Recap: $\rho_1(r,r') = \bar{N}^{-1}\langle \psi^\dagger(r)\,\psi(r')\rangle$

$$= \sum_i \left(\frac{N_i}{\bar{N}}\right) \chi_i^*(r)\,\chi_i(r')$$

$\Psi(r) = \sqrt{N_0}\,\chi_0(r) = |\Psi| e^{i\varphi(r)}$ $\quad N_0 > N_i$ biggest eval.

BEC : $N_0 \sim \mathcal{O}(N)$. $\quad \varphi \equiv \varphi + 2\pi$.

$$\vec{v}_s = \frac{\vec{j}_s}{\rho_s} = \frac{\hbar}{m}\vec{\nabla}\varphi \quad . \quad \vec{\nabla}\times\vec{v}_s = 0$$

if $|\Psi| \neq 0$

$\oint \vec{v}_s \cdot d\vec{\ell} = \frac{h}{m} n$

$n \in \mathbb{Z}$.

Rel'n to Long Range Order (LRO)

$$\lim_{|r-r'|\to\infty} N\rho_1(r,r') = \Psi^*(r)\,\Psi(r') + \underbrace{N\tilde{\rho}_1(r,r')}$$

$\tilde{\rho}_1$: $\sum_{i\neq 0} N_i\,\chi_i^*(r)\,\chi_i(r')$

survives.

$\to 0$ as $|r-r'|\to\infty$

OFF-DIAGONAL LRO.

vs: an ordinary solid diagonal LRO

visible in $n(r) = \rho_1(r,r)$

[Yang] : tr $\equiv$ Sp "spur".

$$N\rho_1(r,r') = \langle gs|\psi^\dagger(r) \,\underset{\mathbb{1} = |gs\rangle\langle gs| + \cdots}{\wedge}\, \psi(r')|gs\rangle$$

$$\to \underbrace{\langle gs|\psi^\dagger(r)|gs\rangle}_{\Psi^\dagger(r)} \underbrace{\langle gs|\psi(r')|gs\rangle}_{\Psi(r')} + \cdots$$

$$\Psi(r) = \langle \psi(r) \rangle$$

particle "condenses" $\equiv$ expectation value for its creation op. or annihilation

Ψ is like a classical field

like $\vec{E}_{classical}(r) = \langle z|\hat{\vec{E}}(r)|z\rangle$ $\quad$ $a|z\rangle = z|z\rangle$.

HOWEVER: $[H_{EM}, \sum_{k,s} a^\dagger_{ks} a_{ks}] \neq 0$.

$\vec{E} \to e^{i\alpha}\vec{E}$ is not a symmetry.

$\Rightarrow \langle \vec{E} \rangle \neq 0$ breaks no symmetry (besides rotations)

$$N = \sum_r \psi_r^\dagger \psi_r$$

Quantum Noether Thm:

$$[N, \psi] = -\psi$$

$$[H, N] = 0.$$

$$\Longrightarrow [H, U] = 0, \quad \hat{U} = e^{i\alpha \hat{N}}$$

$$\psi \to U \psi U^\dagger = e^{-i\alpha\, \mathrm{ad}_N} \psi$$

$$= e^{i\alpha} \psi.$$

IF $\langle G | \psi | G \rangle \neq 0$

$$\to \langle G | U \psi U^\dagger | G \rangle = e^{i\alpha} \langle G | \psi | G \rangle$$

$$\Longleftrightarrow U^\dagger | G \rangle \neq | G \rangle.$$

ie $U(1)$ symmetry generated by N is spontaneously BROKEN.

$$\psi(r) \, | n\text{-particle state} \rangle = | n-1 \text{ particle state} \rangle.$$

$$\langle \Theta_n | \psi | \Theta_n \rangle = 0 .$$

any n-particle state

$$\langle r_1 .. r_n | r_1' -- r_{n'}' \rangle = \delta_{nn'} \left(f_{r_1 \pi(r_1')} \cdots \right)$$

$\Rightarrow$ ONLY IF

$$|\Theta\rangle = \sum_n a_n |\Theta_n\rangle$$

$$= a_n |\Theta_n\rangle + a_{n+1} |\Theta_{n+1}\rangle + \ldots$$

$$\langle \Theta | \psi | \Theta \rangle \neq 0$$

Recall: training field $\Delta H = -h \sum_j Z_j$

explicitly breaks sym.

$$\lim_{N\to\infty} \lim_{h\to 0} \neq \lim_{h\to 0} \lim_{N\to\infty}$$

$$\Delta H_{here} = -\lambda \int d^d r \left(\psi^\dagger(r) + h.c. \right)$$

source of particles

sink of particles

$$\lim_{N\to\infty} \lim_{\lambda\to 0} \neq \lim_{\lambda\to 0} \lim_{N\to\infty}$$

Use grand canonical ensemble

$$\Delta N \equiv \sqrt{\langle N^2\rangle - \langle N\rangle^2}$$

$$\frac{\Delta N}{N} \sim \frac{1}{\sqrt{N}}. \qquad N \sim 10^{26}.$$

3.3 Interactions & BEC

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i<j} V(\underbrace{r_i - r_j}_{\equiv r_{ij}}) + \sum_i U(r_i)$$

What is the gs wavefunction $\Phi_0(r_1 \ldots r_N)$?

- $\Phi_0(r_1 \ldots r_N) = \Phi_0(r_{\pi(1)} \ldots r_{\pi(N)})$.
- $\Phi_0(r_1 \ldots r_N) = \Phi_0^*(r_1 \ldots r_N)$.

$$\left(\left(-\frac{\hbar^2}{2m} \sum_i \nabla_i^2 + \sum_{i<j} V(r_{ij}) \right) \Phi_0 = E\, \Phi_0 \right)^*$$

$$\left(-\frac{\hbar^2}{2m} \sum_i \nabla_i^2 + \sum_{i<j} V(r_{ij}) \right) \Phi_0^* = E\, \Phi_0^*$$

$\longrightarrow \quad \Phi_0 + \Phi_0^*$ does too.

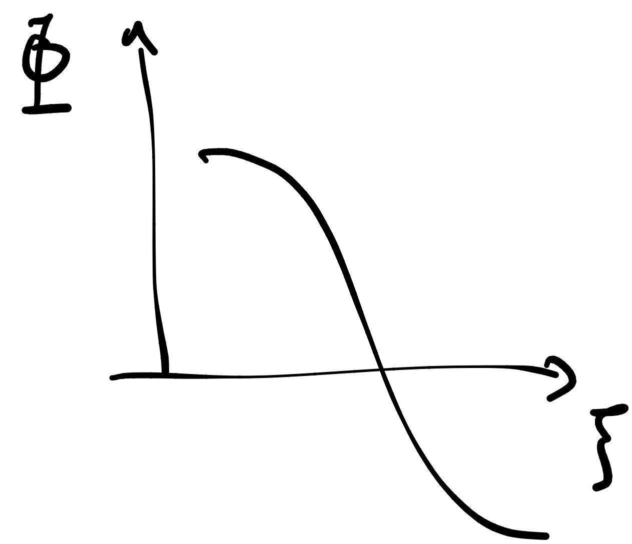
[time-reversal symmetry of H].

True for all eigenstates.

- Φ_0 has no nodes, $\Phi_0 > 0$.

$$\Phi(\xi) \equiv \Phi_0(\xi, r_2 \ldots r_N)$$

Suppose otherwise:

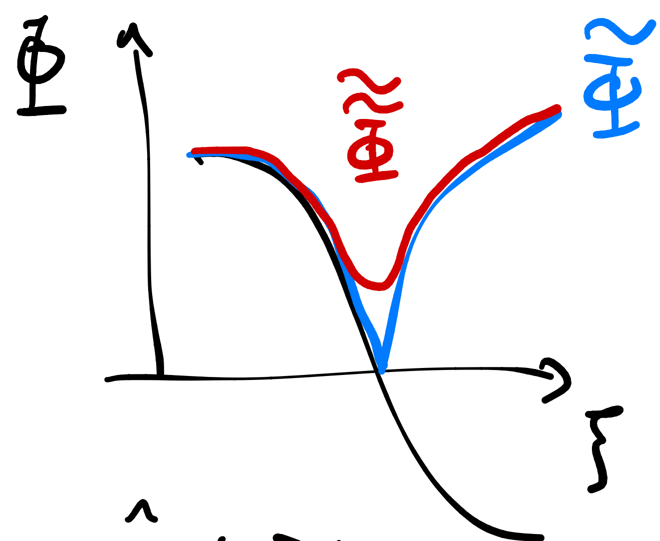


Φ^2, $(\partial_3 \Phi)^2$, $(\partial_n \cdot \Phi)^2$

$$E[\Phi] \equiv \frac{\int \Phi H \Phi}{\int \Phi^2} \overset{IBP}{=} \frac{\int \left(\frac{(\nabla\Phi)^2}{2m} + V\Phi^2\right)}{\int \Phi^2}$$

$\tilde{\Phi} = |\Phi|$. $E[\Phi] = E[\tilde{\Phi}] > E[\tilde{\tilde{\Phi}}]$

Contradiction!



$$E[\Phi] = \frac{\langle \Phi | \hat{H} | \Phi \rangle}{\langle \Phi | \Phi \rangle}$$

- Φ_0 is non-degenerate.

can choose c_0, c_1

$c_0 \Phi_0 + c_1 \Phi_1$

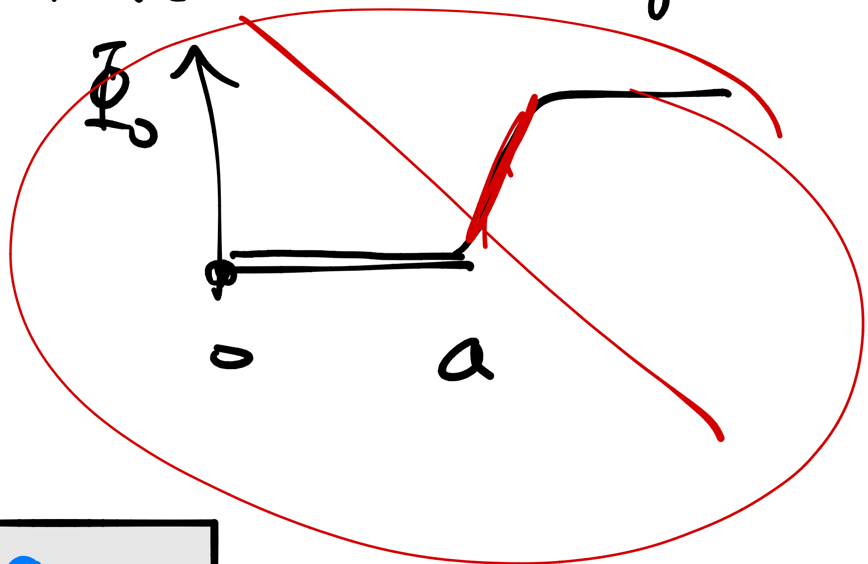
would have a node.

$$0 = \langle \Phi_0 | \Phi_1 \rangle = \int_{\text{space}} \Phi_0(r) \Phi_1(r)$$

requires one to be negative somewhere!

• $\Phi_0(r_1 \ldots r_N) \simeq 0$ if $|r_1 - r_2| < a$.

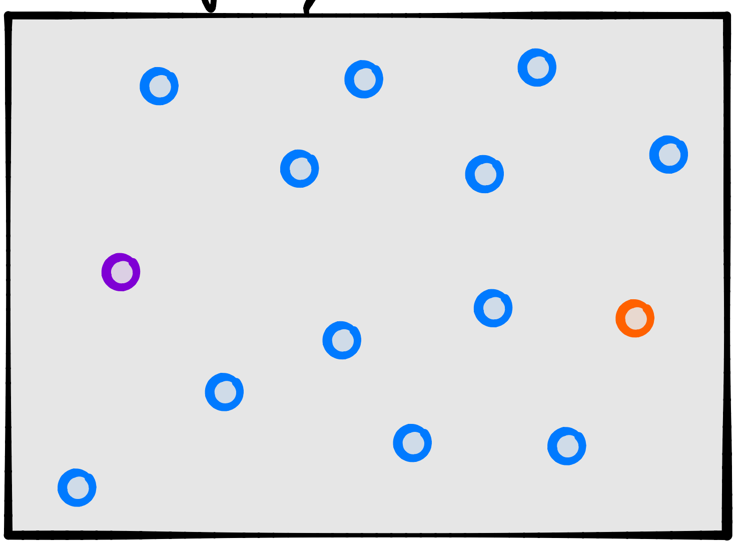
$\Leftarrow$ repulsive hardcore of V.



$\Rightarrow \Phi_0(r_1 \ldots r_N) \sim 0$

if $|r_1 - r_2| \lesssim a$.

configs like:

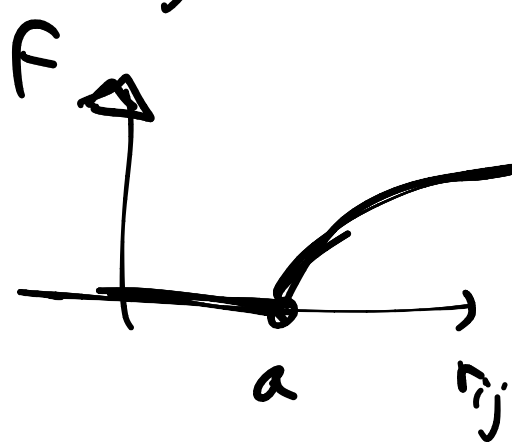
$\Rightarrow$ 

have largest support of $|\Phi_0|$.

"Jastrow".

eg: $\Phi = e^{-\sum_{ij} f(r_{ij})} = \prod_{ij} F(r_{ij})$

$$F(r) = \begin{cases} 0 & r < a \\ \to 0 & r \to a \\ \to 1 & r \to \infty \end{cases}$$



MFT in free space: SIMPLE ANSATZ: simple-particle $\boxed{\frac{(b_\chi^\dagger)^N}{\sqrt{N!}}|0\rangle}$

Put every particle in one state, $\chi(r)$.

$$\Rightarrow \quad \Psi(r) = \sqrt{N}\,\chi(r).$$

$$E[\Psi] = \langle \Psi | H | \Psi \rangle$$

- Is invariant under $\Psi(r) \to e^{i\alpha}\Psi(r)$
- involves at most 2 derivatives
- analytic in $|\Psi|^2$.
- local in space

$$\Rightarrow E[\Psi] = \int d^d r \left(r|\Psi|^2 + u|\Psi|^4 + \cdots + \rho_s \vec{\nabla}\Psi^* \cdot \vec{\nabla}\Psi \right)$$

(LG "free" energy for a U(1)-symmetric magnet)

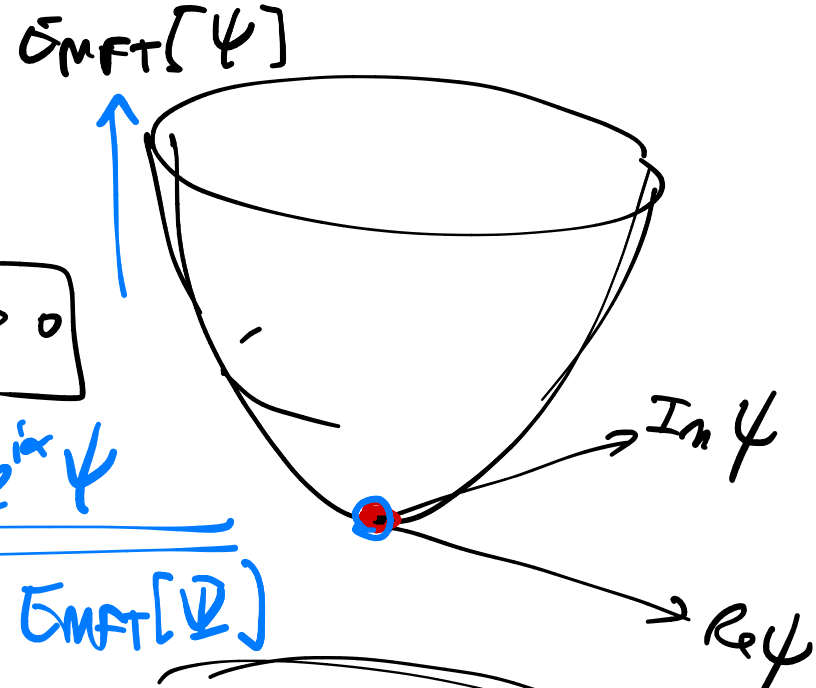
If $\rho_s > 0$ $\Psi = \text{const}$ is better.

Assume $\rho_s > 0$, $u > 0$.

$\psi = 0 \Leftarrow$

no BEC.

$\boxed{r > 0}$

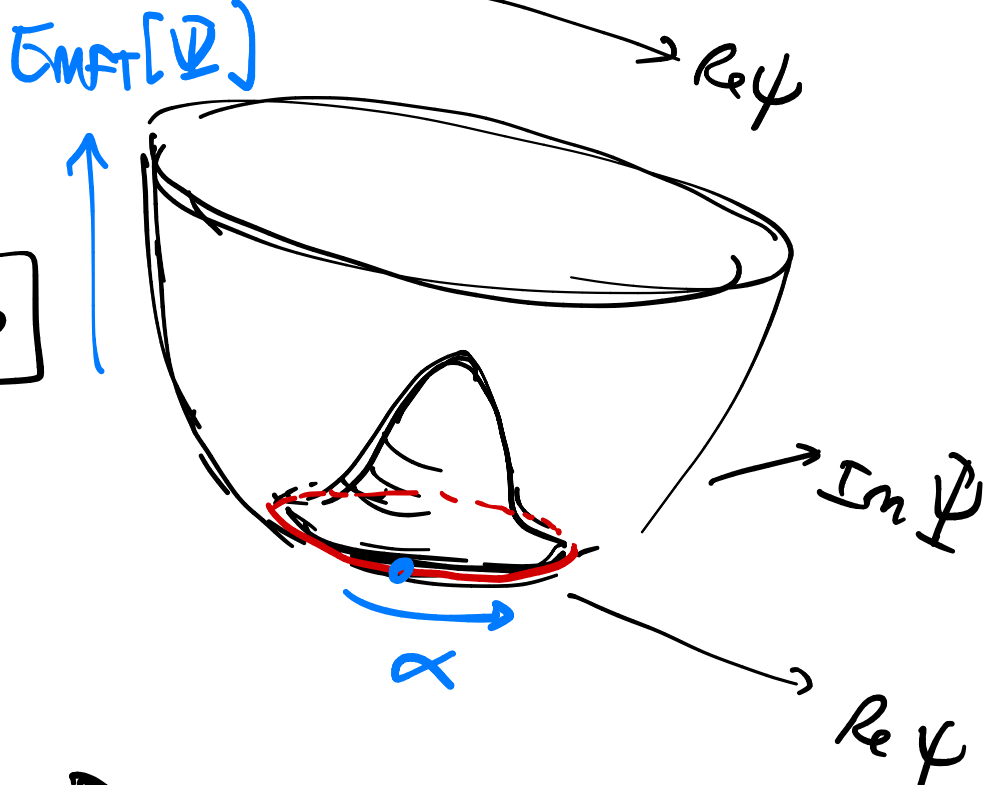


$\psi = 0$ is preserved by $\psi \to e^{i\alpha}\psi$

$\langle \psi \rangle \neq 0$

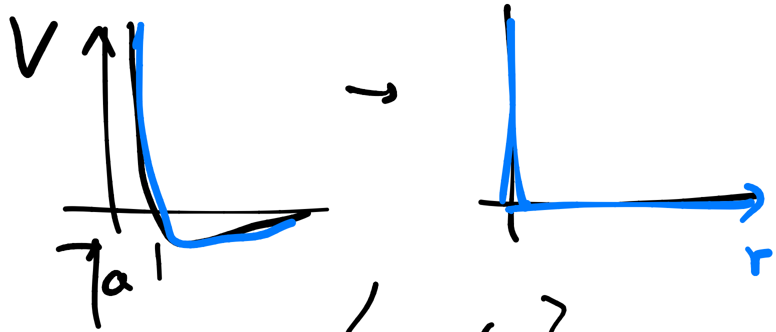
BEC

$\boxed{r < 0}$



SSB.

r, u (parameters in H) ?



$$\underline{V(r_{ij}) = U_0\, \delta^d(r_{ij}).}$$

let $\{\chi_i(r)\}$ ON 1-particle wavefn's.

$\boxed{N=2}$ $\quad \Psi(r_1, r_2) = \Psi(r_2, r_1)$

$$E_{int} = \langle \Psi | H_{int} | \Psi \rangle = U_0 \int d^d r_1 \int d^d r_2\, \delta^d(r_1 - r_2)\, |\Psi(r_1, r_2)|^2$$

$$= U_0 \int d^d r\, |\Psi(r,r)|^2 = U_0 \int d^d r\, \rho_1(r,r).$$

$\underline{\text{2 cases}}$:

① $\underline{\text{SAME}}$: $\quad \Psi(r_1, r_2) = \chi(r_1)\chi(r_2)$

$$E_{int} = U_0 \int d^d r\, |\chi(r)|^4.$$

② $\underline{\text{DIFFERENT}}$: $\quad \Psi(r_1, r_2) = \dfrac{\chi_1(r_1)\chi_2(r_2) + \chi_2(r_1)\chi_1(r_2)}{\sqrt{2}}$

$$E_{int} = U_0 \int d^d r\, \frac{1}{2} \cdot 4\, |\chi_1(r)|^2 |\chi_2(r)|^2 = 2 U_0 \int d^d r\, |\chi_1|^2 |\chi_2|^2.$$

$\boxed{N >> 2}$

$$E_{int} \cong \frac{1}{2} U_0 \sum_{ij} N_i N_j \, (2 - \delta_{ij}) \int d^d r \, |\chi_i(r)|^2 |\chi_j(r)|^2$$

eg: $N_i \simeq 1$: $\frac{U_0}{2}\left(\sum_{i=j} 2 - \sum_i 1\right) = \frac{U_0}{2} \cdot (2N^2 - N)$

$$\simeq U_0 N^2$$

$N_0 = N, N_{i>0} = 0$: $\frac{1}{2} U_0 \, N^2 (2-1) = \frac{1}{2} U_0 N^2$

if $U_0 > 0$ (REPULSIVE) SIMPLE BEC wins.

$N_0 = N/2, N_1 = N/2, N_{i>1} = 0$:

$$\frac{1}{2} U_0 \left((\tfrac{N}{2})^2 + (\tfrac{N}{2})^2 + 2 (\tfrac{N}{2})^2 \cdot 2 \right)$$

$$\simeq \frac{1}{2} U_0 \cdot \frac{3}{2} N^2 .$$