Casimir Effect, cont'd:

scalar in 1d.

$$\phi(0) = \phi(d) = \phi(L) = 0.$$

$$\phi(x) = \sum_{n=1,2,3} \sin \frac{2\pi j x}{d} \qquad k_j = \frac{2\pi j}{d}$$

$$| \leftarrow d \rightarrow | \leftarrow L-d \rightarrow |$$

$$E_0(d) \longrightarrow F = -\frac{\partial}{\partial d} E_0(d)$$

$$\langle 0 | \vec{E} | 0 \rangle = \langle 0 | \sum_k \cdots (a - a^\dagger) | 0 \rangle = 0.$$

$$\langle 0 | E_k^2 | 0 \rangle \neq 0.$$

$$\Rightarrow \Delta_0 E = \sqrt{\langle 0 | E_k^2 | 0 \rangle - \left(\langle 0 | E_k | 0 \rangle\right)^2} \neq 0.$$

$$E_0(d) = f(d) + f(L-d)$$

$$f(d) = \frac{1}{2} \sum_k \hbar \omega_k = \frac{1}{2} \frac{2\pi}{d} \hbar c \sum_{j=1}^{\infty} j = \infty.$$

$$\omega_k = c|k| = c\frac{2\pi j}{d}$$

Declare: $\omega_j > \frac{\pi}{a}$ is "large".

$$f(d) \overset{\text{fiction}}{\rightsquigarrow} f(a,d) = \frac{\hbar\pi c}{2d}\sum_{j=1}^{\infty} j\, e^{-\frac{a\omega_j}{\pi}}$$

$$\boxed{\sum_j j\, e^{-\frac{aj}{d}} = -d\frac{\partial}{\partial a}\sum_{j=0}^{\infty} e^{-\frac{aj}{d}}}$$

$$= \frac{\hbar\pi c}{2d}\sum_{j=1}^{\infty} j\, e^{-\frac{aj}{d}}$$

$$= -\frac{\pi\hbar c}{2}\,\partial_a\Big(\underbrace{\sum_{j=0}^{\infty} e^{-\frac{aj}{d}}}_{\frac{1}{1-e^{-a/d}}}\Big)$$

$$= \frac{\pi\hbar c}{2d}\frac{e^{a/d}}{(e^{a/d}-1)^2}.$$

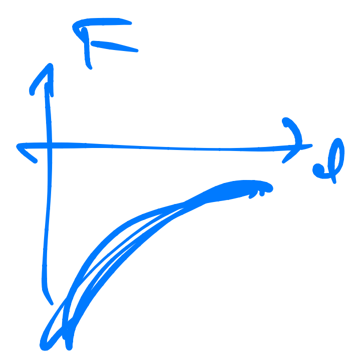
$$\overset{a\ll d}{\simeq} \hbar c\left(\underbrace{\frac{\pi d}{2a^2}}_{\to\infty \text{ as } a\to 0} - \underline{\frac{\pi}{24d}} + \underbrace{\frac{\pi a^2}{480d^3}}_{\to 0 \text{ as } a\to 0} + \dots\right)$$

$$F = -\partial_d E_0 = -\left(\underline{f'(d)} - f'(L-d)\right)$$

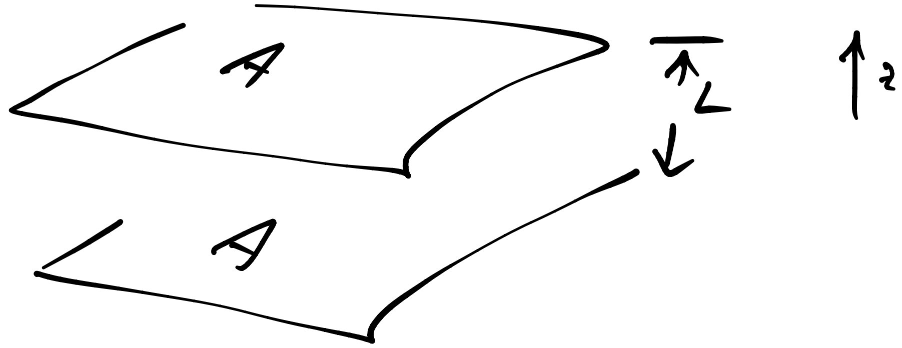
$$= -\hbar c\left(\left(\underline{\frac{\pi}{2a^2}} + \frac{\pi}{24d^2} + \mathcal{O}(a^2)\right) - \left(\underline{\frac{\pi}{2a^2}} + \frac{\pi}{24(L-d)^2} + \mathcal{O}(a^2)\right)\right)$$

$$\overset{a\to 0}{=} -\frac{\pi\hbar c}{24}\left(\frac{1}{d^2} - \frac{1}{(L-d)^2}\right)$$

$$\overset{L\gg d}{\simeq} -\frac{\pi\hbar c}{24d^2}\left(1 + \mathcal{O}\left(\frac{d}{L}\right)\right)$$



Real Thing:



dim'l analysis:

$$\frac{F}{A} = P = \alpha \frac{\hbar c}{L^4}.$$

$A = L_x L_y$

claim: $\alpha \neq 0$.

- PBC in x, y. $\quad \vec{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, k_z \right)$

$n_{x,y} \in \mathbb{Z}$.

- perfect conductor:

$$E_{x,y}(z=0, z=L) = 0. \qquad (E \perp \text{conductor}).$$

if $k_z = 0$: $\quad E \propto \hat{z} \quad$ (one polarization) $n=0$

if $k_z \neq 0$: $\quad E \propto \sin \frac{2\pi n z}{L} \quad n = 1, 2, \ldots$

$k_z = \frac{2\pi n}{L}$. (2 polarizations)

$$\omega_n(k) = c\sqrt{\frac{\pi^2 n^2}{L^2} + \vec{k}^2} \qquad n = 0, 1, 2.$$

$$E_0(L) = \frac{\hbar}{2}\left(2 \sum_{n,k}{}' \omega_n(k)\right)$$

$$\sum_{n,k}{}' = \frac{1}{2}\sum_{n=0,k} + \sum_{\substack{n=1,2\ldots \\ \vec{k}}}.$$

$$\sum_{n,k}{}' \cdot \rightsquigarrow$$

$$\sum_{n,k}{}' \cdot e^{-a\omega_n(k)/\pi}.$$

1.5 Identical particles

$$|n \text{ photons in } \vec{k}, \alpha\rangle = \frac{(a^\dagger_{k,\alpha})^n}{\sqrt{n!}}|0\rangle$$

wave #

polarization

$$\underset{A_1}{p(k_1\alpha_1}\ \underset{A_2}{k_2\alpha_2}\ \ldots\ \underset{A_n}{k_n\alpha_n)} \overset{!}{=} p(\underset{A_{\pi_1}}{k_{\pi_1}\alpha_{\pi_1}},\ \underset{A_{\pi_2}}{k_{\pi_2}\alpha_{\pi_2}}\ \ldots\ \underset{A_{\pi_n}}{k_{\pi_n}\alpha_{\pi_n})}$$

$\pi \in S_n$ permutation

$$(1 2 \ldots n) \longrightarrow (\pi_1\ \pi_2 \ldots \pi_n)$$

A = single-particle labels.

$$p(A_1 \ldots A_n) = |\psi(A_1 \ldots A_n)|^2$$

$$\Rightarrow \psi(A_1 \ldots A_n) = e^{i\theta}\,\psi(A_{\pi_1} \ldots A_{\pi_n})$$

2d quantized:

$$\Psi(A_1 \ldots A_n) = \langle A_1 \ldots A_n|\Psi\rangle$$

$$= \langle 0|a_{A_1} \ldots a_{A_n}|\Psi\rangle$$

i.e. $|A_1 A_2 \ldots A_n\rangle = a^\dagger_{A_1} a^\dagger_{A_2} \ldots a^\dagger_{A_n}|0\rangle$.

$$[a^\dagger_A, a^\dagger_B] = 0 \Rightarrow \text{bosons.}$$

$$\psi(A_2 A_1 \ldots A_n) = a\, \psi(A_1, A_2 \ldots A_n) \qquad \underline{a \in U(1)}$$

$$\psi(A_1 A_2 \ldots A_n) = a^2\, \psi(A_1 A_2 \ldots A_n)$$

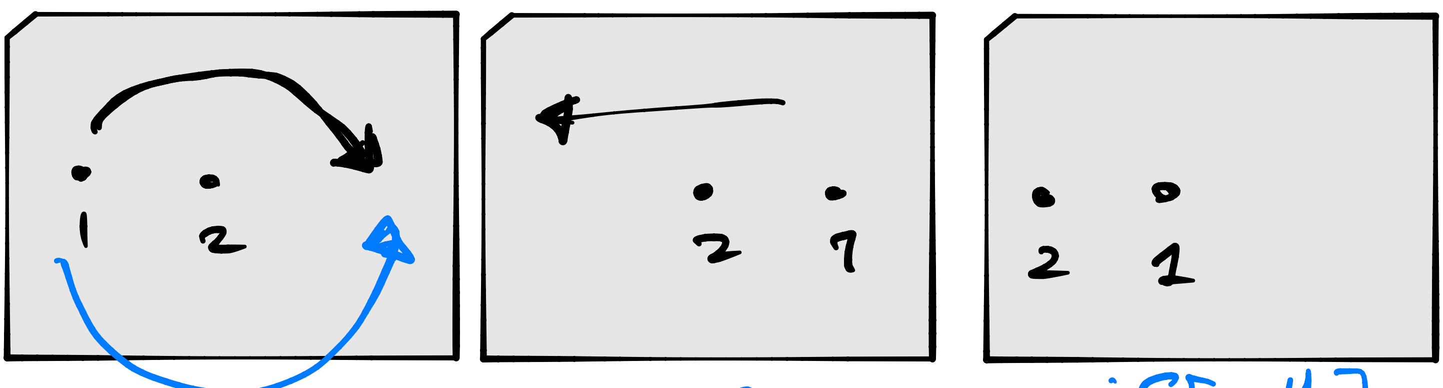
$$\Rightarrow a^2 = 1 \quad \Rightarrow \underline{a = \pm 1}.$$

$a = +1$ Bosons $\quad a = -1$ Fermions.

$\boxed{D = 2+1}$

ANYONS.

exchange 1+2 =
- move 1 by π around 2
- translate.



Path Integral $\quad Z_2 = \int (\text{paths})\, e^{iS[\text{path}]}$

$$= \sum_n \int \left(\substack{\text{paths} \\ \text{w/} \\ \text{n braids}}\right) e^{iS}\, e^{in\theta}$$

$$\psi(x_1, x_2) = \sum_n \int_{\substack{x_1(0)=x_1 \\ x_2(0)=x_2}} D\text{paths}\; e^{iS}\, e^{in\theta}$$

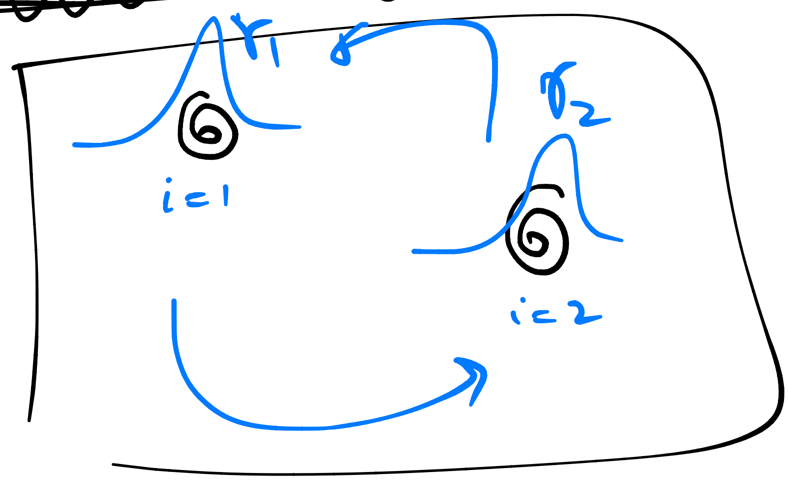
$$\psi(x_2, x_1) = e^{i\theta}\, \psi(x_1, x_2).$$

if $\exists$ degenerate states of n particles $\gamma = 1..D$

$$|A_2 A_1 \cdots A_n \gamma\rangle = \hat{U}_{\gamma\gamma'} |A_1 A_2 \dots A_n \gamma'\rangle$$

$$[U_{12}, U_{34}] \neq 0 \Rightarrow$$ non-abelian anyons.

eg: Ψ:



$$\{\hat\gamma_i, \hat\gamma_j\} = \delta_{ij}$$
$$= \hat\gamma_i \hat\gamma_j + \hat\gamma_j \hat\gamma_i$$
$$\hat\gamma = \hat\gamma^\dagger$$ ← Majorana zeromodes

$$\Psi_B(A_1 \dots A_N) = \Psi_B(A_{\pi_1} \dots A_{\pi_N})$$

$$\Psi_F(A_1 \dots A_N) = \underbrace{(-1)^\pi}\, \Psi_F(A_{\pi_1} \dots A_{\pi_N})$$

-1 for odd permutations = odd # of 2-particle exchanges

eg: $123 \to 213$ is odd

$123 \to 231$ is even.

$\underline{\mathcal{H}^{(N)}_{B,F}} \subset \underline{\mathcal{H}_1^{\otimes N}}$ If $D \equiv \dim \mathcal{H}_1$.

$\dim(\Psi) = D^N$

$\mathcal{H}_1 = \mathrm{Span}\{|A\rangle_1\}$
$A = 1..D$

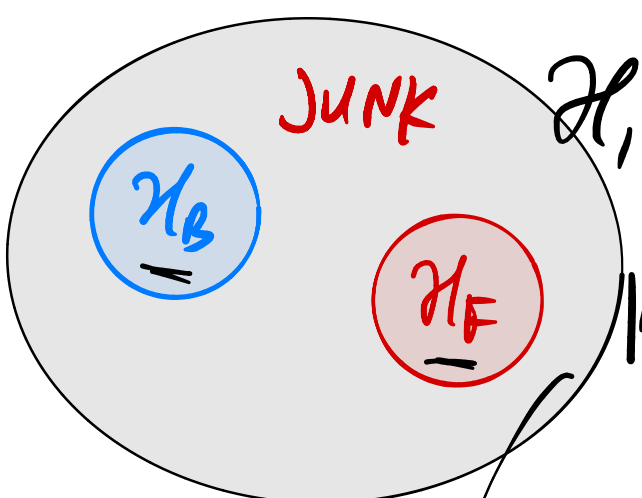
eg: $= \mathrm{span}\{|x=1..L\rangle \otimes |\sigma\rangle,\ \sigma = \uparrow,\downarrow\}$

$$\dim \mathcal{H}_F = \binom{D}{N} = \frac{D!}{N!(D-N)!}.$$

Pauli exclusion principle: $\boxed{\Psi_F(A_1, A_1) = 0}$.

$$\dim \mathcal{H}_B = \frac{(N+D-1)!}{N!(D-1)!} = \binom{N+D-1}{N}$$

$$\dim \mathcal{H}_B + \dim \mathcal{H}_F \underset{N>1}{<} D^N$$



1st quantized POV:

Q: which particle is in in which state?

$$|A,B\rangle_{B/F} = |A\rangle_1 \otimes |B\rangle_2 \pm |B\rangle_1 \otimes |A\rangle_2$$

2nd quantized POV:

Q: How many particles are in each state?

$$|n_A = 1, n_B = 1\rangle_{B/F}$$

(Newspeak.)

$$\Psi^{AB}_{B/F}(x_1, x_2) = \langle x_1 x_2 | A, B \rangle_{B/F}$$

$$= u_A(x_1) u_B(x_2) \pm u_B(x_1) u_A(x_2)$$

$\langle x | A \rangle = u_A(x)$ 1-particle wave fn.

N=3 : $\Psi^{ABC}_F(x_1, x_2, x_3) = \det M$ "Slater determinant"

$$M = \begin{pmatrix} u_A(x_1) & u_A(x_2) & u_A(x_3) \\ u_B(x_1) & u_B(x_2) & u_B(x_3) \\ u_C(x_1) & u_C(x_2) & u_C(x_3) \end{pmatrix}$$

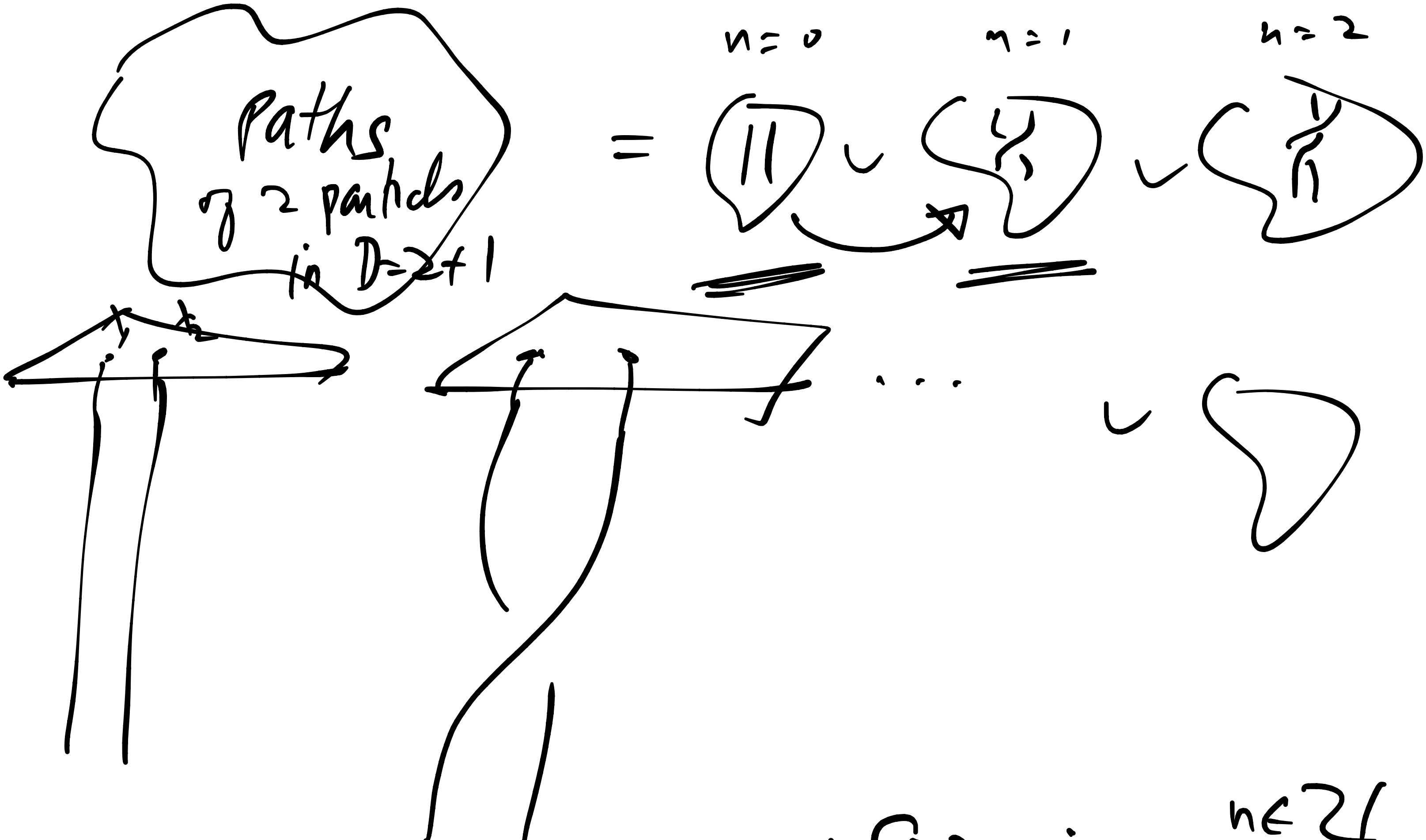
$$\det M = \sum_{\pi} (-1)^{\pi} M_{1\pi_1} M_{2\pi_2} \cdots M_{N\pi_N}$$

$\Longrightarrow$ ✓

$$\Psi^{ABC}_B(x_1, x_2, x_3) = \sum_{\pi} M_{1\pi} \cdots M_{N\pi_N}$$

$$= \operatorname{per}(M).$$

"permanent"



$$\psi(x_1,x_2) = \int_{\substack{x_1(t)\to x_1 \\ x_2(t)\to x_2}} (\text{paths})\, e^{iS[x]}\, e^{in\theta} \qquad \begin{array}{l} n\in\mathbb{Z} \\ \theta\in[0,2\pi) \end{array}$$

$$S[x] = \int m\dot{x}^2 + \dots \quad n^2$$

$$\psi(x_1, x_2) = e^{-in\theta}\,\psi(x_2, x_1).$$

Berry's phase.

$$H = \sum_x \left(\underbrace{\pi^2 + (\phi_x - \phi_{x-1})^2}_{= a^\dagger_i M_{ij} a_j} + \lambda \phi^4 \right)$$

$$a^4 + a^\dagger a^3 + \ldots$$

M_{ij} is $L \times L$.

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$\leftarrow L \rightarrow$

$$a^\dagger_k a_k |n_k\rangle = n_k |n_k\rangle$$

$$n_k = 0, 1, 2 \ldots N_{max}$$

$$\dim \mathcal{H} = L^{N_{max}}$$