Recap: field operator $\psi_s^\dagger(r) = \sum_p u_p^*(r)\, a_{ps}^\dagger$

$$u_p(r) = \frac{e^{ipr}}{\sqrt{V}}.$$

F: $\{\psi(r), \psi(r')^\dagger\} = \delta_{r,r'}$.

$$H = \sum_x \frac{\nabla\psi^\dagger \cdot \nabla\psi}{2m} \Rightarrow \text{g.s. is}$$

$$|\Phi_0\rangle = \prod_{p<p_F} a_{ps}^\dagger |0\rangle$$

$a_{ps}|0\rangle = 0$

for N fermions, $p_F \propto \left(\frac{N}{V}\right)^{1/d}$.

$$= n^{1/d}.$$

$$n_{ps} = \langle \Phi_0 | a_{ps}^\dagger a_{ps} | \Phi_0 \rangle = \begin{cases} 1 & p < p_F \\ 0 & p > p_F. \end{cases}$$

Correlations.

$$\langle \rho(r) \rangle_{\Phi_0} = \sum_s \langle \Phi_0 | \psi_s^\dagger(r) \psi_s(r) | \Phi_0 \rangle$$

$$= \sum_{s p p'} \underbrace{u_p^*(r) u_{p'}(r)} \underbrace{\langle \Phi_0 | a_{ps}^\dagger a_{p's} | \Phi_0 \rangle}_{\delta_{pp'} n_{ps}}$$

$$= \frac{1}{V} \sum_{ps} n_{ps} = n.$$

equal-time Green's function $\equiv$ oneparticle density matrix:

$$G_s(r,r') \equiv \langle \Phi_0 | \psi_s^+(r)\, \psi_s(r') | \Phi_0 \rangle$$

$$= \frac{1}{V} \sum_{pp'} e^{-ipr + ip'r'} \underbrace{\langle \Phi_0 | a_{ps}^+ a_{p's} | \Phi_0 \rangle}_{\delta_{pp'} n_{ps}}$$

$$= \frac{1}{V} \sum_p e^{-ip\cdot(r-r')} n_{ps}$$

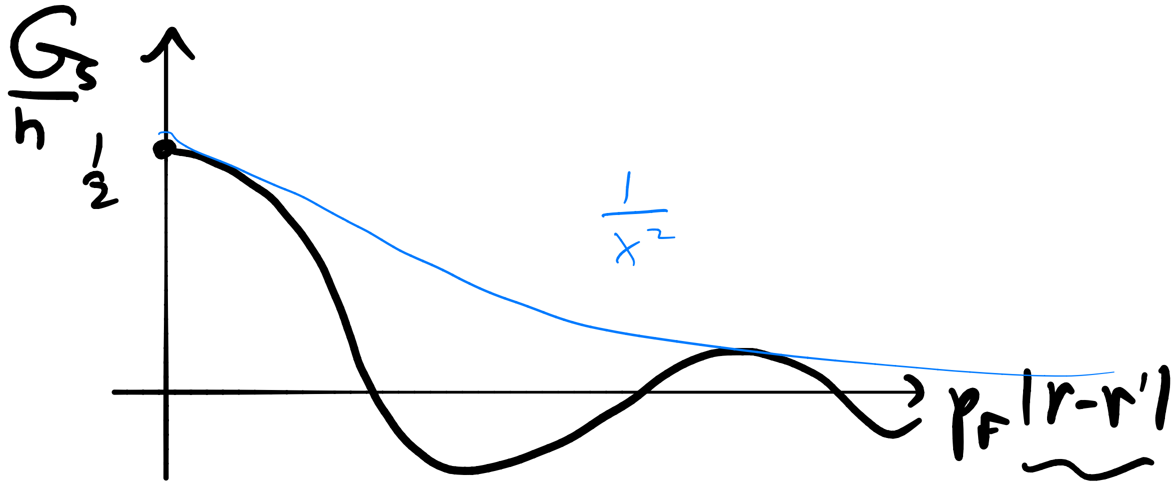
$$\overset{V\to\infty}{=} \int_0^{p_F} d^d p \; e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}$$

$$= \int_0^{p_F} dp\, p^2 \cdot \frac{2\pi}{(2\pi)^3} \underbrace{\int_{-1}^{1} d\mu \; e^{-ip|r-r'|\mu}}_{= 2\frac{\sin p|r-r'|}{p|r-r'|}}$$

$$dp\, p^2\, d\theta\, d\varphi \sin\theta$$
$$= dp\, p^2\, d\varphi\, d\cos\theta$$
$$\mu = \cos\theta$$

$$= \frac{3n}{2} \left(\frac{\sin x - x\cos x}{x^3} \right)$$

$x \equiv p_F |r - r'|$.



Pair correlation f'ns:

$$\mathrm{Prob}_{\Phi_0}\left(\text{particle at } r' \,\middle|\, \text{particle at } r\right) = ?$$

$$|\Phi(r,s)\rangle_{n-1} = \psi_s(r)\,|\Phi_0\rangle_n$$

$$= \mathrm{Prob}_{\Phi(rs)}\left(\text{particle at } r' \text{ with spin } s'\right)$$

$$\mathrm{Prob}_{\Phi}\left(\begin{matrix}\text{particles}\\ \text{at } r' r_2 \ldots r_n\\ \text{w/ } s' s_2 \ldots s_n\end{matrix}\right) = \left|\langle \underset{s' \;\ldots\; s_n}{r' r_2 \ldots r_n} | \Phi \rangle\right|^2$$

$$= \sum_{r_2 \ldots r_n} \left|\langle \underset{s' \;\ldots\; s_n}{r' r_2 \ldots r_n} | \Phi(rs)\rangle\right|^2$$

$$= \sum_{r_2 \cdots r_n} \langle \Phi(rs) | \underbrace{r' r_2 \cdots r_{n-1} \rangle \langle r' r_2 \cdots r_{n-1}}_{\psi^{\dagger}_{s'}(r') | r_2 \cdots r_{n-1} \rangle \langle r_2 \cdots r_{n-1} | \psi_{s'}(r')} | \Phi(rs) \rangle$$

$$= \langle \Phi(rs) | \psi^{\dagger}_{s'}(r') \underbrace{\sum_{\substack{r_2 \cdots r_n \\ s \cdots s}} | r_2 \cdots r_{n-1} \rangle \langle r_2 \cdots r_{n-1} |}_{= \mathbb{1}_{n-2}} \underbrace{\psi_{s'}(r') | \Phi(rs) \rangle}_{n-2}$$

$$= \langle \Phi(rs) | \psi^{\dagger}_{s'}(r') \psi_{s'}(r') | \Phi(rs) \rangle$$

$$= \langle \Phi_0 | \psi^{\dagger}_{s}(r) \psi^{\dagger}_{s'}(r') \psi_{s'}(r') \psi_{s}(r) | \Phi_0 \rangle$$

$$= \left(\frac{n}{2}\right)^2 g_{ss'}(r - r')$$

$$= \frac{1}{V^2} \sum_{pp'qq'} e^{-i(p-p')\cdot r - i(q-q')\cdot r'} \times$$

$$\underbrace{\langle \Phi_0 | a^{\dagger}_{ps} a^{\dagger}_{qs'} a_{q's'} a_{p's} | \Phi_0 \rangle}$$

$\boxed{s \neq s'}$: $p = p'$, $q = q'$.

$$\langle \Phi_0 | a^\dagger_{ps} a^\dagger_{q s'} a_{q' s'} a_{p' s} | \Phi_0 \rangle$$

$$= \delta_{pp'} \delta_{qq'} \langle \Phi_0 | a^\dagger_{ps} a_{ps} \, a^\dagger_{qs'} a_{qs'} | \Phi_0 \rangle$$

$$= \delta_{pp'} \delta_{qq'} \, n_{ps} n_{qs'}$$

$$g_{ss'}(r - r') = \left(\frac{2}{n}\right)^2 \frac{1}{V^2} \sum_{ps} n_{ps} n_{qs'}$$

$$= \left(\frac{2}{n}\right)^2 \cdot \frac{1}{V^2} N_s N_{s'} = 1 .$$

$$N_s = \frac{N}{2} .$$

Boring.

$\boxed{s = s'}$: $p = p', q = q'$ or $p = q', q = p'$ $\} \; p \neq q$

$$\langle \Phi_0 | a^\dagger_{ps} a^\dagger_{qs'} a_{q's'} a_{p's} | \Phi_0 \rangle$$

$$= \delta_{pp'} \delta_{qq'} \langle a^\dagger_{ps} a^\dagger_{qs} a_{qs} a_{ps} \rangle + \delta_{pq'} \delta_{qp'} \langle a^\dagger_{ps} a^\dagger_{qs} a_{ps} a_{qs} \rangle$$

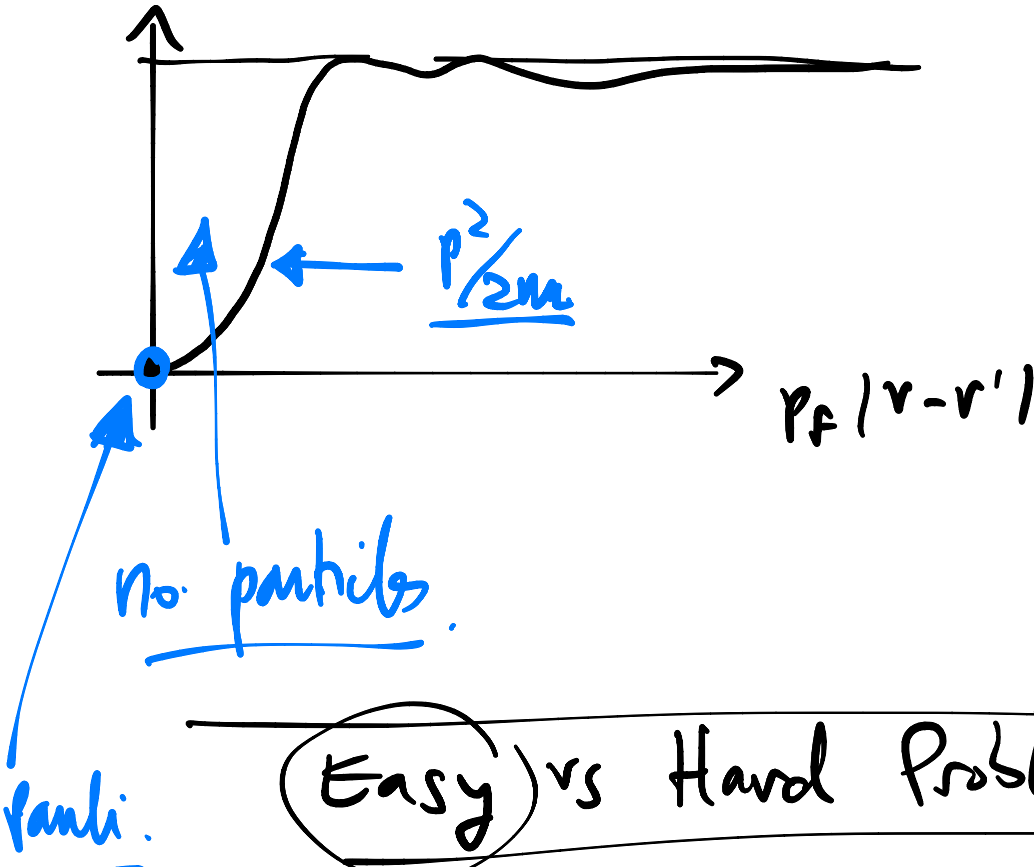
$$= \left(\delta_{pp'} \delta_{qq'} - \delta_{pq'} \delta_{qp'}\right) \langle \underbrace{a^\dagger_{ps} a_{ps} a^\dagger_{qs} a_{qs}}_{= n_{ps} n_{qs}} \rangle$$

$$\to \left(\frac{n}{2}\right)^2 g_{ss}(r-r') = \frac{1}{V^2} \sum_{pq} n_{ps} n_{qs} \left(1 - e^{-i(p-q)(r-r')}\right)$$

$$= \left(\frac{n}{2}\right)^2 - \left(G_s(r-r')\right)^2$$

$$g_s(r-r') = 1 - \frac{9}{x^6}\left(\sin x - x\cos x\right)^2$$

$$x = p_F|r-r'|$$



Fermions avoid each other.

$$H = \sum_i^n \frac{p_i^2}{2m}$$

Easy vs Hard Problems

Non-interacting/Free/Gaussian systems:

$$H = \sum_{ij} a_i^\dagger a_j h_{ij} \qquad i,j = 1\ldots D = \dim \mathcal{H}_1.$$

(not nec. transl. inv't)

with $[a_i, a_j^\dagger]_\pm = a_i a_j^\dagger \mp a_j^\dagger a_i = \delta_{ij}$.

and $[a_i, a_j] = 0$. Diagonalize h_{ij} (small):

$$h_{ij} u_\alpha(i) = \epsilon_\alpha u_\alpha(i)$$

$\{u_\alpha(i) \text{ are ON}\} \Longleftrightarrow u_\alpha^i = u_\alpha(i)$ is unitary.

$$\begin{cases} a_i = a(i) = \sum_\alpha u_\alpha(i) a(u_\alpha) \\ a(u_\alpha) = \sum_i u_\alpha^+(i) a_i \end{cases}$$ "normal modes".

$$\Rightarrow H = \sum_\alpha \epsilon_\alpha a^+(u_\alpha) a(u_\alpha)$$

$$\boxed{a(u_\alpha)|0\rangle = 0 \ \forall \alpha.}$$

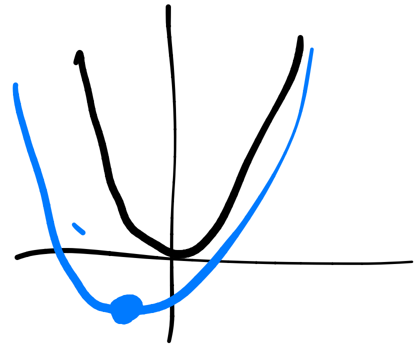
eigenstats of H:

$$|\psi\rangle = a^+(u_1) \cdots a^+(u_n)|0\rangle$$

$$H|\psi\rangle = \sum_{\alpha=1}^{n} \epsilon_\alpha |\psi\rangle.$$

$P_N H P_N = H_N$ is a $D^N \times D^N$ matrix.

$$\Delta H = \sum_i \left(\underline{J_i a_i + a_i^{\dagger} J_i^{*}} \right)$$



also easy.

$$\Delta H = \sum \left(\lambda_i a_i^3 + h.c. \right)$$

<u>hard</u>.

<u>Wick's theorem</u>.

$$a_x = \sum_\alpha u_\alpha(x)\, a(u_\alpha)$$

$$G_\psi(x,y) = \langle \psi | a_x^{\dagger} a_y | \underline{\psi} \rangle$$

$$= \sum_{\alpha\beta} \underbrace{\langle \psi | a_\alpha^{\dagger} a_\beta | \psi \rangle}_{\delta_{\alpha\beta} n_\alpha} u_\alpha^{*}(x)\, u_\beta(y)$$

$$= \sum_{\alpha \in \psi} u_\alpha^{*}(x)\, u_\alpha(y).$$

$$C = \langle \psi | a_x^\dagger a_{x'}^\dagger a_y a_{y'} | \psi \rangle$$

$$= \sum_{\alpha\beta\gamma\delta} \langle \psi | a_\alpha^\dagger a_\beta^\dagger a_\gamma a_\delta | \psi \rangle \, u_\alpha^*(x) u_\beta^*(x') u_\gamma(y) u_\delta(y')$$

$\alpha = \gamma$
$\beta = \delta$

$\alpha = \delta$
$\beta = \gamma$

$\alpha \neq \beta$

$(a_\alpha^\dagger)^2 \overset{F}{=} 0$

$$= \sum_{\alpha \neq \beta} \Big[\langle \psi | a_\alpha^\dagger a_\beta^\dagger a_\alpha a_\beta | \psi \rangle \, u_\alpha^*(x) u_\beta^*(x') u_\alpha(y) u_\beta(y')$$

$$+ \langle \psi | a_\alpha^\dagger a_\beta^\dagger a_\beta a_\alpha | \psi \rangle \, u_\alpha^*(x) u_\beta^*(x') u_\beta(y) u_\alpha(y') \Big]$$

$$= \sum_{\alpha \neq \beta \in \psi} \Big(- u_\alpha^*(x) u_\alpha(y) u_\beta^*(x') u_\beta(y') + u_\alpha^*(x) u_\alpha(y') u_\beta^*(x') u_\beta(y) \Big)$$

$$= - \left(\boxed{\sum_{\alpha \in \psi} u_\alpha^*(x) u_\alpha(y)} \right) \left(\sum_{\beta \in \psi} u_\beta^*(x') u_\beta(y') \right)$$

$$+ \left(\sum_{\alpha \in \psi} u_\alpha^*(x) u_\alpha(y') \right) \left(\sum_{\beta \in \psi} u_\beta^*(x') u_\beta(y) \right)$$

$$C = -G_\psi(xy)G_\psi(x'y') + G_\psi(xy')G_\psi(x'y)$$

Gaussian distr: $\langle ijkl\rangle = \langle ij\rangle\langle kl\rangle + \dots$

Wick's thm: In states like $|\psi\rangle$

$$\langle\psi|\, a^\dagger a \quad a^\dagger a\, |\psi\rangle$$

$$= \sum (-1)^{\#\text{ of crossings}} \left[\text{contractions}\right]$$

$$= \langle a^\dagger \; a^\dagger a \; a\rangle + \langle a^\dagger a^\dagger a \; a\rangle$$

Correlators of bosons (no Wick!)

$$|\psi\rangle = \frac{1}{\prod_\alpha \sqrt{n_\alpha^\psi!}}\, b^\dagger(u_1)\dots b^\dagger(u_n)|0\rangle$$

$b(u_\alpha)|0\rangle = 0.$

$$\hat N_\alpha|\psi\rangle = \hat b^\dagger_\alpha \hat b_\alpha|\psi\rangle = n^\psi_\alpha|\psi\rangle.$$

$\boxed{0! = 1.}$

$b_\alpha \equiv b(u_\alpha).$

$$\left(|n\rangle = \frac{(a^\dagger)^n|0\rangle}{\sqrt{n!}}\right)$$

$$\langle b_x^\dagger b_y \rangle_\Psi = \sum_{\alpha\beta} u_\alpha^*(x)\, u_\beta(y) \underbrace{\langle \Psi | b_\alpha^\dagger b_\beta | \Psi \rangle}_{\delta_{\alpha\beta} n_\alpha^\Psi}$$

$$= \sum_\alpha u_\alpha^*(x)\, u_\alpha(y)\, n_\alpha^\Psi .$$

$\uparrow$ can be > 1.

same as Fermi.

$$B = \langle \Psi | b_x^\dagger b_{x'}^\dagger b_y b_{y'} | \Psi \rangle$$

$$= \sum_{\alpha\beta\gamma\delta} \langle \Psi | b_\alpha^\dagger b_\beta^\dagger b_\gamma b_\delta | \Psi \rangle\, u_\alpha^*(x)\, u_\beta^*(x')\, u_\gamma(y)\, u_\delta(y').$$

(i) $\alpha = \beta = \gamma = \delta$

$$= \sum_\alpha \underbrace{\langle \Psi | (b_\alpha^\dagger)^2 b_\alpha^2 | \Psi \rangle}_{n_\alpha^\Psi (n_\alpha^\Psi - 1)} u_\alpha^*(x)\, u_\alpha^*(x')\, u_\alpha(y)\, u_\alpha(y')$$

(ii) $\alpha \neq \beta$, $\alpha = \gamma$, $\beta = \delta$

$$+ \sum_{\alpha \neq \beta} \underbrace{\langle \Psi | b_\alpha^\dagger b_\beta^\dagger b_\alpha b_\beta | \Psi \rangle}_{n_\alpha^\Psi n_\beta^\Psi} u_\alpha^*(x)\, u_\beta^*(x')\, u_\alpha(y)\, u_\beta(y')$$

(iii) $\alpha \neq \beta$, $\alpha = \delta$, $\beta = \gamma$

$$+ \sum_{\alpha \neq \beta} \underbrace{\langle \Psi | b_\alpha^\dagger b_\beta^\dagger b_\beta b_\alpha | \Psi \rangle}_{n_\alpha^\Psi n_\beta^\Psi} u_\alpha^*(x)\, u_\beta^*(x')\, u_\beta(y)\, u_\alpha(y')$$

$$\langle n|(b^\dagger)^2 b^2|n\rangle = \langle n|b^\dagger b^\dagger b b|n\rangle$$
$$= \langle n|b^\dagger (b b^\dagger - 1) b|n\rangle$$
$$= \langle n|b^\dagger b (b^\dagger b - 1)|n\rangle$$
$$= n(n-1).$$

$$\sum_{\alpha\neq\beta} n_\alpha n_\beta f_{\alpha\beta} = \sum_{\alpha,\beta} n_\alpha n_\beta f_{\alpha\beta} - \sum_\alpha n_\alpha^2 f_{\alpha\alpha}.$$

$$B = \left(\sum_\alpha u_\alpha^*(x) u_\alpha(y) n_\alpha^\psi\right)\left(\sum_\beta u_\alpha^*(x') u_\alpha(y') n_\alpha^\psi\right)$$
$$+ \left(y \leftrightarrow y' \right)\left(y \leftrightarrow y' \right)$$
$$- \sum_\alpha n_\alpha (n_\alpha + 1) u_\alpha^*(x) u_\alpha^*(x') u_\alpha(y) u_\alpha(y').$$

$$= G_\psi(x,y)\, G_\psi(x'y') \overset{F\to B}{+} G_\psi(xy')\, G_\psi(x'y)$$

extra term $\rightarrow$ $- \sum_\alpha n_\alpha (n_\alpha + 1)\, u\, u\, u\, u$ NO wick! except $n_\alpha^\psi = 0$.