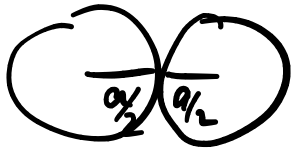
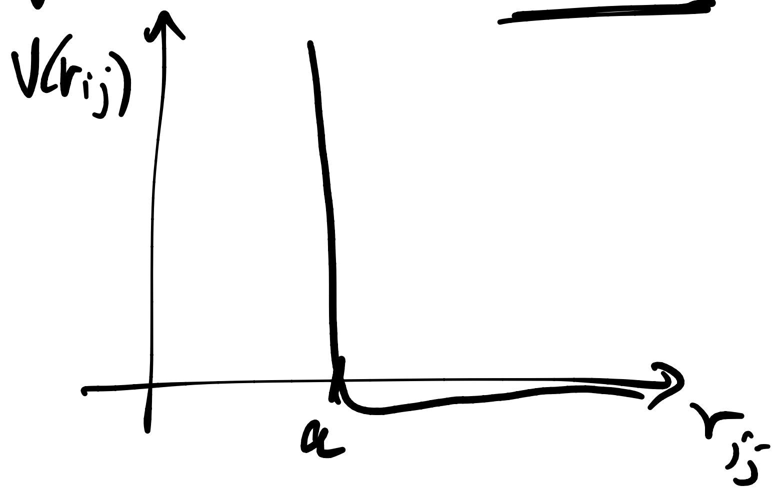
$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i<j} V(r_{ij}) + \underline{\sum_i U(r_i)}$$

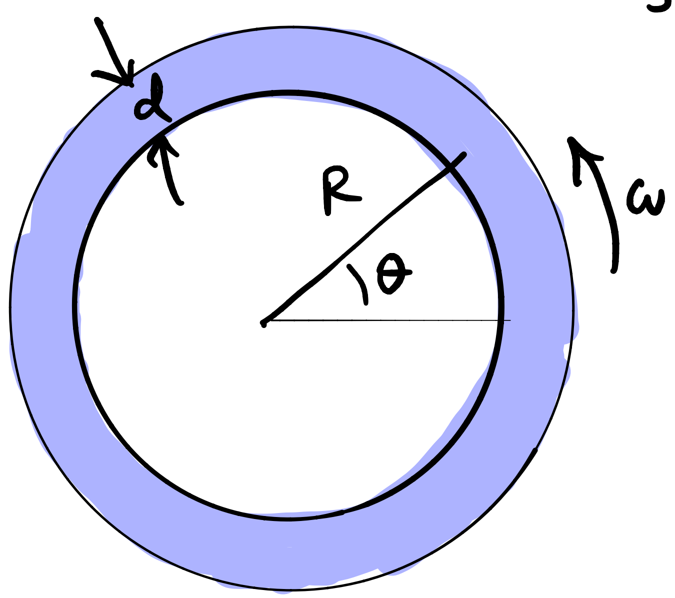
$$r_{ij} = |r_i - r_j|$$





Phenomena

$d \ll R$.



expt #1 : Rotate the fluid.

CLAIM :

$$SF : \boxed{L = f_s(T) I_{cl} \tilde{\omega}}$$

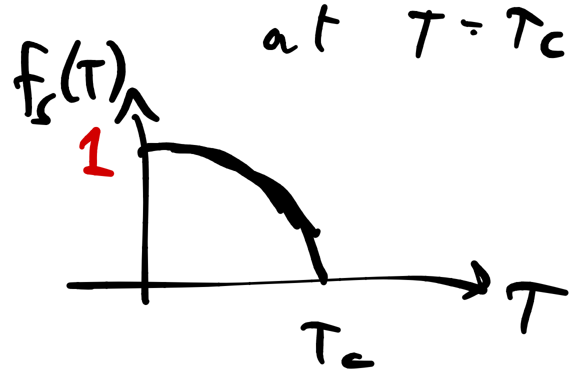
$$I_{cl} = NmR^2$$

$$\omega_c \equiv \frac{\hbar}{mR^2}$$

$\frac{\tilde{\omega}}{\omega_c}$ = integer closest to $\frac{\omega}{\omega_c}$.

ω = initial freq at $T = T_c$.

$$f_s(T) \to \begin{cases} 1 & T \to 0 \\ 0 & T \to T_c \end{cases}$$



expt #2: Rotate the container.

Neutral

$$H \to H_\omega = H - \vec{\omega}\cdot\vec{L}$$
$$= H - \vec{\omega}\cdot\sum_{i=1}^{N}\vec{r}_i\times\vec{p}_i$$

NORMAL: $\vec{L} \sim I_{cl}\vec{\omega}$

SF: If $\omega < \omega_c/2$ $\quad \vec{L} = \underbrace{(1-f_s(T))}_{\equiv f_n(T)} I_{cl}\vec{\omega}$

"normal fluid fraction"

#1 is not an eqbm phenomenon

claim: The gs cannot have $L > N\hbar/2$.

pf: suppose it did. $\Psi(r_1 .. r_N)$

$$\Psi' \equiv e^{-i\sum_{i=1}^{N}\theta_i}\,\Psi$$

$$\langle\Psi'|V|\Psi'\rangle = \langle\Psi|V|\Psi\rangle \qquad \left(\text{same for } u\right)$$

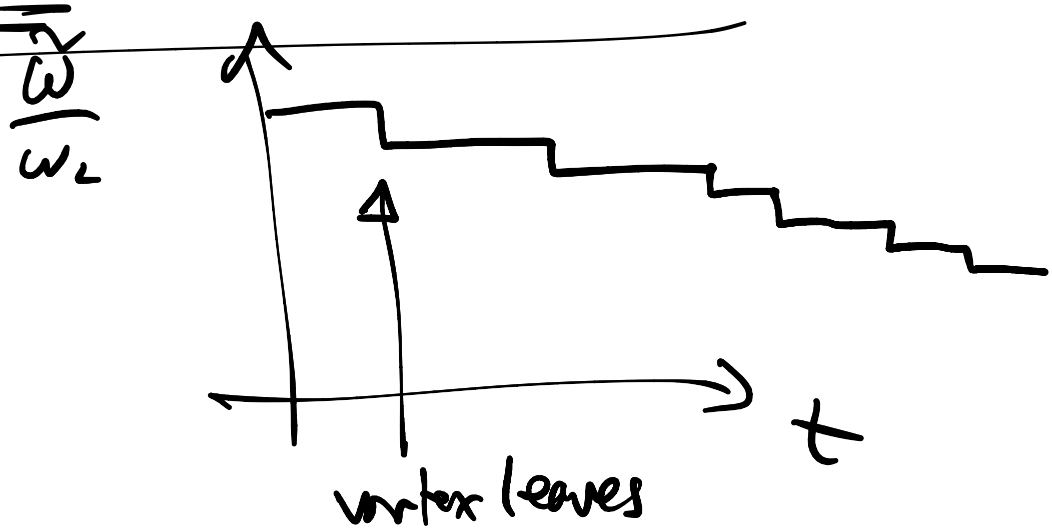
$$\langle\Psi'|K|\Psi'\rangle = \langle\Psi|K|\Psi\rangle \underbrace{- \omega_c L + \tfrac{1}{2}I_{cl}\omega_c^2}_{= -\frac{\hbar}{mR^2}\left(L - \frac{N}{2}\right).}$$

$$L = \langle\Psi|\left(-i\hbar\sum_i \frac{\partial}{\partial\theta_i}\right)|\Psi\rangle.$$

Contradiction if $L > N/2$. $\quad \omega = \tilde{\omega} \gg \omega_c$

vs: lifetime of SF flow ~ 10^{15} years.

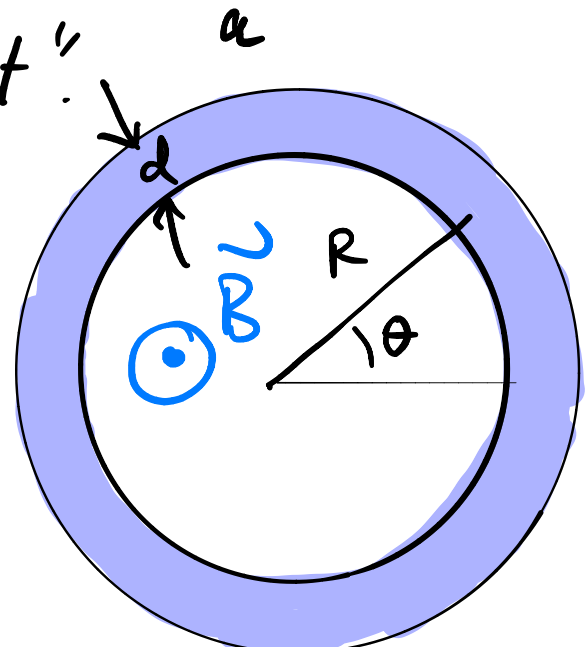
metastable state!



expt 2 for charged case:

$$\frac{\partial}{\partial t}\int \vec{B}(t)\cdot d\vec{A} = \oint \vec{E}\cdot d\vec{l}$$

"Meissner effect":



$$H \to H_A = \frac{1}{2m}\sum_{i=1}^{N}\left(\vec{p}_i - e\vec{A}(r_i)\right)^2 + \sum U + \sum_{i<j} V(r_{ij})$$

$$\vec{A}(r) = m\tilde{\omega}\times\tilde{r}$$

$$\tilde{B} = m\tilde{\omega} = \text{const}$$

$$\dot{\rho} + \vec{\nabla}\cdot\vec{j} = 0 \qquad \rho(r) = e\sum_i \delta^d(r-r_i)$$

$$\vec{j}(r) = \frac{\delta H}{\delta \vec{A}(r)} = \frac{e}{m}\sum_{i=1}^{N}\left(p_i - eA(r_i)\right)\delta^d(r-r_i)$$

Normal fluid: $\langle j\rangle = 0.$ if $\partial_t B = 0.$

superconductor:
(London eq'n)

$$\vec{j}(r) = -\Lambda(T)\,\vec{A}(r)$$

$$\Lambda(T) = \frac{ne^2}{m^*} f_s(T).$$

Faraday:

$$c^2\vec{\nabla}\times\vec{B} = \vec{j} + \partial_t\vec{E}$$

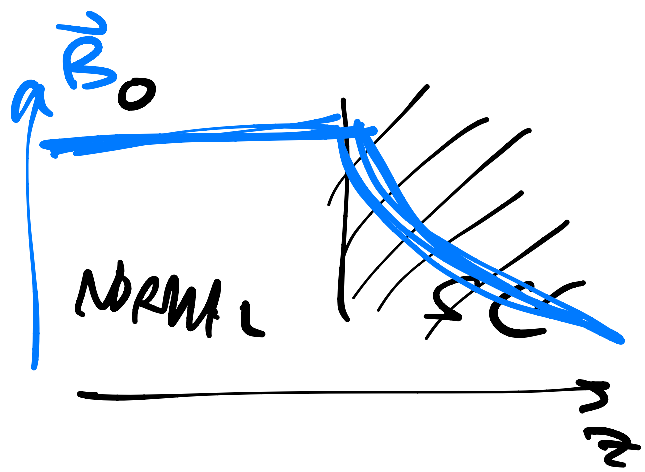
$$\rightarrow \partial_t^2\vec{A} + c^2\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\Lambda\vec{A}$$

$$\partial_t^2\vec{A} - c^2\nabla^2\vec{A} = -\Lambda\vec{A}$$

$\nabla\cdot A = 0$:

STATIC, ONE-dim'l

$$c^2\left(\frac{\partial}{\partial z}\right)^2 A = -\Lambda A$$



$$\Rightarrow \vec{B}(z) = e^{-z\sqrt{\Lambda}/c}\,\vec{B}_0$$

penetration depth $\sim \frac{1}{\sqrt{\Lambda}} \sim 100$ Å.

3.2 Robust def^{n} of BEC.

Q: Given a $\Psi(r_1 \ldots r_N)$ is it a BEC?

OR: $\{\Psi_s(r_1 \ldots r_N)\}$ w/ probs p_s.

$$\rho = \sum_s p_s \, |\Psi_s\rangle\langle\Psi_s|$$

$$\langle \mathcal{O} \rangle = \operatorname{tr} \rho \mathcal{O}. \qquad \operatorname{tr} \rho = 1.$$

A: one-particle density matrix

$$\rho_1(r, r') \equiv \sum_s p_s \sum_{r_2 \ldots r_N} \Psi_s^*(r, r_2 \ldots r_N) \, \Psi_s(r', r_2 \ldots r_N)$$

$$= \langle \psi^\dagger(r) \psi(r') \rangle = \operatorname{tr} \rho \, \psi^\dagger(r) \psi(r')$$

$$\rho_1(r, r') = \rho_1^*(r', r) \quad \text{ie } \rho_1 = \rho_1^\dagger$$

$$\Longrightarrow N \rho_1(r, r') = \sum_{i=1} N_i \, \chi_i^*(r) \chi_i(r')$$

$\frac{N_i}{N}$ evals $\qquad \chi_i(r)$ evecs. of ρ_1

$\mathrm{tr}\,\rho = 1 \Rightarrow \mathrm{tr}\,\rho_1 = 1 \Rightarrow \sum_i N_i = N$

$\Rightarrow \{N_i\}$ are occupation #s!

of the ON orbitals $\chi_i(r)$

$\int d^d r\, \chi_i^*(r)\, \chi_j(r) = \delta_{ij}$.

IF $|\Psi\rangle = b_1^\dagger \cdots b_N^\dagger |0\rangle$

$\chi_\alpha(r) = \langle r | b_\alpha^\dagger | 0 \rangle$

N_i is a histogram of their occupation.

Def: If $N_i \sim O(N^0)$

$\equiv$ "normal state"

$\lim_{N\to\infty} \frac{N_i}{N} = 0$.

If any $N_i \sim O(N)$

$\equiv$ "BEC".

exactly one $\equiv$ simple BEC.

eg: N free bosons at $T < T_c$

$$\lim_{N\to\infty} \frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{\alpha}$$ is finite

$\Rightarrow$ BEC.

ORDER PARAMETER — suppose simple BEC.

let $\Psi(r) \equiv \sqrt{N_0}\,\chi_0(r) = |\Psi(r)| e^{i\varphi(r)}$

$$\sum_r |\Psi|^2 = N_0.$$

Density & Current of the condensate:

$$\rho_c = N_0 |\chi_0|^2 = |\Psi|^2$$

$$\vec{J}_c = N_0\left(\frac{-i\hbar}{2m}\chi_0^* \vec{\nabla}\chi_0 + h.c.\right) = |\Psi|^2 \frac{\hbar}{m}\vec{\nabla}\varphi$$

Let $\vec{V}_s(r) \equiv \dfrac{\vec{J}_c(r)}{\rho_c(r)} = \dfrac{\hbar}{m}\vec{\nabla}\varphi(r)$

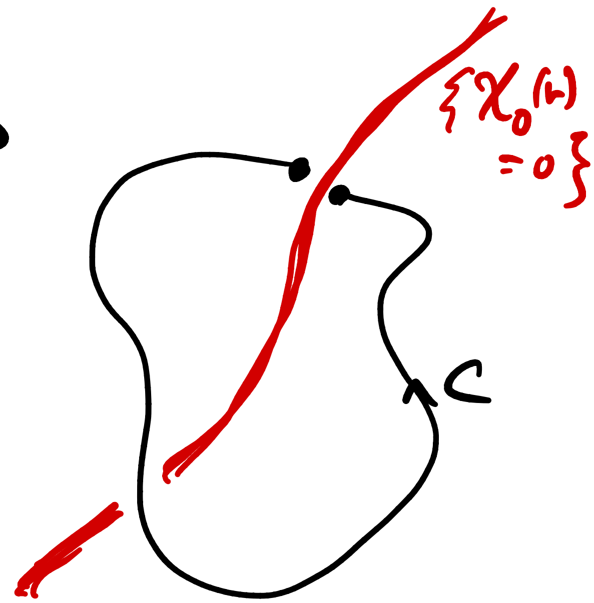
superfluid velocity.

2 properties of $\vec{v}_s$:

① If $\chi_0(r) \neq 0$ then $\underline{\vec{\nabla} \times \vec{v}_s = 0}$.

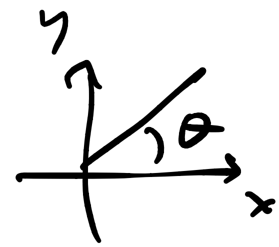
$(|\Psi| \neq 0)$

② $\oint_C \vec{v}_s \cdot d\vec{\ell} = \frac{\hbar}{m} \underbrace{\oint_C \vec{\nabla}\varphi \cdot d\vec{\ell}}_{=\Delta\varphi}$



$\boxed{\begin{matrix}\varphi \equiv \varphi + 2\pi n \\ n \in \mathbb{Z}.\end{matrix}}$ $= \frac{\hbar}{m} 2\pi n$

$$\boxed{\oint_C \vec{v}_s \cdot d\vec{\ell} = \frac{nh}{m} \quad n \in \mathbb{Z}}$$



eg: vortex

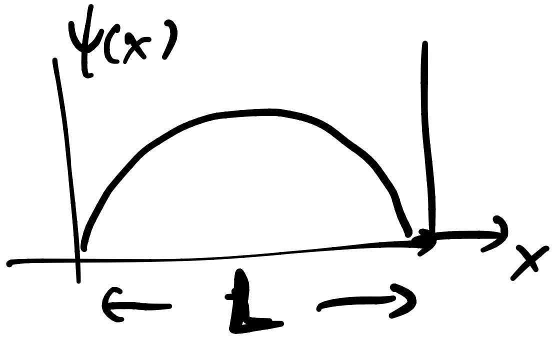
$\Psi(\underbrace{x+iy}_{\equiv z}) \propto \underline{z\, f(|z|)}$

$e^{i\varphi} \propto e^{i\theta}$

$\boxed{\varphi = \theta}$

VORTICITY ($\oint \vec{v}_s \cdot d\vec{\ell}$) OF SUPER FLOW IS QUANTIZED why can't we *always* do this?

- One particle in a box in 1d.



$$\psi_1(x) = \sqrt{\tfrac{2}{L}} \sin \tfrac{\pi x}{L}.$$

$$= e^{i\varphi_1} |\psi_1|.$$

$$\vec{v}_1 \equiv \frac{\vec{J}_1}{\rho_1} = \frac{\hbar}{m} \vec{\nabla}\varphi_1$$

$$\Longrightarrow \text{①}\ \vec{\nabla}\times\vec{v}_1 = 0$$

$$\text{②}\ \oint \vec{v}_1 \cdot d\vec{\ell} = \frac{hn}{m},\ n\in\mathbb{Z}$$

BUT: $\varphi_1 = 0.$

measure: $\hat{v} = \frac{\hat{p}}{m} = \frac{-i\hbar\partial_x}{m}$ get: $\pm \frac{\hbar\pi}{mL}.$

$\rightsquigarrow \ p = 1/2$

BIG FLUCTUATIONS.

• Normal fluid of many particles

$$\tilde{\vec{v}}_{hydro} = \frac{\vec{J}_{total}}{\rho_{total}} = \frac{\hbar}{m}\left(\frac{\sum_i N_i |\chi_i|^2 \vec{\nabla}\varphi_i}{\sum_i N_i |\chi_i|^2}\right)$$

— small fluctuations ✓

$\sim \frac{1}{\sqrt{N}}$. (Central Limit thm)

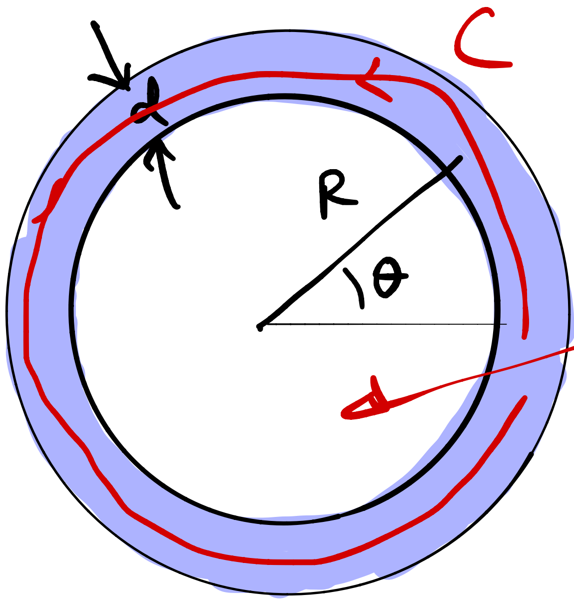
BUT :

$$\begin{cases} \nabla \times v_{hydro} \neq 0 \\ \oint \tilde{v}_{hydro} \cdot d\tilde{l} \notin n \frac{h}{m}. \end{cases}$$

(UNLESS SIMPLE BEC w/ $N_0 = N$.)

SF : small fluctuations : $\frac{\sqrt{\langle v^2\rangle - \langle v\rangle^2}}{\langle v\rangle} \sim \frac{1}{\sqrt{N}}$.

But : quantum. $\oint \tilde{v}_s \cdot dl \in \mathbb{Z}\, \frac{h}{m}$.



$$\oint_C \vec{v}_s \cdot d\vec{\ell} \in \frac{h}{m}\mathbb{Z}.$$

$\underline{d \ll R}.$

$\underline{\Psi = 0.}$

$$S[\theta] = \int dt \left[\frac{1}{2} m R \dot{\theta}^2 + \omega m R^2 \dot{\theta}\right]$$

(Pf of Bohr-van Leeuwen thm)

$$\vec{\omega} \times \vec{L} = \omega m R^2 \dot{\theta} \hat{z}$$

(total derivative does not change eom.)

$$\hat{\pi} = \frac{\partial L}{\partial \dot{\theta}} = m R \dot{\theta} + \omega m R^2.$$

$I = mR^2.$ $\rightsquigarrow$ $$H = \frac{\hbar^2}{2I}\underline{\underline{(\hat{\pi} - I\omega)^2}}$$

$$\underline{[\theta, \hat{\pi}] = i\hbar.}$$

$$= \frac{\hbar\omega_c}{2}\left(n - \frac{\omega}{\omega_c}\right)^2$$

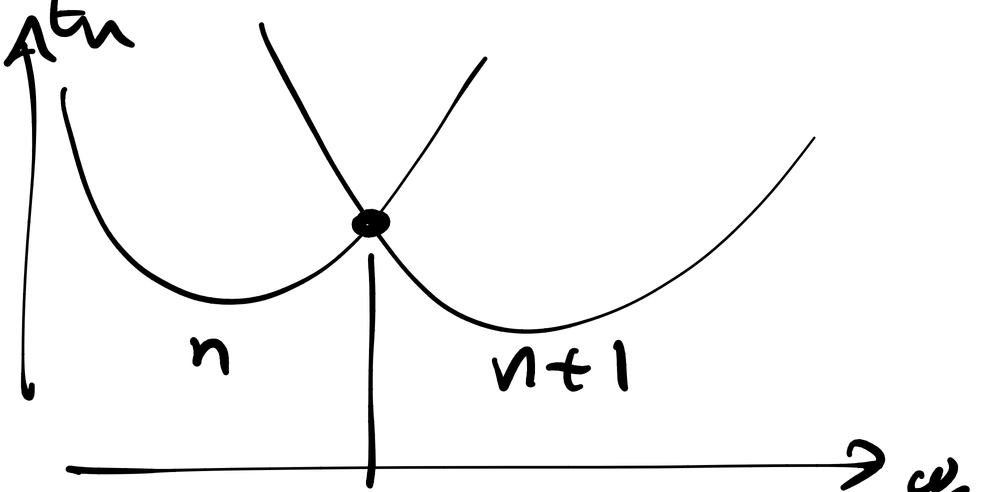
$$\theta \equiv \theta + 2\pi \Rightarrow e^{i\hat{\pi}\theta} = 1.$$

$$\Rightarrow \underline{\pi \in \mathbb{Z}}.$$

GS is when n = integer closest to ω/ω_c.

At $\omega = \omega_c (n + \frac{1}{2}) \Rightarrow$ degeneracy.

between n & $n+1$



$\omega = \omega_c (n + \frac{1}{2})$

$$\boxed{\omega = \frac{\omega_c}{2}}$$

allows a vortex to enter.

$$\hat{N} = \sum_r \psi_r^\dagger \psi_r$$

$$[\hat{N}, H] = 0 \Rightarrow U = e^{i\alpha \hat{N}}$$

$$[U, H] = 0.$$

<u>U(1):</u> $\psi_r \rightarrow U \psi_r U^\dagger = e^{-i\alpha} \psi_r$.