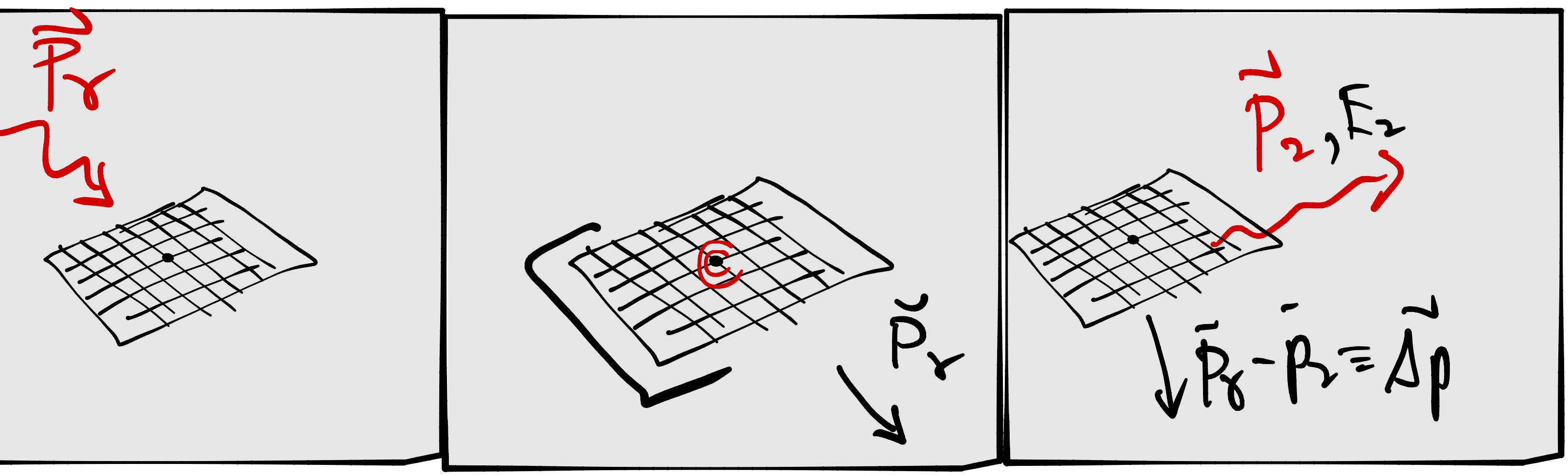
Zero-phonon scattering:



$M_{solid} \sim N_{atoms}$. $\quad E_{solid} = \dfrac{(\Delta p)^2}{2 M_{solid}}$

What is Prob(NO phonons)?

H_{int} small.

Fermi Golden Rule

$$W(N_i L_i \to N_f L_f) \propto \left| \langle \underline{f} | H_{int} | \underline{i} \rangle \right|^2$$

internal state of atoms

positions of atoms.

Dynamics of nuclei are fast $\longrightarrow$

$$W(N_i L_i \longrightarrow N_f L_i) \propto |\langle L_f | H_L | L_i \rangle|^2$$

claim: $H_L = \sum_n A\, e^{i\vec{K}\cdot\vec{X}_n}$

$\tilde{X}_n$ = pos'n of n'th atom.

$\tilde{K}$ = wavevector of incident photon.

$(\vec{P}_\gamma = \hbar\tilde{K}.)$

$$\hat{X}_n = na + \hat{q}_n$$

$$= na + \sum_k N_k \left(e^{ikna} a_k + e^{-ikna} a_k^\dagger\right)$$

$$N_k = \sqrt{\frac{\hbar}{2mN\omega_k}}$$

$|L_i\rangle = |0\rangle$

$|L_f\rangle = |0\rangle.$

$$P(0\text{ phonon}) \propto \left|\langle 0| e^{i\tilde{K}\cdot(\vec{n}a + \tilde{q}_n)} |0\rangle\right|^2$$

gaussian systems : $\underline{\langle 0| e^{i\vec{K}\cdot\vec{q}} |0\rangle = e^{-K_\alpha K_\beta \langle 0| q_\alpha q_\beta |0\rangle}}$

$\Leftarrow$ $\equiv$ ① path integral is gaussian

$$\int Dq\, e^{\int \dot{q}^2 - q^2}.$$

② $H = M_{ij} P_i P_j + V_{ij} Q_i Q_j$

$\langle 0| q_\alpha q_\beta |0\rangle \equiv$ Green's f'n of q_α.

$$\langle 0| q_n^2 |0\rangle = \sum_k \frac{\hbar}{2m\omega_k N}$$

$$= \frac{\hbar}{2mN} \sum_k \frac{1}{\omega_k}$$

$\omega_k \overset{\text{small } k}{\simeq} v_s |k|$

$k = \frac{2\pi}{Na} j$

$\boxed{d=1}$

$$\simeq \frac{\hbar}{N 2m} \left(\sum_{j=1}^{N} \frac{1}{j} \right) \frac{Na}{2\pi}$$

$$= \frac{\hbar a}{2m \cdot 2\pi} \underbrace{\sum_{j=1}^{N} \frac{1}{j}}_{\sim \log N} \propto \log N$$

$\boxed{d=1}$

$$\text{Prob (0 phonons)} \propto e^{-K^2 \cdots \log N} \sim N^{-K^2} \xrightarrow[N\to\infty]{} 0$$

In $d>1$:

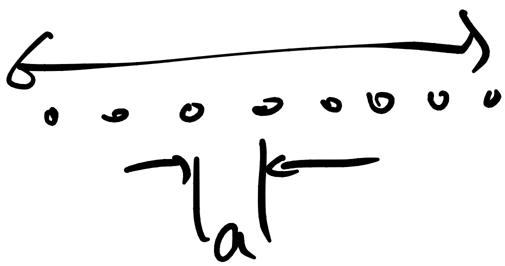
$$\langle q^2\rangle = \frac{\hbar}{2mL^d}\sum_k \frac{1}{\omega_k} \sim \int_{1/Na}^{1/a} d^dk\, \frac{1}{k}$$

finite as $L\to\infty$.

$d=1$

$$\sum_k \frac{1}{k} \simeq \int_{1/Na}^{1/a} \frac{dk}{k} \sim \log(N)$$

$L = Na$.



$$e^{-K^2\langle q^2\rangle}$$

"Debye-Waller factor".

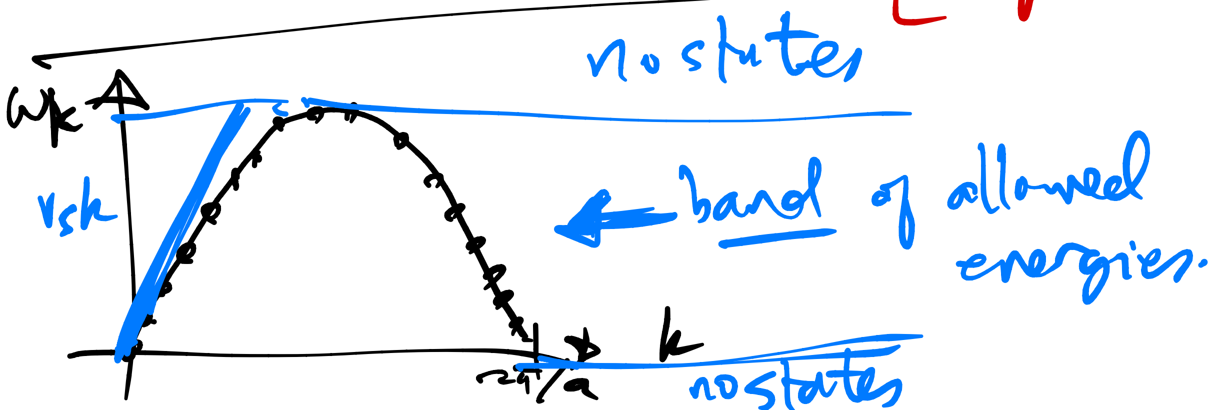
U.V. $\omega_k \sim |\sin ka/2|$

$$\sum_k \frac{1}{\omega_k} \simeq \int_{1/Na}^{1/a} \frac{d^dk}{k}$$

IR $\omega_k \sim k$.

finite if $d>1$

$$\sum_{j_x,j_y,j_z=1}^{N} \frac{1}{\sqrt{j_x^2+j_y^2+j_z^2}} \xrightarrow{N\to\infty} <\infty.$$



1.3 Path Integral Reminder

$$H = \frac{p^2}{2m} + V(q)$$

$$\underline{\langle q| e^{-iHt}} \, |\underline{q_0}\rangle = \int_{q(0)=q_0}^{q(t)=q} [dq]\, e^{iS[q]}$$

$$S[q] = \int_0^t ds \left[m\frac{\dot{q}^2}{2} - V(q) \right]$$

$$\underbrace{[dq] = \prod_{i=1}^{M} dq_i}$$

$$e^{-iHt} = \underbrace{e^{-iH\Delta t}} \wedge \cdots\cdots \wedge \underbrace{e^{-iH\Delta t}} \quad \mathbb{1} = \underline{\int dq_1 |q_1\rangle\langle q_1|}$$

$\underline{\Delta t M = t}$.

$$\mathbb{1} = \int dq_M\, |q_M\rangle\langle q_M|$$

$$\iiint \langle q| e^{-iH\Delta t} |q_M\rangle\langle q_M| e^{-iH\Delta t} |q_{M-1}\rangle\langle q_{M-1}| \cdots \langle q_1| e^{-iH\Delta t} |q_0\rangle$$

$$\underline{\langle q_{i+1}| e^{-i\Delta t H} |q_i\rangle}$$

$$\langle q_{i+1}| \underbrace{e^{-i\Delta t H}} |q_i\rangle$$

$$e^{-i\Delta t\left(\frac{p^2}{2m}+V(q)\right)} \simeq e^{-i\Delta t \frac{p^2}{2m}}\, e^{-i\Delta t V(q)} + O(\Delta t^2)$$

$$\langle q_{i+1}| e^{-i\Delta t \frac{p^2}{2m}} \; e^{-i\Delta t V(q_i)} |q_i\rangle$$

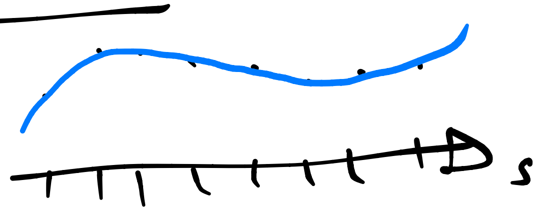
$$\mathbb{1} = \int đp_i\, |p_i\rangle\langle p_i|$$

$$\boxed{\int đp \equiv \int \frac{dp}{2\pi}}$$

Applications: ① Explains E.o.M as stationary phase $\quad 0 = \frac{\partial S}{\partial q_i}$

$q(s_i) \equiv q_i$.

ie: $\boxed{0 = \frac{\delta S}{\delta q(s)}}$



$$\frac{\partial q_i}{\partial q_j} = \delta_{ij} \longrightarrow \frac{\delta q(t)}{\delta q(s)} = \delta(t-s).$$

② Euclidean-time path integral

$$\underbrace{|g.s.\rangle \propto e^{-HT}} \underbrace{|any\rangle}_{\sum_n c_n |E_n\rangle}$$

$H|E_n\rangle = E_n|E_n\rangle$

$$= \sum_n c_n \, e^{-E_n T} |E_n\rangle$$

$$\frac{\text{state } n}{g.s.} = e^{-(E_n - E_0)T} \xrightarrow{T\to\infty} 0$$

$$\Longrightarrow \langle 0|f(q)|0\rangle \propto^{ex} \int [dq] \, f(q(0)) \, e^{-\underbrace{\int_{-T/2}^{T/2} d\tau \left(\frac{m\dot{q}^2}{2} + V(q)\right)}}$$

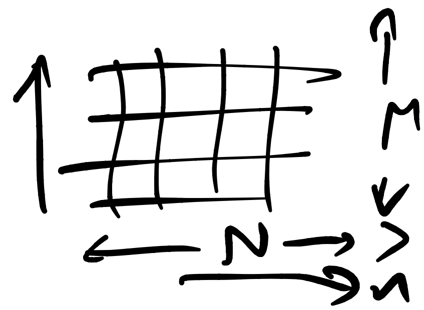
"euclidean action"! $\tau = is$.

eg: $f(q) = e^{iKq}$

1.4 Scalar Field Theory.

$$Z = \int [dq_{n=1}\, dq_{n=2}\, dq_{n=3} \dots dq_N]\, e^{i S[q]}$$

$$[dq_n] \equiv \prod_{i=1}^{M} dq_{n,i}$$



$$S[q] = \int dt \left[\sum_{n=1}^{N} \frac{1}{2} m_n \dot{q}_n^2 - V(q) \right]$$

$$V(q) = \sum_n \frac{1}{2} K (q_n - q_{n-1})^2$$

eom:

$$0 = \frac{\delta S}{\delta q_n(t)} = -m_n \ddot{q}_n - \frac{\partial V}{\partial q_n}$$

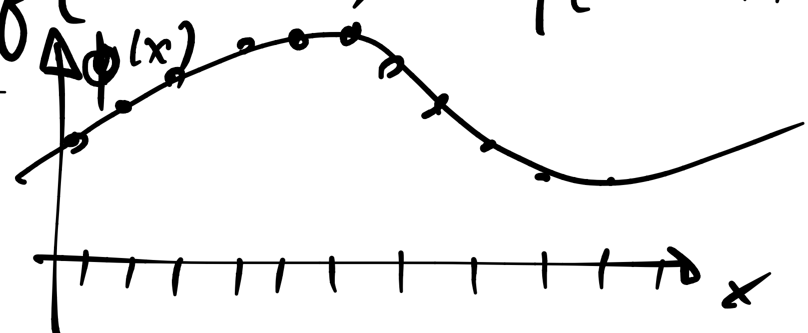
$$= -m_n \ddot{q}_n - K(2q_n - q_{n-1} - q_{n+1})$$

Continuum limit:

$Ka \ll 1$

$N \to \infty.$

$$q_n \equiv q(x = na) \equiv \phi(x = na)$$



"$\underline{ka} \ll 1$" : $\underline{a\partial_x q} \gg (a\partial_x)^2 q$.

$$\begin{cases} (q_n - q_{n-1})^2 \simeq a^2(\partial_x q)^2\Big|_{x=na} + O(a\partial_x)^3 \\ 2q_n - q_{n-1} - q_{n+1} \simeq a^2 \partial_x^2 q. + O(a\partial_x)^3 \end{cases}$$

$$\xrightarrow{\text{eom}} \quad \ddot{q} = \# q'' . \quad \text{(wave equation)}$$

$$Z \to \int [Dq]\, e^{iS[q]}$$

$$Dq \equiv \prod_n^N \prod_i^M dq_{in} .$$

$$S[q] = \int dt \int dx \left[\frac{1}{2}\mu(\partial_t q)^2 - \mu v_s^2 (\partial_x q)^2 - V(q) \right]$$

$$= \int dt\, L$$

$$= \int dt\, dx\, \mathcal{L}$$

Lagrange density.

$$\boxed{V(q) = rq^2 + uq^4 + \ldots}$$

harmonic, $= 0$

translation-invt chain.

$r = u = 0$.

$$\underline{V = V(q_n - q_{n-1})}.$$

$$a\sum_n \rightsquigarrow \int dx$$

Hamiltonian descr. of scalar F.T. : $p_n = \frac{\partial L}{\partial \dot{q}_n}$.

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial_t q(x)} = \mu \partial_t q$$

$$H = \sum_n \left(p_n \dot{q}_n - L_n\right) \rightsquigarrow \int_0^L dx \left(\pi(x)\dot{q}(x) - \mathcal{L}\right)$$

$$= \int dx \left(\frac{\pi(x)^2}{2\mu} + \mu v_s^2 (\partial_x q)^2 + r q^2 + \cancel{v q^4 + \dots} \right)$$

$$q(x) = \sqrt{\frac{\hbar}{2\mu L}} \sum_k \frac{1}{\sqrt{\omega_k}} \left(e^{ikx} a_k + e^{-ikx} a_k^\dagger\right)$$

$$\pi(x) = \frac{1}{i}\sqrt{\frac{\hbar\mu}{2L}} \sum_k \sqrt{\omega_k} \left(e^{ikx} a_k - e^{-ikx} a_k^\dagger\right)$$

$$[q(x), \pi(y)] = i\delta(x-y) \quad \leftrightarrow \quad [a_k, a_{k'}^\dagger] = \delta_{k,k'}.$$

$$\boxed{H = \sum_k \left(a_k^\dagger a_k + \tfrac{1}{2}\right)\hbar\omega_k.}$$

Kronecker.

About $L \to \infty$: $\frac{1}{L}\sum_k \overset{L\to\infty}{\rightsquigarrow} \int dk = \int \frac{dk}{2\pi}$

$[a_k, a_{k'}^\dagger] = \delta_{k,k'} \overset{?}{\rightsquigarrow} 2\pi\delta(k-k')$

Requires replacement $\begin{cases} a_k \longrightarrow \sqrt{L}\, a_k. \\ a_k^\dagger \longrightarrow \sqrt{L}\, a_k^\dagger. \end{cases}$

Continuum free Real scalar field theory in $d+1$ dims :

$$S[\phi] = \int dt \int d^dx \left[\mu (\partial_t \phi)^2 - \mu v_s^2 (\vec{\nabla}\phi \cdot \vec{\nabla}\phi) - V(\phi) \right].$$

$\phi = \phi^\dagger$

$$H = \int d^dx \left(\frac{\pi^2}{2\mu} + \frac{1}{2}\mu v_s^2\, \nabla\phi\cdot\nabla\phi + V(\phi) \right).$$

If $V(\phi) = \frac{m^2}{2}\phi^2$. $\quad 0 = \frac{\delta S}{\delta\phi(x)} = -\partial_t^2\phi - \vec{\nabla}^2\phi - m^2\phi.$

$$\begin{cases} \phi(x) = \frac{1}{\sqrt{L^d}} \sum_k e^{-i\vec{k}\cdot\vec{x}} \phi_k \\ \pi(x) = \frac{1}{\sqrt{L^d}} \sum_k e^{-i\vec{k}\cdot\vec{x}} \pi_k. \end{cases}$$

$$H = \sum_k \left(\frac{1}{2\mu} \pi_k \pi_{-k} + \frac{1}{2}\mu v_s^2 k^2 \phi_k \phi_{-k} + \frac{m^2}{2} \phi_k \phi_{-k} \right)$$

$$\int d^dx \, e^{-i\vec{x}\cdot(\vec{k}+\vec{k}')} = (2\pi)^d \delta^d(\vec{k}+\vec{k}')$$

$$k^2 = (-i\vec{k})\cdot(+i\vec{k}) = \vec{k}\cdot\vec{k}.$$

$$\begin{cases} \phi_k = \sqrt{\dfrac{\hbar}{2\mu\omega_k}} \left(a_k + a_k^\dagger \right) \\ \pi_k = \dfrac{1}{i} \sqrt{\dfrac{\hbar\omega_k\mu}{2}} \left(a_k - a_{-k}^\dagger \right) \end{cases}$$

$$\Rightarrow H = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + 1/2 \right)$$

$$\omega_k^2 = v_s^2 \vec{k}\cdot\vec{k} + m^2.$$

$$\phi(x) = \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k V}} \left(e^{i\vec{k}\cdot\vec{x}} a_k + h.c. \right)$$

$$\pi(x) = \dots.$$

$$[\phi(x), \pi(y)] = i\hbar \mathbb{1} \delta^d(x-y)$$

$$\left([q_n, p_m] = i\hbar \mathbb{1} \delta_{nm}. \right)$$