Many fermions

$$\mathcal{H}_1 = \text{span}\{|A\rangle\}$$

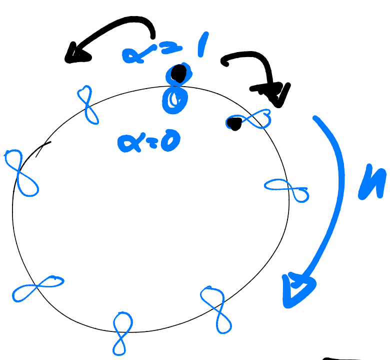
$$\Psi_F(A_1, A_2, \dots A_n) = -\Psi_F(A_2, A_1 \dots A_n)$$

$\Rightarrow$ Pauli principle $\quad \Psi_F(A, A, \dots) = 0.$

$\Rightarrow$

Band structure : metals vs Insulators

Consider $\mathcal{H}_1 = \text{span}\{|n\rangle \otimes |\alpha\rangle \;\; n = 1..N \;, \alpha = 0, 1\}$



$$|n+N\rangle = |n\rangle$$

α

$$H_1 = -t \sum_n \left(|n+1\rangle\langle n| + |n\rangle\langle n+1| \right) \otimes \mathbb{1}$$

$$+ \sum_n |n\rangle\langle n| \otimes \left(\epsilon |1\rangle\langle 1| + 0 |0\rangle\langle 0| \right)$$

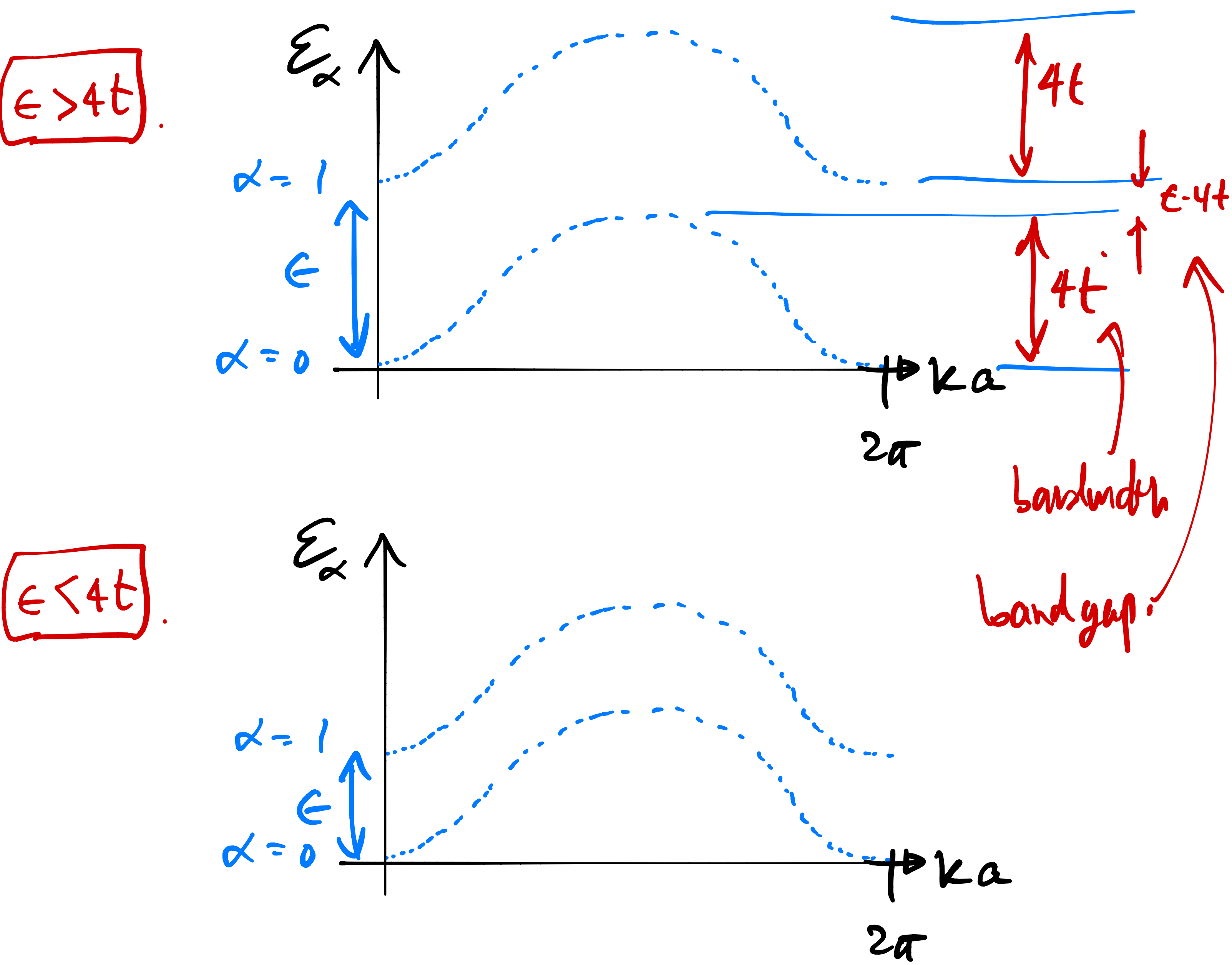
$$= H_t + H_\epsilon .$$

$[H_t, H_\epsilon] = 0.$

eigenstates: $|k\rangle\otimes|\alpha\rangle = \frac{1}{\sqrt{N}}\sum_n e^{ikan}|n\rangle\otimes|\alpha\rangle$

evals: $\varepsilon_\alpha(k) = (2-2\cos ka)t + \epsilon\,\delta_{\alpha,1}$

$N\to\infty$: $k_j = \frac{2\pi j}{Na}$, $j=1..N$



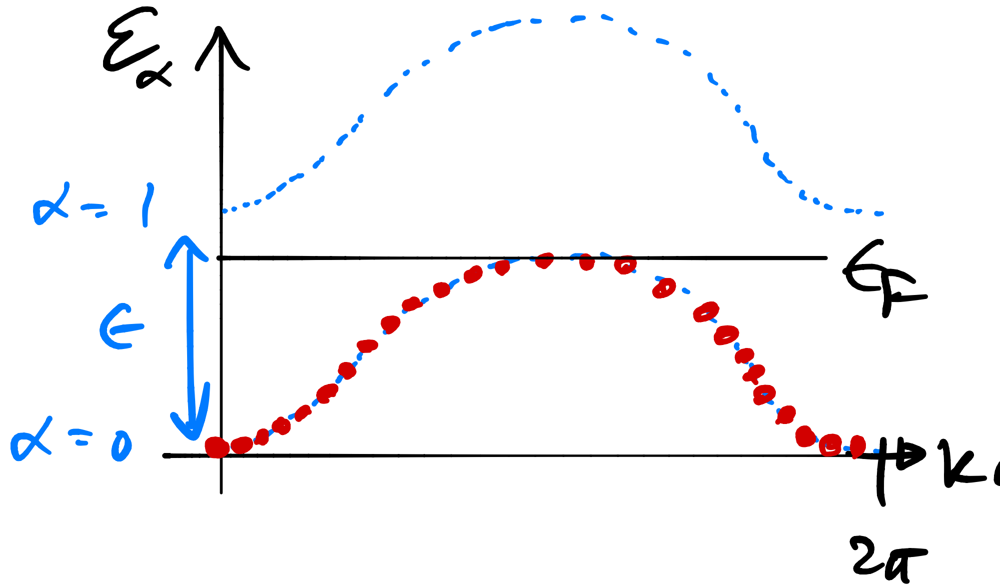
Many fermions:

$$H = \sum_i H_1(i)$$

eigenvalues are

$$\sum_{i=1}^{n} \varepsilon_{\alpha_i}(k_i)$$

$$\{k_1\alpha_1 \ldots k_n\alpha_n\}$$



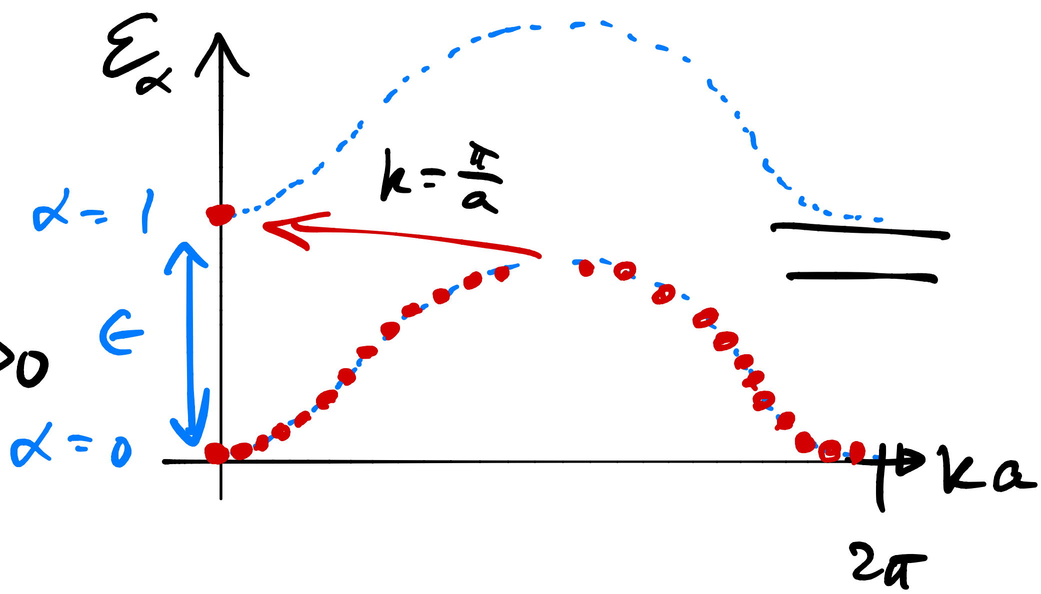
Half filling: # of electrons = # of sites = N.

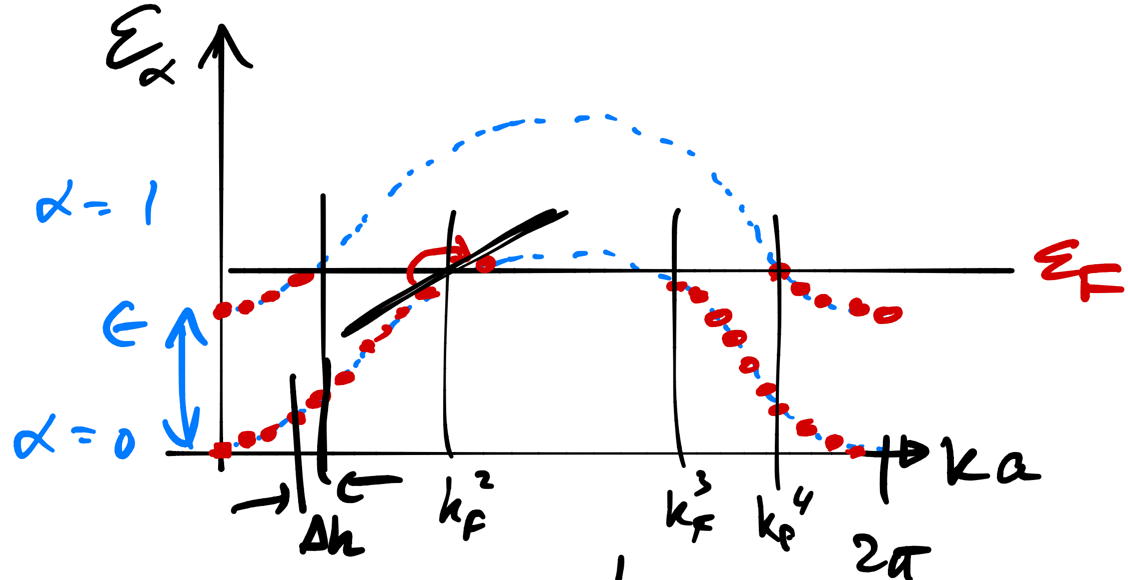
First excited state:

$$\Delta E \xrightarrow{N\to\infty} \epsilon - 4t > 0$$

ind. of N

ie **energy gap**.

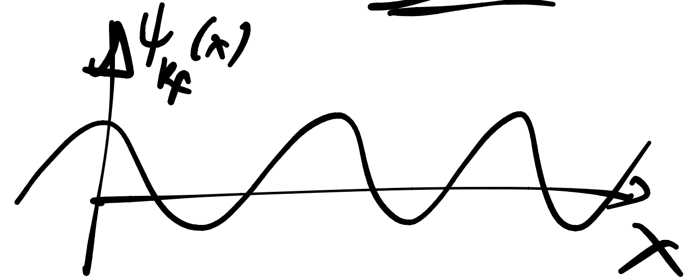




$$\Delta E = \underbrace{\Delta k}_{=\frac{2\pi}{Na}} \underbrace{\partial_k \mathcal{E}(k)\Big|_{\mathcal{E}_F}}_{v_F} \propto \frac{1}{N} \xrightarrow{N\to\infty} 0$$

gapless

Conduction requires that the states near ϵ_F are **extended**. → **metal**.



$$j_x = \mathrm{Im}\, \psi^*_{k_F} \partial_x \psi_{k_F} \neq 0$$

So far $\sigma_{DC} = \infty$.

To fix this:
- phonons
- impurities in the lattice.

$$\frac{d\vec{p}}{dt} = q\vec{E} \qquad \vec{j} = \frac{e\vec{p}}{m} \overset{?}{=} \sigma\vec{E}$$

$$\underbrace{}\ -i\omega\vec{p} = q\vec{E} \qquad \Rightarrow \vec{j} = \frac{e}{m}\frac{q}{i\omega}\cdot\vec{E}$$

$$p = \frac{qE}{i\omega} \qquad \sigma_{DC} = \lim_{\omega\to 0}\left(\frac{eq}{m\omega}\right) = \infty$$

Fermionic creation & annihilation operators

$$|0\rangle \equiv |\text{no electrons}\rangle$$

$$c_{n\alpha}|0\rangle \overset{!}{=} 0. \qquad c^{\dagger}_{\underline{n\alpha}}|0\rangle \overset{!}{=} \underline{|n\rangle\otimes|\alpha\rangle}$$

Recall: $f_{AB} = [a_A, a_B^{\dagger}] = \cancel{a_A a_B^{\dagger} - a_B^{\dagger} a_A}$.

$[a_A^{\dagger}, a_B^{\dagger}] = 0 \qquad a_A^{\dagger} a_B^{\dagger}|0\rangle = +a_B^{\dagger} a_A^{\dagger}|0\rangle.$

Pauli: $(c^{\dagger}_{n\alpha})^2 = 0. \quad c^2_{n\alpha} = 0. \quad \underline{\forall n,\alpha}$

$f_{AB} = c_A c_B^{\dagger} + c_B^{\dagger} c_A.$

$\underbrace{\{c_A^\dagger, c_B^\dagger\}}_{\text{anti-commutator}} = 0 \implies c_A^\dagger c_B^\dagger|0\rangle = -c_B^\dagger c_A^\dagger|0\rangle$

$= c_A^\dagger c_B^\dagger + c_B^\dagger c_A^\dagger.$

$$\begin{cases} \{c_{n\alpha}, c^\dagger_{n'\alpha'}\} = \delta_{nn'}\delta_{\alpha\alpha'}. \\ \{c_{n\alpha}, c_{n'\alpha'}\} = 0. \end{cases}$$

<u>One Mode</u>. $\mathcal{H}_1 = |\cdot\rangle$

$c^2 = 0, \ \{c, c^\dagger\} = 1.$

$0 = c|0\rangle = c|\downarrow\rangle.$ $\quad c^\dagger|\downarrow\rangle = |1\rangle = |\uparrow\rangle$

$\mathcal{H}_{many} = \text{span}\{|0\rangle, |1\rangle\} = \text{span}\{|\downarrow\rangle, |\uparrow\rangle\} = $ a <u>single qbit</u>.

$(c^\dagger)|\uparrow\rangle = (c^\dagger)^2|\downarrow\rangle = 0.$

$[c^\dagger c, c] = -c, \quad [c^\dagger c, c^\dagger] = +c^\dagger.$

$N = c^\dagger c$

$c^\dagger c |\downarrow\rangle = 0 |\downarrow\rangle$

$c^\dagger c |\uparrow\rangle = 1 |\uparrow\rangle$

eigenstates of $N = c^\dagger c$

<u>ex:</u>

Basis of hermitian ops on a qbit : $\mathbb{1}, \sigma^x, \sigma^y, \sigma^z$.

$\sigma^x = c + c^\dagger$, $\sigma^y = \frac{c - c^\dagger}{i}$, $\sigma^z = 2c^\dagger c - 1$
$= 2N - 1$

$\sigma^i \sigma^j = i \varepsilon^{ijk} \sigma^k$.

$N = \frac{\sigma^z}{2} - \mathbb{1}$.

$\sigma^+ = c^\dagger$, $\sigma^- = c$.

$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

More generally $\qquad D = \dim \mathcal{H}_1.$

$$\dim \mathcal{H}_{many} = 2^D = \sum_{N=0}^{D} \dim \mathcal{H}_F^{(N)}$$

$$= \sum_{N=0}^{D} \binom{N}{D}$$

$$= (1+1)^D .$$

Back to <u>tight-binding model</u>:

$$H_\epsilon = \epsilon\left(\#\text{ of electrons w/ } \alpha=1\right)$$

$$= \epsilon \cdot \sum_n c^\dagger_{\alpha n} c_{\alpha n} \delta_{\alpha=1}$$

$$\underline{|n+1\rangle\langle n|} = c^\dagger_{n+1} |0\rangle\langle 0| c_n$$

$$H_t = -t \sum_n \left(\underline{c^\dagger_{n\alpha}}\, \underline{c_{n+1,\alpha}} + c^\dagger_{n+1,\alpha} c_{n,\alpha} \right)$$

$$H_{many} = -t \sum_{n,\alpha} \left(c^\dagger_{n\alpha} c_{n+1\alpha} + h.c. \right)$$

$$+ \sum_{n,\alpha\beta} \epsilon_{\alpha\beta}\, c^\dagger_{n\alpha} c_{n\beta}$$

interaction

$$= \sum_{AB=1}^{D} c_A^\dagger h_{AB} c_B \quad + \sum_{ABCD} c^\dagger_A c_B c^\dagger_C c_D \; V_{ABCD}$$

$$H_{many} = H_{many}^\dagger \iff h_{AB} = (h^\dagger)_{AB}$$

Diagonalize:

$$h_{AB} = U_{AK}\, \varepsilon_K\, U^\dagger_{KB}$$

$$\text{ie} \quad h_{AB} U_{BK} = \varepsilon_K U_{AK} .$$

$$\Rightarrow H_{many} = \sum_K \varepsilon_K c^\dagger_K c_K$$

$$c_K \equiv \sum_A (U^\dagger)_{KA} c_A .$$

eg:

$$c_{k\alpha} = \sum_n \frac{e^{iknа}}{\sqrt{N}} c_{n\alpha}$$

$$\boxed{\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}} \Rightarrow H_{many} = \sum_{k\alpha} c^{\dagger}_{k\alpha} c_{k\alpha} \varepsilon_{\alpha}(k)$$

$$E(k_1\alpha_1 \dots k_n\alpha_n) = \sum_{i=1}^{n} \varepsilon_{\alpha_i}(k_i).$$

Groundstate:

$$|gs\rangle = \prod_{\{k,\alpha \text{ with the smallest } \varepsilon_\alpha(k)\}}^{n} c^{\dagger}_{k\alpha} |0\rangle$$

$$\Psi(n_1\alpha_1 \dots n_N\alpha_N) = \langle n_1\alpha_1 \dots n_N\alpha_N | gs \rangle$$

$$= \langle 0 | c_{n_1\alpha_1} \dots c_{n_N\alpha_N} | gs \rangle$$

$$\overset{\dots}{=} \det \begin{pmatrix} u_{k_1\beta_1}(n_1\alpha_1) & u_{k_1\beta_1}(n_2\alpha_2) & \dots \\ u_{k_2\beta_2}(n_1\alpha_1) & u_{k_2\beta_2}(n_2\alpha_2) & \dots \\ \vdots & & \end{pmatrix}$$

$$= \det_{\ell j}\left(\frac{e^{i k_j \ell a}}{\sqrt{N}} \right).$$

$$u_{k\beta}(n\alpha) = \frac{e^{ikna}}{\sqrt{N}} f_{\alpha\beta}$$

$$M_{\ell j} = \frac{e^{i k_j \ell a}}{\sqrt{N}}.$$

1.8 2d quantiz'n, from scratch

creation ops for general 1-particle states

$$\varphi(k) = \langle k | \varphi \rangle \qquad \mathbb{1} = \sum_k |k\rangle\langle k|$$

$$|\varphi\rangle = \sum_k |k\rangle\langle k|\varphi\rangle = \sum_k \varphi(k)|k\rangle.$$

let $a^\dagger(\varphi) \equiv \sum_k a_k^\dagger \varphi(k)$.

$$|k\rangle = a_k^\dagger |0\rangle \Rightarrow a^\dagger(\varphi)|0\rangle = \sum_k \varphi(k)\underbrace{a_k^\dagger|0\rangle}_{|k\rangle} = |\varphi\rangle.$$

If $[a_k, a^\dagger_{k'}]_s \equiv a_k a^\dagger_{k'} - s\, a^\dagger_{k'} a_k = \underline{\underline{\delta_{kk'}}}$

$s = \pm 1$ for $\frac{\text{Bosons}}{\text{Fermions}}$. $= \langle k' | k \rangle$.

then $a(\varphi_1)\, a^\dagger(\varphi_2) - s\, a^\dagger(\varphi_2)\, a(\varphi_1)$

$$= \langle \varphi_2 | \varphi_1 \rangle .$$

and $[a(\varphi_1), a(\varphi_2)]_s = 0.$

One-body operators.

$\mathcal{H}_1 = \text{span}\{ |A\rangle \}$ ON basis.

$$\mathcal{O}_1 = \sum_{AB} |A\rangle\langle B|\, \mathcal{O}_{AB}$$

$\mathbb{1}_1 = \sum_A |A\rangle\langle A|$ $\quad \mathbb{1}_1 = \sum_B |B\rangle\langle B|$ $\qquad \mathcal{O}_{AB} = \langle A | \mathcal{O}_1 | B \rangle$

$$|u_1 \dots u_N\rangle = \sum_\pi s^\pi \, |u_{\pi_1}\rangle \otimes |u_{\pi_2}\rangle \otimes \dots |u_{\pi_N}\rangle$$

$$\mathcal{O}|u_1 \ldots u_N\rangle = \sum_{\pi} s^{\pi} \Big[\underbrace{\mathcal{O}_1 |u_{\pi_1}\rangle}\, |u_{\pi_2}\rangle \cdots |u_{\pi_N}\rangle$$
$$+ |u_{\pi_1}\rangle\, \mathcal{O}_1 |u_{\pi_2}\rangle \cdots$$
$$+ \cdots$$
$$+ |u_{\pi_1}\rangle \otimes |u_{\pi_2}\rangle \cdots \mathcal{O}_1 |u_{\pi_N}\rangle \Big]$$

$$\mathcal{O}_1 |u\rangle$$
$$= \sum_{AB} \mathcal{O}_{AB} |A\rangle\langle B|u\rangle$$

$|A\rangle\langle B|$

Replace B with A.

$a^{\dagger}(A)\,a(B)$

$$\Rightarrow \boxed{\mathcal{O} = \sum_{AB} a^{\dagger}(A)\,a(B)\,\mathcal{O}_{AB}}$$

eg: $H_1 = \frac{p^2}{2m}$. $\quad \mathcal{H}_1 = \operatorname{span}\{|p\rangle\}_{p \in \mathbb{R}}$

$$H_n = \sum_{i=1}^{n} \frac{p_i^2}{2m}. = \Pi_n \mathbf{H}\, \Pi_n$$

$$\mathbf{H} = \sum_p a_p^{\dagger} a_p \frac{p^2}{2m}.$$

Π_n = projector onto $\mathcal{H}_n$. //

end 826

$$|AB\rangle = c_A^\dagger c_B^\dagger |0\rangle$$

$$|\text{one particle in } A, \text{ one particle in } B\rangle = -c_B^\dagger c_A^\dagger |0\rangle$$

$$\overset{!}{=} -|BA\rangle$$

$$\Longleftarrow \quad c_A^\dagger c_B^\dagger + c_B^\dagger c_A^\dagger = 0.$$

$$= \{c_A^\dagger, c_B^\dagger\}$$

$$\mathcal{H}_1 = \text{span}\{|A\rangle\}$$

$$(c^\dagger)^2 = 0 \qquad\qquad N = c^\dagger c.$$

$$\Rightarrow \begin{cases} [c^\dagger c, c] = -c \\ [c^\dagger c, c^\dagger] = +c^\dagger \end{cases} \qquad \begin{cases} c|0\rangle = 0 \\ c^\dagger |0\rangle = |1\rangle. \end{cases}$$

$$A = 1..D \rightarrow \quad 2^D = \dim \mathcal{H}.$$

$$\boxed{\{c, c^\dagger\} = \mathbb{1}} \Rightarrow cc^\dagger = \mathbb{1} - \underline{c^\dagger c}.$$

$$\underline{c^\dagger c} + \underline{c c^\dagger}$$

Demand:

$|0\rangle \to 0$ $\quad$ $|0\rangle \to |0\rangle|\lambda|^2$

$|1\rangle \to |1\rangle|\lambda|^2$ $\quad$ $|1\rangle \to 0$

$$= |1\rangle\langle 1| + |0\rangle\langle 0| = \mathbb{1}.$$

$$(c)^\dagger = c^\dagger.$$

$$\left\{ \begin{array}{l} c \to \lambda c \\ c^\dagger \to \lambda^* c \end{array} \right.$$

$$\{c, c^\dagger\} = |\lambda|^2 \mathbb{1}$$

$$N = \frac{c^\dagger c}{|\lambda|^2}.$$

$|0\rangle$. $\quad$ $\underline{c|0\rangle = 0}$. $\quad$ $(c^\dagger)^2 = 0$.

$c^\dagger|0\rangle = |1\rangle$.

$$\Rightarrow \mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}.$$

$$\langle 1|0\rangle \overset{?}{=} 0 \qquad \langle 1|0\rangle = \langle 0|c|0\rangle = 0 \checkmark$$

$$\underline{\langle 1|1\rangle = 1}. \qquad \langle 1|1\rangle = \langle 0|cc^\dagger|0\rangle$$

$$= \langle 0|(\mathbb{1} - \underline{c^\dagger c})|0\rangle$$

$$= \underline{\langle 0|0\rangle}.$$

$$\underbrace{\langle 0| e^{ikq(0)} |0\rangle} = \int [Dq]\, e^{-S[q]}\, e^{ikq(0)}$$

$$= \int \prod_{i=-M}^{M} dq_i \; e^{-\sum_i (q_i - q_{-i})^2 - \sum_i^M V(q)}$$

$$e^{ikq_0}$$

if

$$V(q) = k_{ij}\, q_i q_j$$

$$Z[J] = \int \prod dq \; e^{-q_i M_{ij} q_j + q_i J_i}$$

$$\boxed{J_i = \delta_{i,0}\, ik. \qquad M_{ij} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & & \end{pmatrix} + k_{ij}}$$

$$Z[J] = \frac{1}{\sqrt{\det M}}\, e^{-\frac{1}{2} J M^{-1} J}$$

$$\langle e^{ikq} \rangle \propto e^{-\frac{1}{2} k^2 \underbrace{(M^{-1})_{00}}}$$

$$\langle q^2(0) \rangle = \left(\frac{\partial}{\partial J_0}\right)^2 \ln Z[J] \Big|_{J=0} = (M^{-1})_{00}$$

$$e^{ikq} = e^{ikN(a+a^\dagger)}$$

$$\overset{BCH}{=} \underline{\underline{e^{ika^\dagger N} e^{ikaN}}}\, e^{-k^2 \underbrace{[q, a^\dagger]}_{\propto N^2}}$$

$$\underbrace{\langle 0 | e^{ika^\dagger}}_{\langle 0 |} \underbrace{e^{ika} | 0 \rangle}_{| 0 \rangle} \underline{e^{-k^2 N^2}} .$$

$$N^2 = \langle q^2 \rangle .$$