University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 211C (239) Phases of Quantum Matter, Spring 2021
Assignment 2 – Solutions

Due 12:30pm Wednesday, April 14, 2021

Thanks for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. $\pi_1(G)$ **is abelian.** [Optional] Complete the proof that for any Lie group (or, if you like, more generally any topological group) $\pi_1(G)$ is abelian by finding a homotopy between $f \star g$ and $g \star f$.

   One way to do it following the hint from lecture is to first show that $f \star g \simeq fg$ (where $(fg)(t) = f(t)g(t)$ is the pointwise multiplication of group elements), and then show $fg \simeq gf$.

   Here is a homotopy between $fg(t)$ and $(f \star g)(t)$:

   $$F_1(t,s) = \begin{cases} f(2t)g(2ts), & 0 \le t \le \frac{1}{2}, \\ g((1-s)2t + 2s - 1), & \frac{1}{2} \le t \le 1 \end{cases}$$

   At $s = 0$ this is $f \star g$, at $s = 1$ this is fg. Notice that it is continuous at $t = \frac{1}{2}$, since $F_1(\frac{1}{2}, s) = g(s)$ from both sides.

   And here is a homotopy between fg and gf:

   $$F_2(t,s) = g(st)f(t)g(t)g^{-1}(st).$$

   At $t = 1$ it is $g(s)g^{-1}(s) = e$. At $s = 1$ it is $g(t)f(t)$.

   A homotopy similar to F_1 then takes gf to $g \star f$.

   Thanks to Ahmed Akhtar, Arghadip Koner, Meng Zeng, and Zhengdi Sun for help with this problem.

2. [This problem is optional] Suppose we have a system with symmetry $G = \mathsf{SU}(2) \times U(1)_Y$ which is broken down to $\mathsf{U}(1)_Q$, where the unbroken subgroup is generated by

   $$Q = pT_3 + rY$$

   for some integers p, r with no common factor. Here $T_3^{(i)}$ means a generator of $\mathsf{SU}(2)$and we normalize T_3 and Y so that their smallest nonzero eigenvalue is one. Show that there are stable codimension-two defects which can disappear in clumps of r.

Further bonus problem: Can you find an example of a linearly-transforming order parameter that produces this pattern of symmetry-breaking?

We need to show that

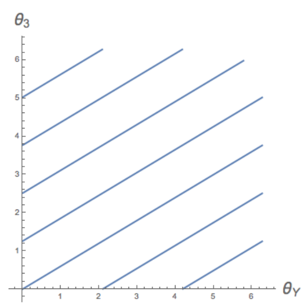
$$\pi_1(V) = \mathbb{Z}_r$$

where $V = G/H$.

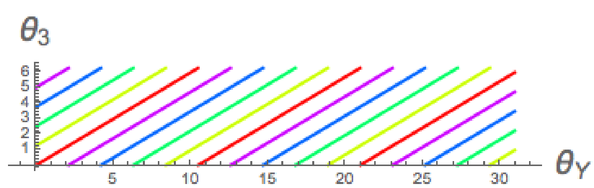
A useful way to think about this problem is focus on the $\mathsf{U}(1)_3 \times \mathsf{U}(1)_Y$ subgroup of $G = \mathsf{SU}(2) \times \mathsf{U}(1)_Y$ generated by T_3 and Q. Draw this $T^2 = S^1 \times S^1$ as a square with opposite sides identified. (The S^1_3 circle is actually contractible in $\mathsf{SU}(2) \simeq S^3$.)

To use our formula $\pi_1(G/H) = \pi_0(H)/\pi_0(G)$ requires that G is simply connected, so let's think about the universal cover $\hat{G}$ of G, which is $\hat{G} = \mathsf{SU}(2) \times \mathbb{R}$, and is connected $\pi_0(\hat{G}) = 0$. In the universal cover of G, the orbit of the two generators T_3 and Y is replaced by $\mathsf{U}(1)_3 \times R$, a whole row of these squares, with only top and bottom identified.

Now draw the image of $\mathsf{U}(1)_Q$ in each of these two spaces. In the single square, its image has a single connected component. It winds p times around one direction and r times around the other, like this (for $p = 3, r = 5$):



In $U(1)_3 \times R$, in contrast, this locus has r connected components:



Therefore

$$\pi_1(V) = \pi_1(\hat{G}/\hat{H}) = \pi_0(\hat{H})/\pi_0(\hat{G}) = \pi_0(\hat{H}) = \mathbb{Z}_r.$$

A more direct method was suggested by Zhengdi Sun: Let's try to gauge fix, $i.e.$choose representatives of the cosets. Write an element of the numerator as $(g, e^{\mathbf{i}\theta_Y})$, which transforms into $(e^{\mathbf{i}pT^3\theta}g, e^{\mathbf{i}(\theta_Y + r\theta)})$ under a $\mathsf{U}(1)_Q$ rotation. So choose $\theta = \frac{\theta_Y}{r} + \frac{2\pi k}{r}$ to set the second entry to unity. Here $k = 0..r-1$ is undetermined by this choice. The representative is then $(e^{\mathbf{i}\frac{pT^3(\theta_Y+2\pi k)}{r}}g, 1)$, which still has a residual $\mathbb{Z}_r$ action from the choice of k. Therefore $V = \mathsf{SU}(2)/\mathbb{Z}_r$, which has $\pi_1(V) = \mathbb{Z}_r$ (and $\pi_2(V) = 0$).

I got this problem and the next from Preskill's 1987 Les Houches lectures.

3. [This problem is also optional] Show that the Standard Model (SM) has no topologically-stable string or point-defect solutions.

For purposes of this question[1], the SM can be regarded as an ordered medium that breaks $G = \mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)_Y$ down to $\mathsf{SU}(3) \times \mathsf{U}(1)_Q$, where

$$Q = T_3 + Y$$

with the same normalization of generators as in the previous problem.

In a previous version of this problem, I had written that $Q = \cos\theta_W \tau^3 + \sin\theta_W Y$, where $\tan\theta_W = g/g'$ is the Weinberg angle, an irrational number. I was thinking that in order for the covariant derivative $(\partial_\mu - igW_\mu^a\tau^a - \frac{1}{2}ig?Y_\mu)\phi$ to actually be covariant, the transformation of ϕ under G must be

$$\phi \to e^{ig\alpha^a\tau^a + \frac{1}{2}ig?\alpha_Y}\phi \tag{1}$$

where α^a, α_Y parametrize the $SU(2) \times U(1)_Y$ transformation. If that?s correct, then the subgroup that?s unbroken when $\phi = (0, v)$ is when $g\alpha^3 + \frac{1}{2}g?\alpha_Y = 0$ (mod 2π). This much is all true.

However, you can see from (1) that the compactness relations of $SU(2)$ and $U(1)$ are satisfied not by α^a and α^Y but by $g\alpha^a$ and $g?\alpha^Y$. That is, we can just absorb g an $g?$ into the definitions of the gauge parameters, $\lambda^a \equiv g\alpha^a \in [0, 4\pi)$ and $\lambda_Y \equiv g?\alpha_Y \in [0, 2\pi)$. The unbroken generator is just $Q = \tau^3 + Y$. I?m sorry for the confusion. There is a very misleading statement in the QFT textbook by Ryder on this subject.

So this is really just the $r = 1$ case of the previous problem on HW 2.

We conclude $\pi_1(G/H) = \pi_0(\hat{H}) = 0$, so there are no strings. $\pi_2(G/H) = \pi_1(\hat{H})$ is also trivial.

[1] In the SM, these groups are actually gauge groups, and not symmetries. However, this only affects the energetics of the topological defects.

A cheap but effective way to arrive at this same answer is to note that the role of the order parameter field is played by the Higgs field $\Phi = (\Phi_1, \Phi_2)$ which is a doublet of $\mathsf{SU}(2)$ and is charged under $\mathsf{U}(1)_Y$. The minima of the potential lie at

$$V = \{v = |\Phi|^2 = \sum_\alpha \big(\mathrm{Re}\,(\Phi_\alpha)^2 + \mathrm{Im}\,(\Phi_\alpha)^2\big)\} = S^3.$$

And $\pi_2(S^3) = 0 = \pi_1(S^3)$.