Recap: $H = \sum_n \frac{p_n^2}{2m_n} + \sum_{nm} \frac{K_{nm}}{2} q_n q_m$

- only $\frac{K_{nm}+K_{mn}}{2}$ appears. $K_{nm} = (K^T)_{nm}$
$K_{nm} \in \mathbb{R}$
$\Rightarrow \underline{K = K^+}$.

- let $\begin{cases} Q_n = \sqrt{m_n}\, q_n \\ P_n = p_n/\sqrt{m_n} \\ V_{nm} = K_{nm}/\sqrt{m_n m_m} \end{cases}$

$[q_n, p_m] = i\delta_{nm}$
$\Leftrightarrow [Q_n, P_m] = i\delta_{nm}$

$$H = \frac{1}{2}\left(\sum_n P_n^2 + \sum_{nm} V_{nm} Q_n Q_m\right)$$

find U s.t. $UVU^+ =$ diagonal.

$$\sum_{nm} U_{\alpha n} V_{nm} (U^+)_{m\beta} = \omega_\alpha^2 \delta_{\alpha\beta}$$

$UU^+ = \mathbb{1}$ i.e. $\sum_n U_{\alpha n} (U^+)_{n\beta} = \delta_{\alpha\beta}$

$U^+U = \mathbb{1}$ i.e. $\sum_\alpha (U^+)_{n\alpha} U_{\alpha m} = \delta_{nm}$ ⊛

normal modes: $\begin{cases} \tilde{Q}_\alpha \equiv \sum_n U_{\alpha n} Q_n \\ \tilde{P}_\alpha \equiv \sum_n U_{\alpha n} P_n \end{cases}$ $\overset{⊛}{\Longrightarrow}$ $\begin{cases} Q_n = \sum_\alpha (U^+)_{n\alpha} \tilde{Q}_\alpha \\ P_n = \sum_\alpha (U^+)_{n\alpha} \tilde{P}_\alpha. \end{cases}$

$$Q_n = Q_n^+ = \sum_\alpha (U^+)^*_{n\alpha}\, \tilde{Q}^+_\alpha$$

P_n^+

$$= \sum_\alpha (U^T)_{n\alpha}\, \tilde{Q}^+_\alpha$$

$$= \sum_\alpha \tilde{Q}^+_\alpha\, U_{\alpha n}$$

$\tilde{P}^+_\alpha$

Look at H:

$$\sum_{nm} V_{nm} Q_n Q_m = \sum_{nm} V_{nm} Q_n^+ Q_m$$

$$= \sum_{\alpha\beta} \underbrace{\sum_{nm} U_{\alpha n} \underline{V_{nm}} (U^+)_{m\beta}}_{= \omega_\alpha^2 \delta_{\alpha\beta}} \tilde{Q}^+_\alpha \tilde{Q}_\beta$$

$$= \sum_\alpha \omega_\alpha^2\, \tilde{Q}^+_\alpha \tilde{Q}_\alpha$$

$$\sum_n P_n^2 = \sum_n P_n^+ P_n$$

$$= \sum_{\alpha\beta} \underbrace{\sum_n U_{\alpha n} (U^+)_{n\beta}}_{= \delta_{\alpha\beta}} \tilde{P}^+_\alpha \tilde{P}_\beta$$

$$= \sum_\alpha \tilde{P}^+_\alpha \tilde{P}_\alpha \, .$$

$$= \sum_n P_n \delta_{nm} P_m$$

special case:

- local
- transl. inv't.: $K_{nm} q_n q_m$ only depends on $(q_n - q_m)$.

$$K_{nm} q_n q_m = \sum_n (q_{n+1} - q_n)^2 = \sum_n ((T - 1) q_n)^2$$

$$u_{kn} = \frac{1}{\sqrt{N}} e^{ikna}$$

$$u^*_{kn} = \frac{1}{\sqrt{N}} e^{-ikna} = u_{-k,n}$$

$$Q_n = Q_n^\dagger \Rightarrow Q^\dagger_k = Q_{-k}, \quad P^\dagger_k = P_{-k}.$$

$$\rightarrow H = \sum_k \left(\frac{\tilde{P}_k \tilde{P}_{-k}}{2} + \frac{1}{2} m \omega_k^2 \tilde{Q}_k \tilde{Q}_{-k} \right)$$

and $\omega_k \sim v_s |k|$ for $k \ll \frac{1}{a}$.

$$\hat{Q}_n = \sum_k U_{nk} \hat{\tilde{Q}}_k$$

$$[Q_n, P_m] = i\delta_{nm}$$

$$[\tilde{Q}_k, \tilde{P}_{k'}] = i\delta_{kk'}$$

$$\mathcal{H} = \bigotimes_n \mathcal{H}^{(n)}_{SHO} = \bigotimes_k \mathcal{H}^{(k)}_{SHO}.$$

<u>for $k \neq 0$:</u>

$$q_k = \sqrt{\frac{\hbar}{2m\omega_k}}\left(a_k + a_{-k}^{+}\right)$$

$$p_k = \frac{1}{i}\sqrt{\frac{\hbar m\omega_k}{2}}\left(a_k - a_{-k}^{+}\right)$$

$$\Leftrightarrow \quad [a_k, a_{k'}^{+}] = \delta_{kk'}\mathbb{1}$$

<u>note</u>: the form of p is fixed by

$$\dot{q}_k = i[H, q_k] = p_k/m.$$

$$\Rightarrow H = \sum_k \hbar\omega_k\left(\underbrace{a_k^{+}a_k}_{\geq 0} + \frac{1}{2}\right) + \underbrace{\frac{p_0^2}{2m}}_{\geq 0}$$

oscillators

c.o.m.

<u>Groundstate</u>: is

$$a_k|0\rangle_{osc} = 0 \quad \forall k.$$

$$\hat{P}_0|p_0\rangle = p_0|p_0\rangle$$

$$|0\rangle = |0\rangle_{osc} \otimes |p_0 = 0\rangle$$

Phonons:

$$a_k^\dagger |0\rangle = \left| \begin{matrix}\text{one phonon} \\ \text{of momentum } \hbar k\end{matrix} \right\rangle$$

has energy $E_0 + \hbar\omega_k$

$|\hbar k\rangle$.

$$|\text{one phonon at position } x\rangle = \sum_k e^{ikx} |\hbar k\rangle$$

"$|x\rangle$".

$$= \underbrace{\sum_k e^{ikx} a_k^\dagger |0\rangle}_{0}$$

$$\left| \begin{matrix}\text{one phonon w/} \\ \text{wavefunction } \psi(x)\end{matrix} \right\rangle = \int dx\, \psi(x) |x\rangle$$

$$= \int dx \sum_k e^{ikx} \psi(x) a_k^\dagger |0\rangle$$

$N_k = a_k^\dagger a_k$ counts phonons w/ k.

$|k, k'\rangle \equiv a_k^\dagger a_{k'}^\dagger |0\rangle$ 2 phonons

$$E = E_0 + \hbar\omega_k + \hbar\omega_{k'}$$

A basis for $\mathcal{H}$ is:

$$(a_{k_1}^+)^{n_{k_1}}(a_{k_2}^+)^{n_{k_2}}\cdots|0\rangle \propto |n_{k_1}, n_{k_2}\cdots\rangle$$

occupation #s.

a_k^+ creates phonons

a_k annihilates phonons.

$$|k,k'\rangle = a_k^+ a_{k'}^+ |0\rangle$$

$$\overset{k\neq k'}{=} a_{k'}^+ a_k^+ |0\rangle = |k',k\rangle$$

- phonons are indistinguishable.
- phonons are bosons.

$$\begin{cases} q_n = \sqrt{\frac{\hbar}{2m}} \sum_k \frac{1}{\sqrt{\omega_k}}\left(e^{ikx} a_k + e^{-ikx} a_k^+\right) + \frac{q_0}{\sqrt{N}} \\ p_n = \frac{1}{i}\sqrt{\frac{\hbar m}{2}} \sum_k \sqrt{\omega_k}\left(e^{ikx} a_k - e^{-ikx} a_k^+\right) + \frac{p_0}{\sqrt{N}}. \end{cases}$$

$$\Delta H = \sum_{nml} \lambda_{nml}\, q_n q_m q_l$$

$$= \sum\left(a^+a^+a^+ + a^+a^+a + a^+aa + aaa\right)$$

$$\Delta H = \lambda_{nmlo}\, q_n q_m q_l q_o$$

$$= \sum (a^\dagger)^4 + \ldots \quad \underbrace{a^\dagger_{k_1} a^\dagger_{k_2}}\, \underbrace{a_{k_3} a_{k_4}} + \ldots$$

$$|k_3, k_4\rangle \longrightarrow |k_1, k_2\rangle.$$

Gaplessness

$a_k^\dagger |0\rangle$ has energy $\hbar\omega_k$

$$\Delta E \equiv E_1 - E_0.$$

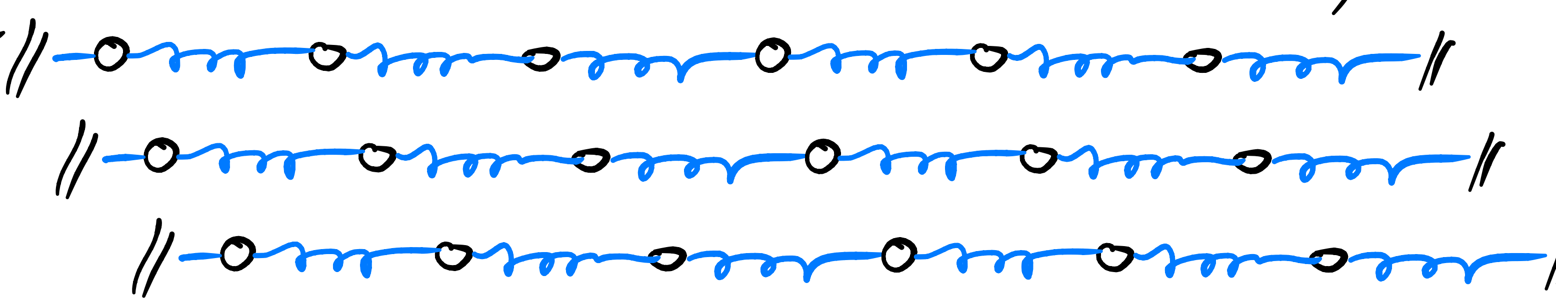
$$= \min_k \hbar\omega_k$$

$$= \frac{\hbar \cdot 2\pi}{L} \xrightarrow{L\to\infty} 0$$

$$L = Na$$

$$k = \frac{2\pi j}{Na} \qquad j = 1..N$$

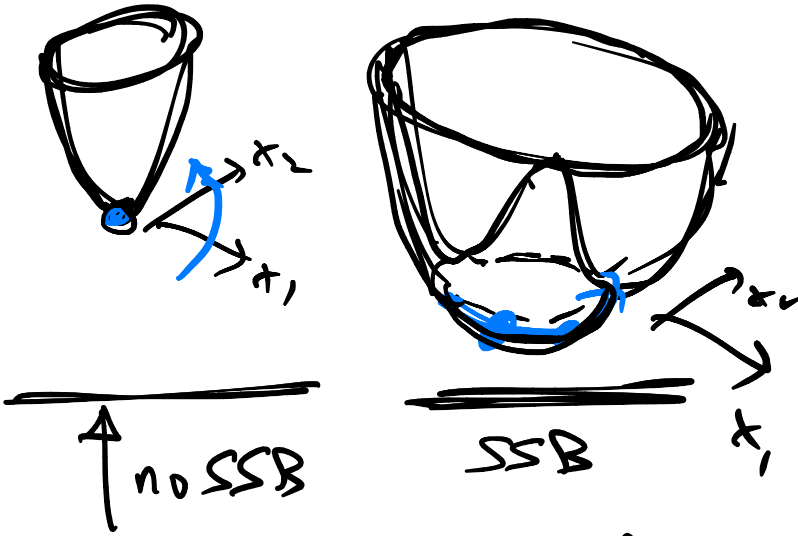
("Energy gap" $\equiv$ ΔE is finite as $L \to \infty$.)



Goldstone's theorem:

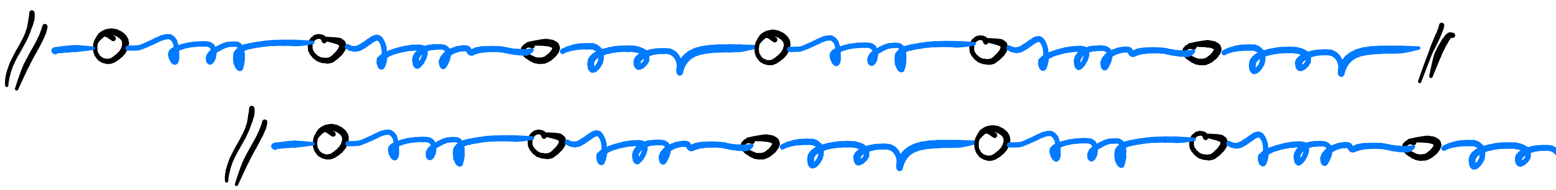
spontaneously breaking a continuous symmetry

$\longrightarrow$ gapless



Tower of states:

q_0, p_0. $\quad \frac{\partial \mathcal{L}}{\partial q_0} = 0$.



$$|\{q_n\}\rangle \equiv |\{q_n + a\}\rangle$$

$$q_0 = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} q_n e^{-i0x_n} \equiv \frac{1}{\sqrt{N}}\left(\sum_{n=1}^{N} q_n + Na\right)$$

$$= q_0 + \sqrt{N} a.$$

$$\Rightarrow e^{ip_0 q_0} \overset{!}{=} e^{ip_0(q_0 + \sqrt{N} a)}$$

$$\Rightarrow p_0 \in \frac{2\pi}{\sqrt{N} a} \mathbb{Z}$$

1st exc. state:

$$\frac{p_0^2}{2m}\Big|_{p_0=\frac{2\pi\hbar}{\sqrt{N}a}} = \frac{1}{2}\left(\frac{2\pi\hbar}{a}\right)^2\frac{1}{Nm}.$$

Nm = total mass

Rigid translation of the whole solid.

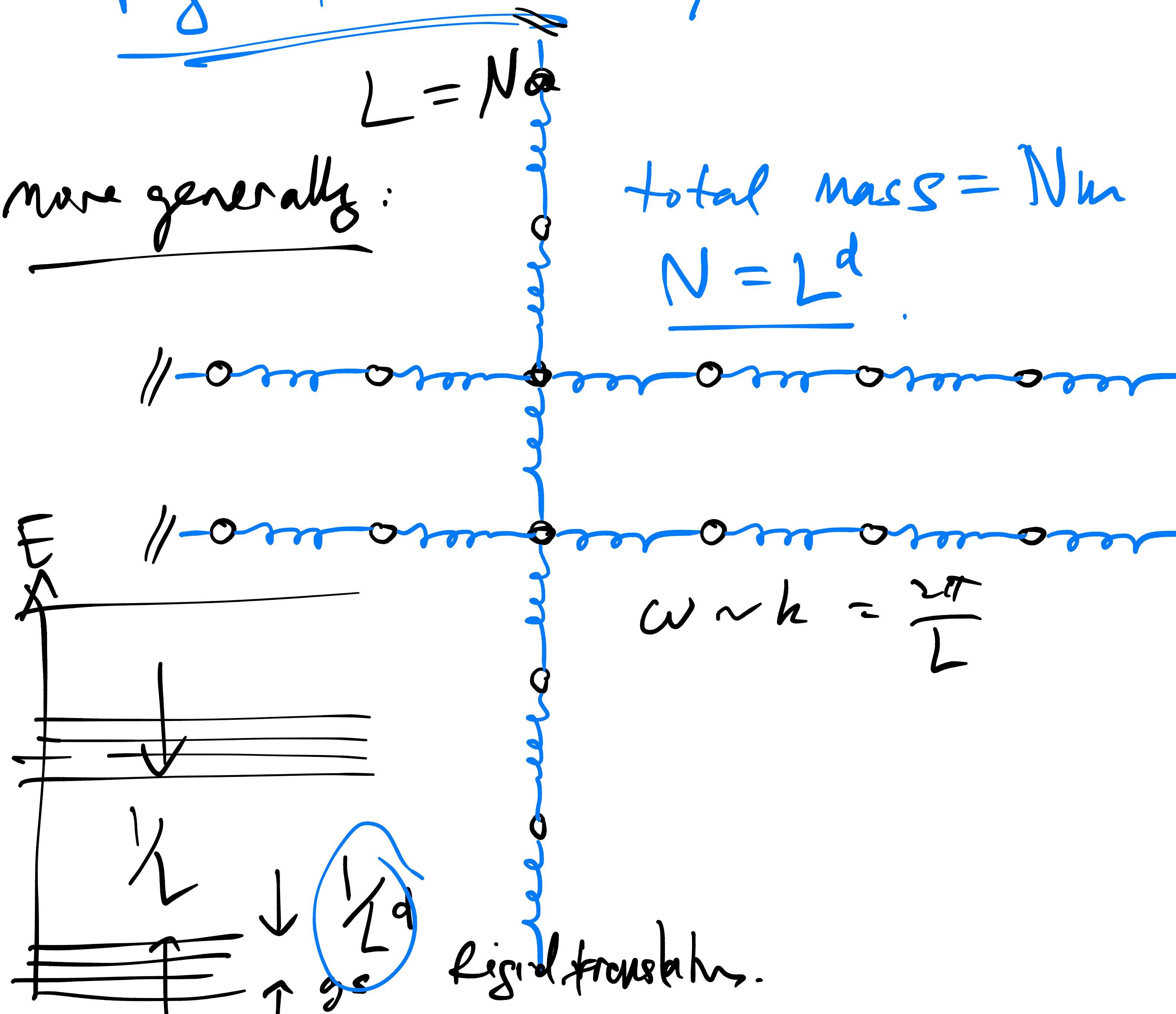
$L = Na$

more generally:

total mass $= Nm$

$N = L^d$.

$\omega \sim k = \frac{2\pi}{L}$

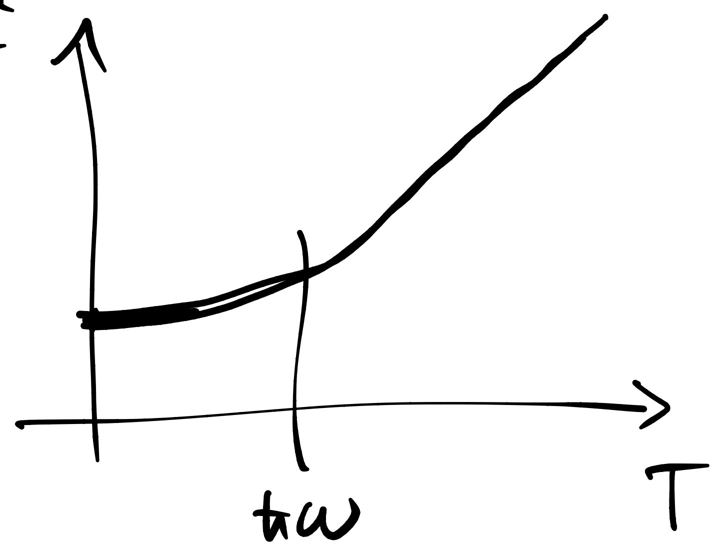


Rigid translation.

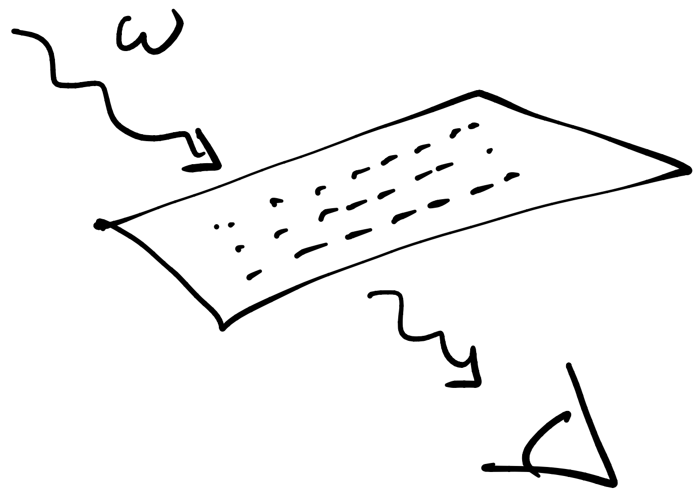
• Heat capacity of solids

$$\langle \hat{H} \rangle_T = \sum_k \langle \hbar\omega a_k^\dagger a_k \rangle + \text{st}$$

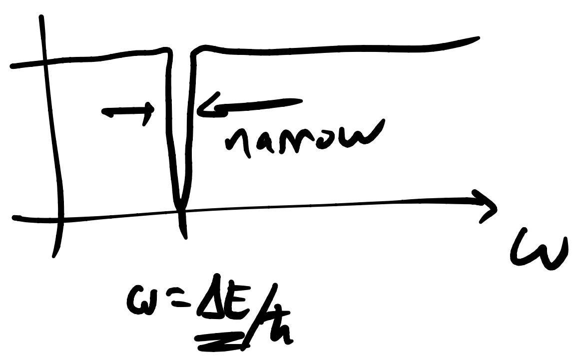
$$C_V = \partial_T E \xrightarrow{T \to 0} 0$$

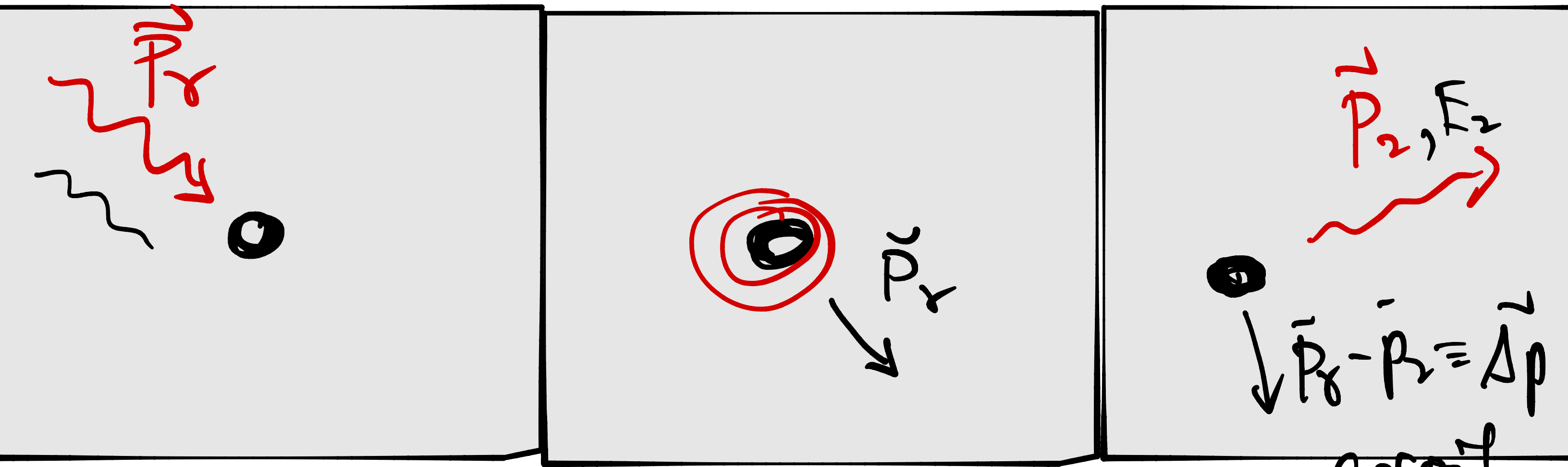


• Mösbauer effect (Bragg scattering...)



Fact:





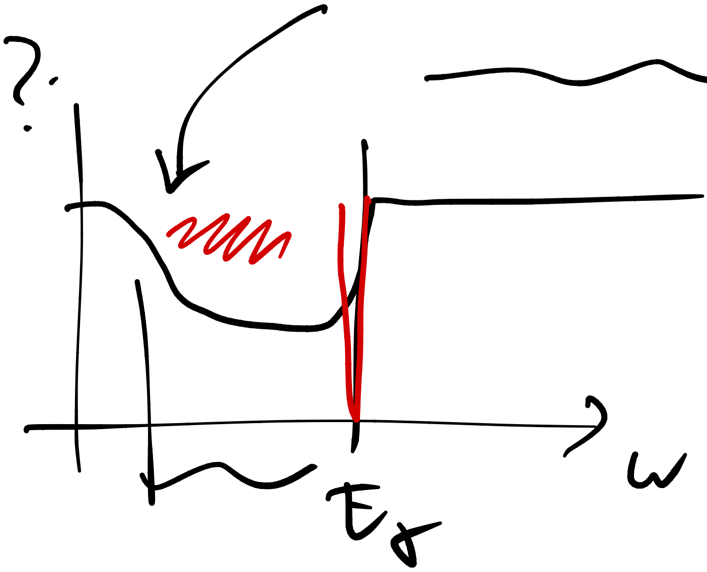
Recoil

$|\vec{p}_\gamma| = E_\gamma / c$

$$E_2 = E_\gamma - \underbrace{\frac{(\Delta p)^2}{2m}}_{\text{KE of atom}}$$

$$|\Delta p|_{max} = 2|p_\gamma|$$

$$E_2 \in \left(E_\gamma - 2\frac{(E_\gamma/c)^2}{m},\ E_\gamma \right)$$



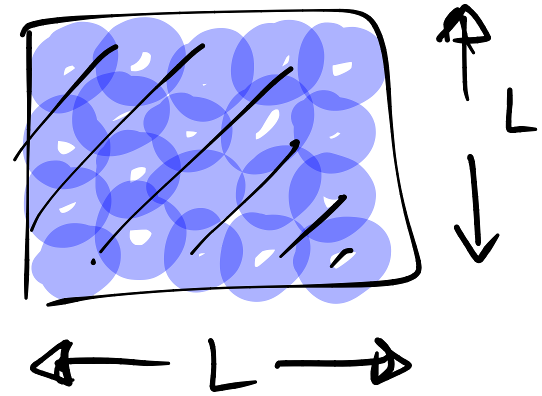
Resolution: Prob(zero phonon process) $\neq 0$.

$M = Nm$

$N \sim 10^{26}$

$$E_2 = E_\gamma - \cancel{\frac{(\Delta p)^2}{2M}}$$

$$\mathcal{H} = \bigotimes_{x\in \text{space}}^{L} \mathcal{H}_x$$



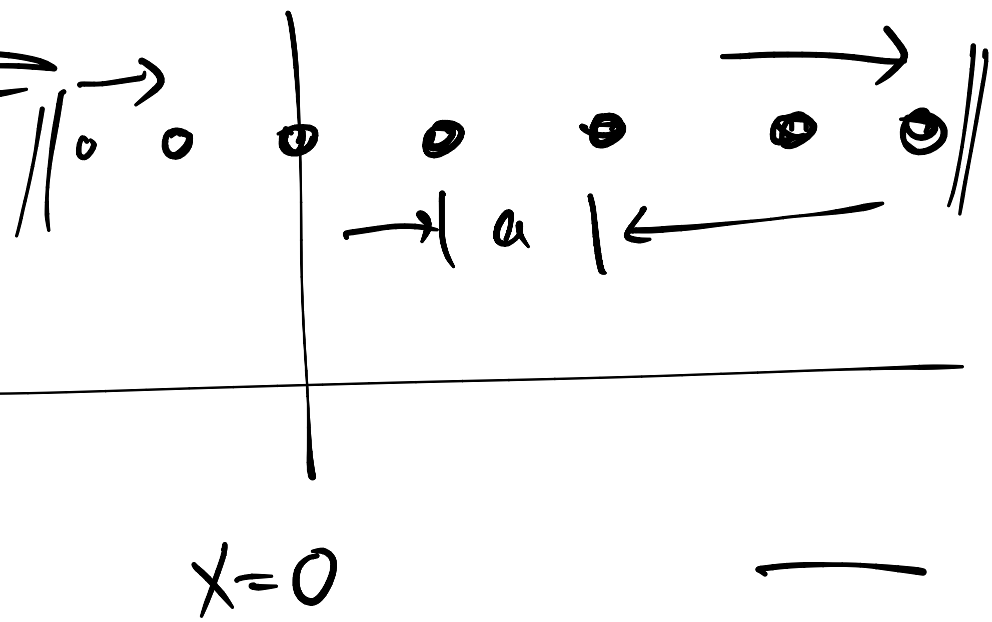
$$H = \sum_{x}^{L} h_x$$

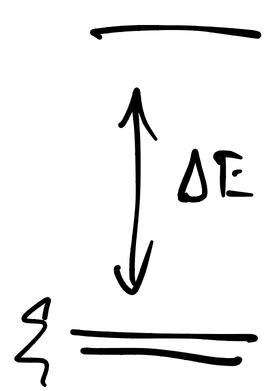
$$\Delta E = E_1 - E_0 \xrightarrow{L\to\infty} \begin{cases} \text{finite, gap} \\ 0, \text{ gapless} \end{cases}$$

$$\Delta E \sim \frac{1}{L} \xrightarrow{L\to\infty} 0$$

$L = Na$.

$N \to \infty$.





$$\sum_n \left((T-1)q_n\right)^2$$

$$q_n = \sum_k \frac{e^{-ikna}}{\sqrt{N}} q_k$$

$$= \sum_{kk'} \boxed{\sum_n \frac{1}{N} e^{-ikna-ik'na} = \delta_{k,-k'}} \underbrace{(T-1)q_k}_{(e^{-ika}-1)q_k} (T-1)q_{k'}$$

$$T q_k = e^{-ika} q_k$$

$$(e^{-ika}-1)q_k (e^{-ik'a}-1)q_{k'}$$

$$= \sum_k q_k q_{-k} \left[(e^{-ika}-1)(e^{+ika}-1)\right]$$

$$\omega_k^2 = \left(e^{-\frac{ika}{2}} - e^{\frac{ika}{2}}\right)\left(e^{\frac{ika}{2}} - e^{-\frac{ika}{2}}\right)$$

$$= -4(i)^2 \sin^2 ka/2$$

$$= 4 \sin^2 \frac{ka}{2}$$

$$[p^2, q] = 2p(-i)$$

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(1) $\dot{q}_n = i[H, q_n] = i\left[\sum_{n'} \frac{p_{n'}^2}{2m}, q_n\right] = p_n/m$

(2) $\dot{p}_n = i[H, p_n] = i\left[(q_n - q_{n-1})^2 + (q_{n+1}-q_n)^2, p_n\right]$

(1) $\downarrow$ $\quad = -\left[2(q_n - q_{n-1}) - 2(q_{n+1} - q_n)\right]$

$= m\ddot{q}_n$