Hard Problems: Transverse-Field Ising Model.

$$H_{TFIM} = -J\left[\sum_x g X_x - \sum_{\langle xy\rangle} Z_x Z_y\right]$$

① symmetries:

- $S = \prod_x X_x$ $\qquad [H_{TFIM}, S] = 0.$

$\mathbb{Z}_2$: $S^2 = \mathbb{1}$. $\qquad [X_x X_y, Z_x Z_z] = 0.$

- translation symm.

② limits. $\quad H_{g\to\infty} = -\sum_x X_x$

$\boxed{g=\infty}$

$$X_x |gs\rangle = +1|gs\rangle \quad \forall x$$

$$\Rightarrow |gs\rangle_\infty = \bigotimes_x |\rightarrow\rangle_x$$

$$X|\rightarrow\rangle = X\left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}\right)$$

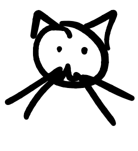
$$S|gs\rangle_\infty = |gs\rangle_\infty \qquad = \frac{|\downarrow\rangle + |\uparrow\rangle}{\sqrt{2}} = |\rightarrow\rangle.$$

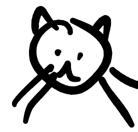
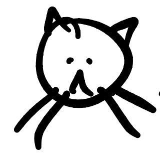
"paramagnet".

$\boxed{g=0}$ $\quad H_0 = -J\sum_x Z_x Z_{x+1}$

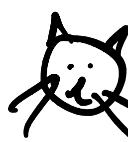
$|+\rangle = |\uparrow \cdots \uparrow\rangle$, $\quad |-\rangle = |\downarrow \cdots \downarrow\rangle$.

$S|\pm\rangle = |\mp\rangle$. Not symmetric.

$|$  $_\pm\rangle \equiv \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right)$

$S|$  $_\pm\rangle = \pm|$  $_\pm\rangle$

even worse: if $g>0$ and $V<\infty$

then $|$  $_+\rangle$ is the groundstate!

BUT: Degenerate pert thy: $|\pm\rangle$.

$$\Delta H = -\sum_x \underline{gJ}\, X_x$$

Q: $\langle -|\ (\Delta H)^n\ |+\rangle \neq 0$?

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requires $n=V$!

$$T = \langle -|(\Delta H)^n|+\rangle \propto (gJ)^V \xrightarrow{V\to\infty} 0.$$

$$\sim e^{-V|\ln g|}$$

- $| 🐱_+ \rangle = \dfrac{|\uparrow \dots \uparrow\rangle \pm |\downarrow\downarrow \dots \downarrow\rangle}{\sqrt{2}}$

is unstable to coupling to any environment

$\longrightarrow |\uparrow \dots \uparrow\rangle$ OR $|\downarrow \dots \downarrow\rangle$.

- Add a longitudinal field $\Delta H = -\sum_x h_x Z_x$

explicitly breaks S. $[S, \Delta H] \neq 0$.

$$\begin{array}{l} \overline{}\ |-\rangle \\ \overline{}\ |+\rangle \end{array} \updownarrow V h_x$$

$N \equiv V$.

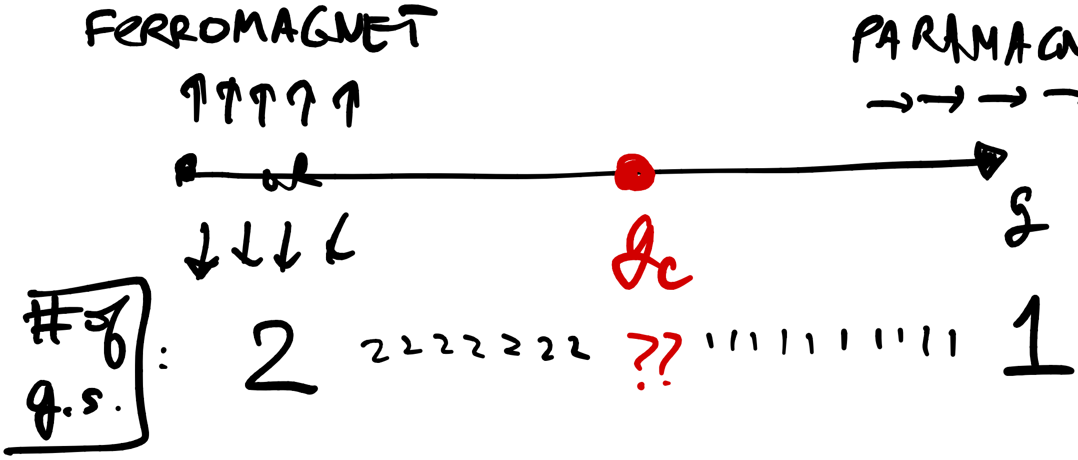
$$\underbrace{\lim_{N\to\infty} \lim_{h\to 0^\pm} |gs\rangle}_{= | 🐱_\pm \rangle} \neq \lim_{h\to 0^\pm} \lim_{N\to\infty} |gs\rangle = |\pm\rangle$$

physics!

SSB happens in the thermodynamic limit.

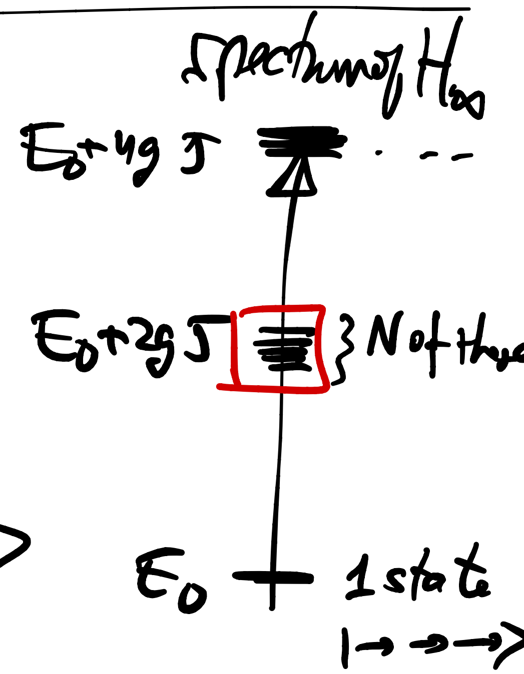
spontaneously breaks (Z_2) spin symmetry but preserves lattice symmetries

$H_{g=0}$ is a ferromagnet



Topological ≡ can't change smoothly.

③ elementary excitations



$\boxed{g \gg 1}$ $$H = \underbrace{H_\infty} - J\sum ZZ$$

$$H_\infty = -gJ\sum_x X_x$$

eigenstates: $|gs\rangle_\infty = |\to\to\to\to\cdots\rangle$

has $E_0 = -gJN$

first excited states: $$|n\rangle \equiv |\to\cdots\to\underset{\substack{\uparrow \\ \text{nth site}}}{\leftarrow}\to\cdots\to\rangle$$

$$(H_\infty - E_0)|n\rangle = 2gJ|n\rangle$$

Degen. Pert Th: Mix $|n\rangle$. by $\Delta H = -J\sum ZZ$

$Z|\rightarrow\rangle = |\leftarrow\rangle$. $\boxed{\text{ASSUME } d=1}$

$$Z_j Z_{j+1} |\rightarrow_j \leftarrow_{j+1}\rangle = |\leftarrow_j \rightarrow_{j+1}\rangle$$

$$Z_j Z_{j+1}|n\rangle \begin{cases} = |\rightarrow \cdots \rightarrow \underset{n}{\rightarrow} \underset{n+1}{\leftarrow} \rightarrow \cdots \rightarrow\rangle = |n+1\rangle \\ \text{if } j=n \\ = |\rightarrow \cdots \underset{n-1}{\leftarrow} \underset{n}{\rightarrow}\rangle = |n-1\rangle \\ \text{if } \underline{j=n-1} \end{cases}$$

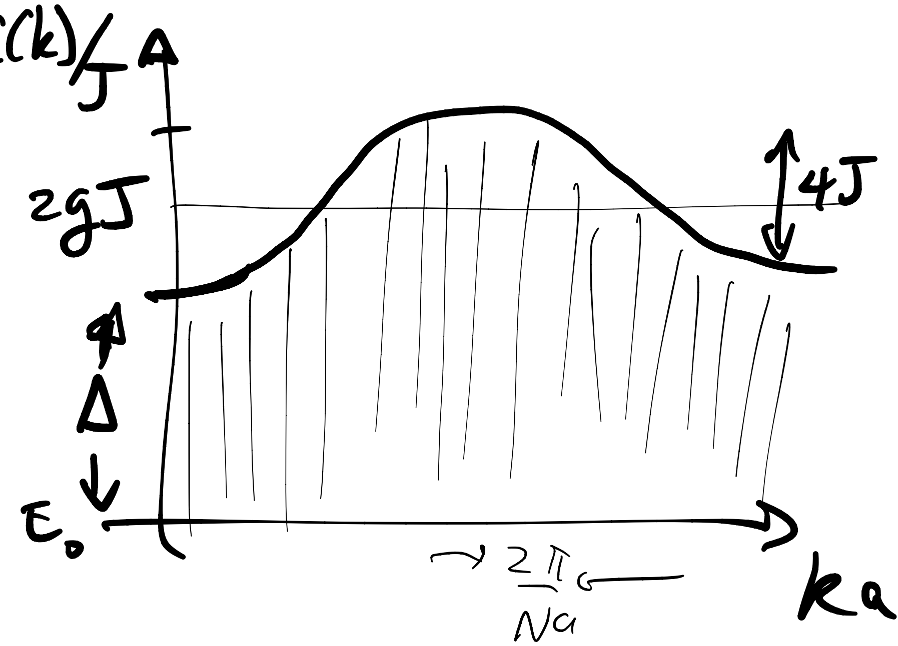
$$\Rightarrow \langle n\pm1| \sum_j Z_j Z_{j+1} |n\rangle = 1.$$

$$H_{eff}|n\rangle = -J\left(|n+1\rangle + |n-1\rangle\right) + (E_0 + 2gJ)|n\rangle.$$

PBC: $|n+N\rangle = |n\rangle$.

$$|n\rangle = \frac{1}{\sqrt{N}} \sum_j e^{-ikx_j} |k\rangle \begin{cases} x_j = ja \\ k = \frac{2\pi m}{Na} \\ m = 1 \ldots N. \end{cases}$$

$$(H - E_0)|k\rangle = (-2J\cos ka + 2gJ)|k\rangle$$



$$\epsilon(k) = 2J(g - \cos ka)$$

$$\underset{k\to 0}{\sim} \Delta + \underline{\underline{J(ka)^2}} + \dots$$

$\Delta = 2J(g-1)$ = <u>energy gap.</u>

$$\epsilon = \Delta + \frac{k^2}{2M}$$

↑ rest energy

M = inertial mass $= (2Ja^2)^{-1}$

These massive particles are called spin waves.

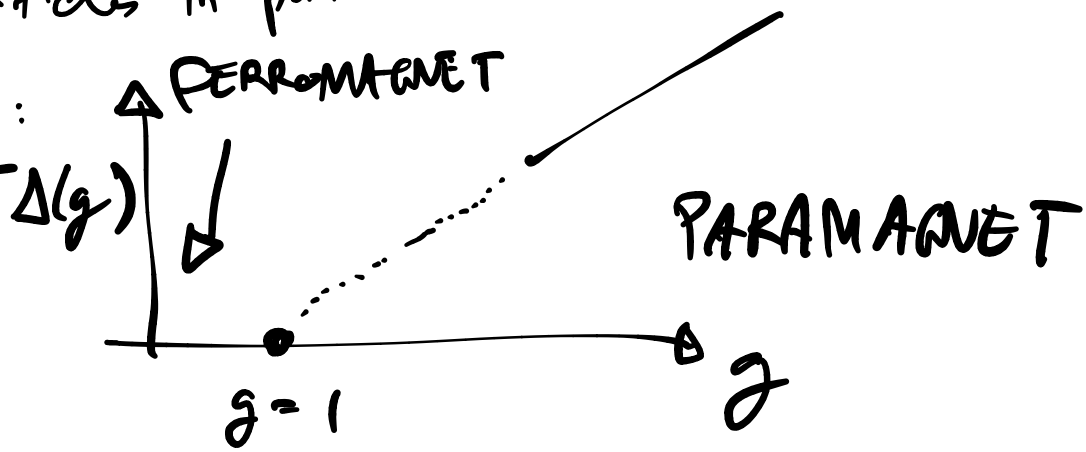
$$|n\rangle = Z_n |gs_\infty\rangle$$

$Z_n^2 = \mathbb{1}$. These particles are their own antiparticles.

Number operator: $\sum_j (-X_j)$

(only conserved mod 2. $H \ni ZZ$ creates particles in pairs.)

Make g smaller:



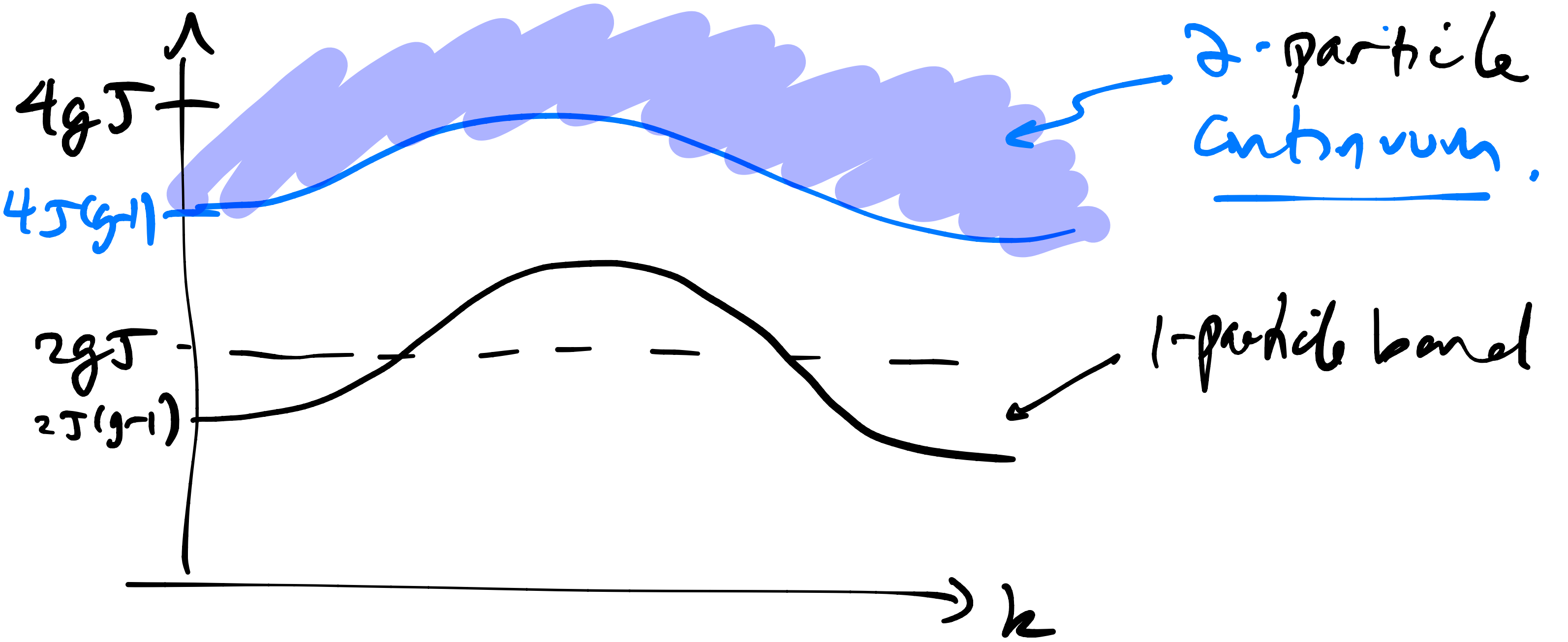
particle X

"Condenses" $\equiv \langle a^{\dagger}_X \rangle \neq 0$.

creates particle X.

Condense spin flips $\Longleftrightarrow \langle Z \rangle \neq 0$.

ferro-magnet.



$\boxed{g \ll 1}$ $|+\rangle$ eigenstate of $H_0 = -J\sum ZZ$

$= |\uparrow \cdots \uparrow\uparrow\uparrow\rangle$ w $E_0 = -JN$

<u>Excited state:</u>

$|\uparrow\uparrow\uparrow \cdots \uparrow|\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow|$

$-JZZ = +J$ $+J$

$|\uparrow \cdots \uparrow \cdot \underset{j}{\downarrow} \cdot \uparrow\uparrow \cdots \uparrow\rangle$

$= X_j |+\rangle.$

$(H - E_0)(\downarrow) = (2J + 2J)(\ )$

$|\uparrow \cdots \uparrow \cdot \downarrow\downarrow \cdot \uparrow\uparrow\uparrow\rangle$

$(H - E_0)(\downarrow) = (2J + 2J)(\ )$ same!

elementary exc: <u>domain wall</u>. rest energy $2J$.

w/ PBC, DWs can only be created in pairs.

perturb by
$$\Delta H = -J\sum_i g X_i .$$

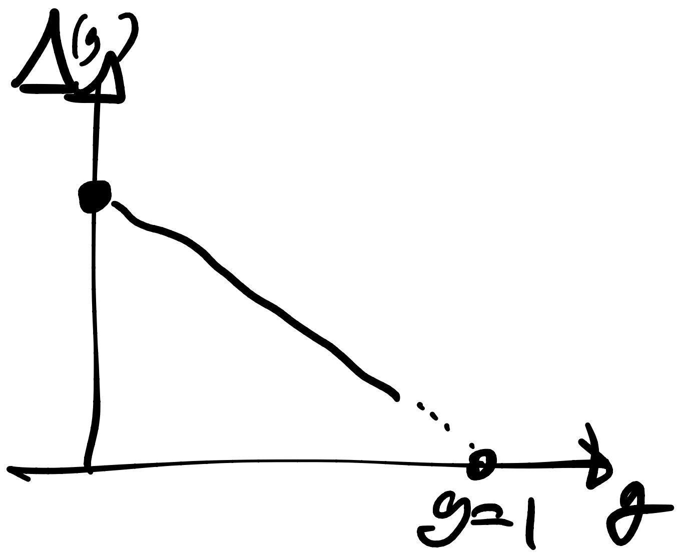
$$X_{j+1} \underbrace{|\cdots \uparrow\uparrow\uparrow_j \cdot \downarrow_{j+1}\downarrow\cdots\downarrow\rangle}_{|\bar{j}\rangle}$$

$$= \underbrace{|\cdots \uparrow\uparrow\uparrow_j \uparrow_{j+1} \cdot \downarrow_{j+2} \cdots\rangle}_{=|\bar{j}+1\rangle}$$

$$(H_{eff}-E_0)|\bar{j}\rangle = -gJ\big(|\bar{j}+1\rangle + |\bar{j}-1\rangle + 2J|\bar{j}\rangle\big)$$

$$\rightsquigarrow \epsilon_{\substack{\text{one}\\ \text{DW}}}(k) = 2J\,(1-g\cos ka).$$

$$\Delta_{DW} = 2J(1-g)$$



actual
$$\Delta \simeq 2\Delta_{DW}$$

Step 4

Mean Field Theory

$\langle\psi|H|\psi\rangle \geq E_{\text{actual gs.}}$

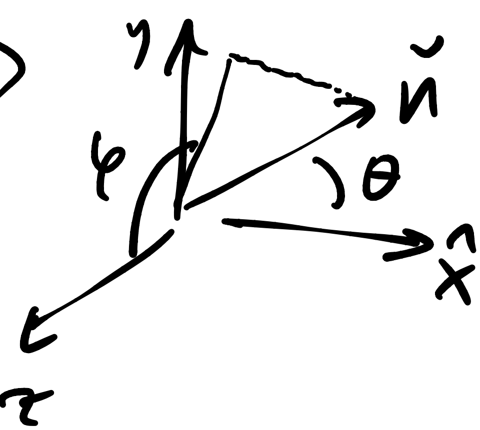
- $|MFT\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle$

<u>ansatz</u>: • assume $\psi_x = \psi \quad \forall x$.

$$|\check{n}\rangle = \bigotimes_x |\uparrow_{\check{n}}\rangle_x$$

$$= \bigotimes_x \left(\cos\tfrac{\theta}{2}\, e^{i\varphi/2}|\rightarrow\rangle + \sin\tfrac{\theta}{2}\, e^{-i\varphi/2}|\leftarrow\rangle\right)_x$$

$\theta = 0 = \varphi$ is $|\mapsto\rangle$



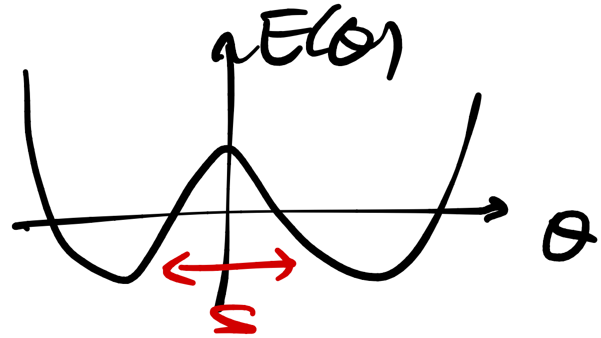
$$\begin{cases} \langle\uparrow_{\check{n}}|X|\uparrow_{\check{n}}\rangle = \cos\theta \\ \langle\uparrow_{\hat{n}}|Z|\uparrow_{\hat{n}}\rangle = \sin\theta\cos\varphi. \end{cases}$$

$$E(\theta,\varphi) \equiv \langle \check{n} | \hat{H}_{TFIM} | \check{n} \rangle$$
$$= -NJ\left(\underline{\sin^2\theta \cos^2\varphi} + g\cos\theta \right)$$

Minimize over φ, θ.

$\underline{\varphi = 0}$

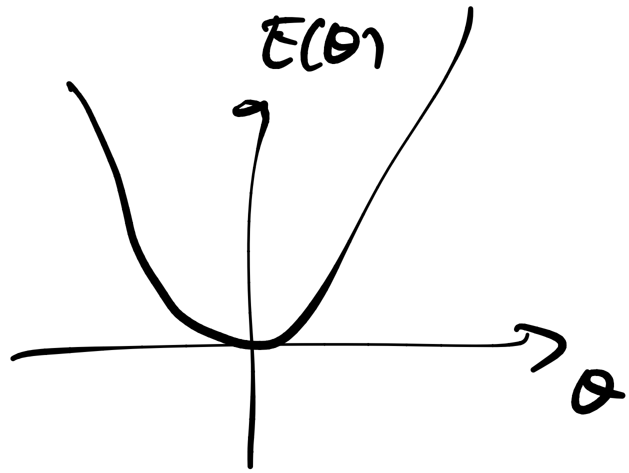
$g<g_c$



$\longrightarrow$

$$\begin{cases} g > g_c = 2 \\ \langle Z_j \rangle = \sin\theta \neq 0 \\ g < g_c \\ \langle Z_j \rangle = 0 . \end{cases}$$

$g>g_c$



Approach $g \to g_c^+$: θ is small.

$$\underline{E(\theta) \simeq NJ\left(-2 + \frac{g-2}{2}\theta^2 + \frac{1}{4}\theta^4\right)} + \dots$$

$$\langle Z_i \rangle \simeq \sin\theta \simeq \theta = \begin{cases} \pm\sqrt{g_c - g} & g<g_c \\ 0 & g>g_c . \end{cases}$$

Notice: $E(m) = r m^2 + u m^4 + c m^6 + \dots$

$m = \langle Z \rangle$
(near transition)

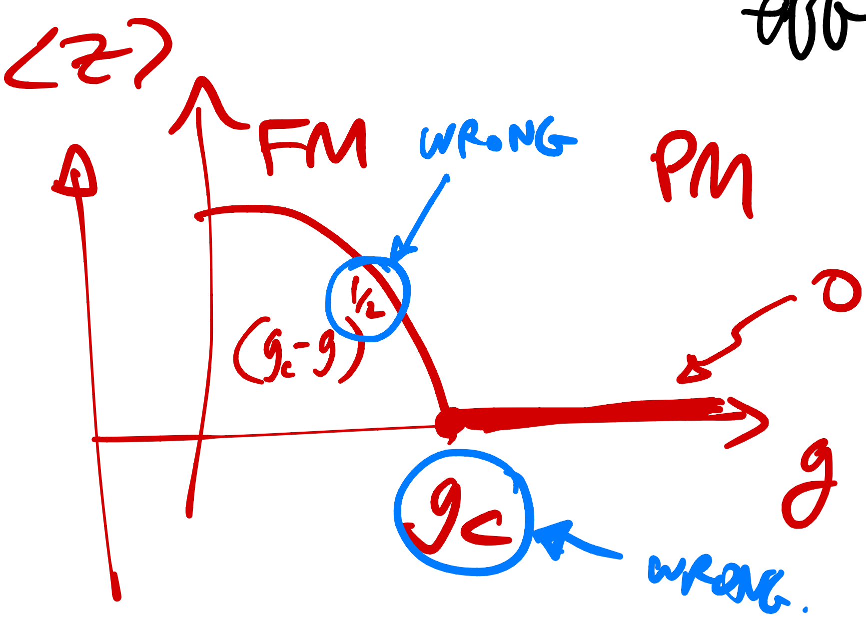
(what else could it be?)

$r(g) \propto g - g_c$

— should be analytic in m

— even in $m \to -m$ $(\mathbb{Z}_2)$

$\rightsquigarrow$ Landau-Ginzburg effective action



"condensate of spin flips"

$$X(Z|\to\rangle)$$

$$= XZ|\to\rangle$$

$$= -ZX|\to\rangle$$

$$= -Z|\to\rangle$$