} + i\epsilon^{abc}\frac{J^c(0)}{z} + ...$$

with $k = 1$. This is the level-$k = 1$ SU(2)Kac-Moody-Bardakci-Halpern algebra.

(d) [Bonus tedium] Defining a mode expansion for a dimension 1 operator,

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

show that

$$[J^a_m, J^b_n] = i\epsilon^{abc}J^c_{m+n} + mk\delta^{ab}\delta_{m+n}$$

with $k = 1$, which is an algebra called Affine $SU(2)$ at level $k = 1$. Note that the $m = 0$ modes satisfy the ordinary $SU(2)$ lie algebra.

For hints (and some applications in string theory) see problem 5 here.