Discussion section: Thursdays 4pm–5pm.

Quantum light aka photons:

Maxwell's eqn:

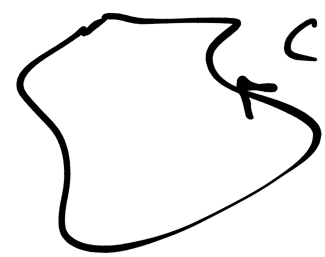
$$\vec\nabla\cdot\vec B = 0 \qquad \vec\nabla\times\vec E = -\partial_t\vec B \qquad \text{constraints}$$

$$\vec\nabla\cdot\vec E = \rho/\varepsilon_0 \qquad c^2\vec\nabla\times\vec B = \partial_t E + \vec J/\varepsilon_0$$

solve constraints: $\vec E = -\vec\nabla\Phi - \partial_t\vec A\,,\quad \vec B = \vec\nabla\times\vec A.$

price: $\begin{cases}\vec A \to \vec A^\lambda = \vec A - \vec\nabla\lambda\,, \\ \Phi \to \Phi^\lambda = \Phi + \partial_t\lambda\end{cases}$ $\longrightarrow$ same $\vec E, \vec B$.

Note: $\oint_C \vec A\cdot d\vec l = \oint_C \vec A^\lambda\cdot d\vec l$



Gauge fix: choose $\vec\nabla\cdot\vec A = 0.$

vacuum: $\rho = \vec J = 0.$ can also choose $\Phi = 0.$

Ampere: $c^2\,\vec\nabla\times(\vec\nabla\times\vec A) = c^2\,\vec\nabla\cdot(\cancelto{0}{\vec\nabla\cdot\vec A}) - c^2\nabla^2\vec A$

$$= -\partial_t^2\vec A$$

$$\boxed{\partial_t^2\vec A - c^2\nabla^2\vec A = 0}$$

Compare to $(\partial_t^2 - v_s^2 \partial_x^2)\phi(x,t) = 0$:

– 3d , – $v_s \to c$, – $\vec{A}$ is a vector

w/ $\vec{\nabla}\cdot\vec{A}$.

linear & transl. inv't $\Rightarrow$ $\vec{A}(x) = \sum_k \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} \vec{A}(k)$

$V = L^3$.

$\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \vec{k}\cdot\vec{A}(k) = 0$.

"$\vec{A}$ is transverse".

energy density:

$$u = \frac{\varepsilon_0}{2}\left(\vec{E}^2 + c^2\vec{B}^2\right)$$

ie

$$H = \frac{\varepsilon_0}{2}\int d^3r\left(\vec{E}^2 + c^2\vec{B}^2\right)$$

$$= \frac{\varepsilon_0}{2}\int d^3r\left(\left(\partial_t\vec{A}\right)^2 + c^2\left(\vec{\nabla}\times\vec{A}\right)^2\right)$$

vs:

$$H = \frac{\mu}{2}\int dx\left[\left(\partial_t\phi\right)^2 + v_s^2\left(\partial_x\phi\right)^2 + \cancel{V(\phi)}\right]$$

$$\phi(x) = \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k V}}\left(a_k e^{i\vec{k}\cdot\vec{x}} + h.c.\right)$$

$[a_k, a^\dagger_{k'}] = \delta_{k,k'}$

$$\vec{A}(x) = \sum_{s,k} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k V}} \Big(\underline{\underline{a_{\vec{k},s}}}\, e^{i\vec{k}\cdot\vec{x}}\, \vec{e}_s(\hat{k})$$

$$+ \underline{\underline{a^\dagger_{\vec{k},s}}}\, e^{-i\vec{k}\cdot\vec{x}}\, \vec{e}_s^{\,*}(\hat{k}) \Big)$$

$\{\vec{e}_s(k),\ s=1,2\}$ ON basis of transverse vectors.

- $\vec{k}\cdot\vec{e}_s(k) = 0 \quad s=1,2.$
- $\vec{e}_s(k)\cdot\vec{e}^{\,*}_{s'}(k) = \delta_{s,s'}$
- $\sum_s (e_s(k))_i\,(e^*_s(k))_j = \underbrace{\overbrace{\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}}^{\Delta_{ij}}}_{\text{Projection onto transverse vectors.}}$

$$\left\{ \begin{array}{l} \Delta_{ij}\Delta_{jk} = \Delta_{ik}. \\ \Delta_{ij} k_j = 0. \end{array} \right\}$$

$$[a_{\vec{k}s}, a^\dagger_{\vec{k}'s'}] = \delta_{kk'}\delta_{ss'}$$

$$[a, a] = 0 = [a^\dagger, a^\dagger].$$

$$\vec{A}(\vec{x},t) = \sum_{s,k} N_k \left(a_{\vec{k},s}\, e^{i\vec{k}\cdot\vec{x} - i\omega_k t}\, \vec{e}_s(\vec{k}) + h.c. \right)$$

classically: integration constants

quantumly: creation operator!

soln of wave eqn. if $\omega_k = c|k|$.

$$\vec{E} = -\partial_t \vec{A} = i \sum_k N_k \omega_k \sum_s \left(a_{\vec{k},s}\, \vec{e}_s(k) e^{ik\cdot x - i\omega_k t} - h.c. \right)$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \sum_{k,s} N_k\, i\vec{k} \times \left(a_{ks}\, \vec{e}_s(k) e^{i(k\cdot x - \omega t)} - h.c. \right)$$

claim:

$$H = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right).$$

check: $-\partial_t^2 \vec{A} = \partial_t \vec{E} = \frac{i}{\hbar}[H, \vec{E}]$

$$\boxed{\partial_t O = \frac{i}{\hbar}[H, O]}$$

$$\underset{\uparrow \text{ HW3}}{=} -c^2 \nabla^2 \vec{A}$$

$$[\phi(x), \pi(y)] = i\hbar\, \delta(x-y)\, \mathbb{1}$$

$\pi = \dot\phi$

$$\rightsquigarrow \quad [A_i(x), E_j(y)] \overset{?}{=} -i\hbar\, \delta^3(x-y)\, \delta_{ij}/\varepsilon_0$$

$$\left(\pi_A = \frac{\partial \mathcal{L}}{\partial \dot A_i} = -\varepsilon_0 E_i \right)$$

actually: $[A_i(x), E_j(y)] = -\frac{i\hbar}{\varepsilon_0} \int d^3k\, e^{i\vec k\cdot(\vec x - \vec y)} \times \underline{(\delta_{ij} - \hat k_i \hat k_j)}$

same when acting on transverse vectors.

Spectrum: $|0\rangle \qquad a_{k,s}|0\rangle = 0 \quad \forall k, s.$

"no photons".

$a^\dagger_{\vec k,s}|0\rangle = |\text{one photon, momentum } \hbar \vec k, \text{ polarization } s\rangle$

$$H(\downarrow) = (E_0 + \hbar\omega_k)\, a^\dagger_{ks}|0\rangle.$$

$$|n_{\vec k_1, s_1}, n_{k_2, s_2} \dots\rangle = \frac{(a^\dagger_{k_1, s_1})^{n_{k_1 s_1}}}{\sqrt{n_{k_1 s_1}!}} \frac{(a^\dagger_{k_2, s_2})^{n_{k_2 s_2}}}{\sqrt{n_{k_2 s_2}!}} \cdots |0\rangle$$

Photons are bosons!

$$a^\dagger_{k_1 s_1} a^\dagger_{k_2 s_2} |0\rangle = |k_1 s_1, k_2 s_2\rangle$$

$$= a^\dagger_{k_2 s_2} a^\dagger_{k_1 s_1} |0\rangle = |k_2 s_2, k_1 s_1\rangle$$

$$E_0 = \sum_k \hbar\omega_k \cdot \frac{1}{2}.$$

photon states

$$H|0\rangle_R = E_0 |0\rangle_R$$

Mossbauer revisited. $\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_L \otimes \mathcal{H}_R$

$$W(N_i L_i R_i \to N_f L_f R_f) \propto |\langle f | H_{int} | i \rangle|^2$$

$$\propto |\langle L_f| \otimes \langle R_f| \, H_{L,R} \, |L_i\rangle \otimes |R_i\rangle|^2$$

$$\langle R_f| = {}_R\langle 0|$$

$$\begin{cases} a_k |0\rangle_R = 0 \\ {}_R\langle 0| a_k^\dagger = 0 . \end{cases}$$

$$|R_i\rangle = |K, s\rangle_R$$

$$= a^\dagger_{K,s} |0\rangle_R .$$

$$H_1 = \frac{1}{2m}(\vec{p} + e\tilde{A}(x))^2 = \frac{1}{2m}(p^2 + epA + eAp + e^2A^2)$$

$$= \frac{p^2}{2m} + H_{LR}$$

$$\hat{A}(\hat{x}_n) \sim \sum_k \left(\hat{a}_k \, e^{ik\cdot\hat{x}_n} + \hat{a}_k^{\dagger} e^{-ik\cdot\hat{x}_n} \right)$$

$$\hat{x}_n = na + \hat{q}_n .$$

$$A^2 \sim \underbrace{aa}_{-2} + \underbrace{a^{\dagger}a + aa^{\dagger}}_{+0} + \underbrace{a^{\dagger}a^{\dagger}}_{+2}$$

$$\langle 0 \text{ photons} | A^2 | 1 \text{ photon} \rangle = 0 .$$

$$\hat{p}\, e^{ik\hat{x}} + e^{ik\hat{x}}\hat{p} = (\hat{p} + \underline{K})\, e^{iK\hat{x}}$$

$$W \propto \left| \langle L_f | \otimes {}_R\langle 0 | \sum_{K'} e^{iK'x} \underbrace{a_{K'} \; a_k^{\dagger}}_{\delta_{kk'}} |0\rangle_R \otimes |L_i\rangle \right|^2$$

$$= \underbrace{\left| \langle L_f | e^{iKx} | L_i \rangle \right|^2}_{e^{-K^2\langle q^2\rangle}} \underbrace{\left| {}_R\langle 0 | 0 \rangle_R \right|^2}_{=1}$$

$$S[x] = \int dt\, \frac{m\dot{x}^2}{2} + e\int dt\, \underbrace{\frac{dx^\mu}{dt} A_\mu(x(t))}_{-\frac{dt}{dt}\Phi + \frac{d\vec{x}}{dt}\cdot\vec{A}}$$

$$A_\mu = (\Phi, \vec{A})_\mu$$

$$= \int dt \left(m\frac{\dot{x}^2}{2} - e\Phi(x(t)) + e\vec{v}\cdot\vec{A} \right)$$

$$\int dt\, \frac{d\vec{x}}{dt}\cdot\vec{A} = \int d\vec{x}\cdot\vec{A}$$

"minimal coupling"

① $0 = \frac{\delta S}{\delta x} \longrightarrow$ Lorentz force law.

② $H = p\dot{x} - L = \frac{(p+eA)^2}{2m}.$

$$m\ddot{x} = e(\vec{E} + \vec{v}\times\vec{B}).$$

$$\dot{x}^2 \sim \sqrt{1-\dot{x}^2}$$

$$H = \sum_{k,s} \hbar\omega_k \left(\underbrace{a^\dagger_{ks} a_{ks}}_{} + \frac{1}{2} \right)$$

↑ # of photons in $\vec{k}, s$.

$\omega_k = c|\vec{k}|$.

$$\partial^\mu \partial_\mu A = 0.$$
$$= (\omega^2 - c^2\vec{k}^2) A_k$$

$$\xrightarrow{L\to\infty} L^3 \sum_s \int d^3k \, \hbar\omega \left(a^\dagger_{k_s} a_{k_s} + \frac{1}{2}\right)$$

$$E_0^{(L)} = \sum_{k,s} \hbar\omega_k \cdot \frac{1}{2}$$
$$\xrightarrow{L\to\infty} \frac{2L^3}{2} \int d^3k \, \hbar\omega_k$$

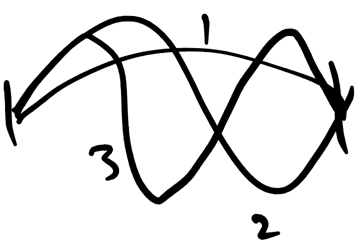
$k = \frac{2\pi l}{L}$, $l \in \mathbb{Z}$

$$F = -\frac{\partial}{\partial L} E_0(L)$$ Casimir force.

1+1 D . scalar. $S[\phi] = \int dt\,dx \left[\frac{1}{2}\left(\dot{\phi}^2 - c^2(\partial_x \phi)^2\right) \right]$

$$|\leftarrow \underbrace{d}_{} \rightarrow | \leftarrow L-d \rightarrow |$$

$L \gg d$ suppose: $\phi(0) = \phi(d) = \phi(L)$.



LEFT: $\phi_j = \sin \frac{j\pi x}{d}$ $j = 1, 2, 3 \ldots$

$\omega_j = \frac{\pi |j|}{d} \cdot c$

RIGHT: $d \to L - d$.

$$E_0(d) = f(d) + f(L-d)$$

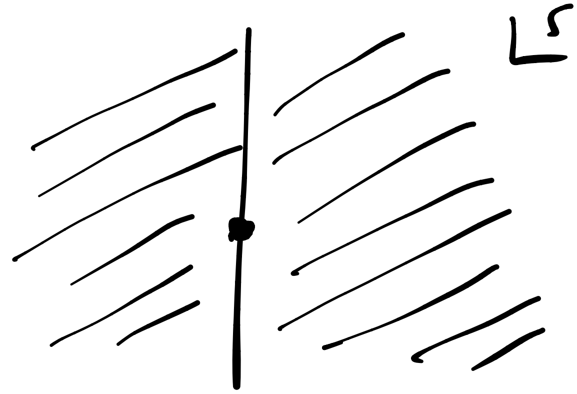
$$f(d) = \frac{1}{2}\hbar \sum_{j=1}^{\infty} \omega_j = \hbar c \frac{\pi}{2d} \sum_{j=1}^{\infty} j = \infty$$

$$f(d) \rightsquigarrow f(a,d)$$

eg: $= \hbar \frac{\pi}{2d} \sum_{j=1}^{\infty} j\, e^{-a\omega_j/\pi}$

...

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$$



$$\zeta(-1) = -\frac{1}{12}.$$