University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 211C (239) Phases of Quantum Matter, Spring 2021
Assignment 3 – Solutions

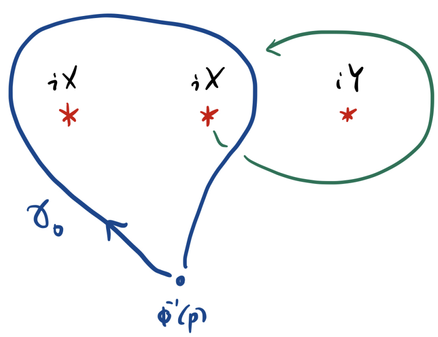
Due 12:30pm Friday, April 23, 2021

Thanks for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

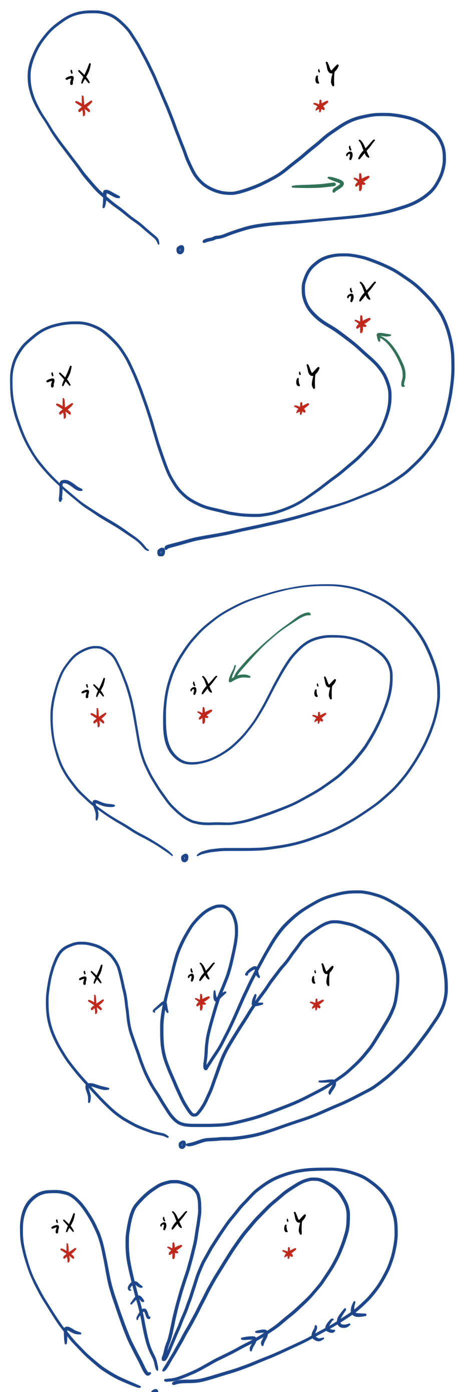
---

1. Consider a biaxial nematic in 2 spatial dimensions, in the presence of two $\mathbf{i}X$ disclinations and a $\mathbf{i}Y$ disclination. (Here I am choosing a base point and measuring the homotopy class of each of the defects by the image of a path starting at the pre-image of the base point, and going around only that defect.)

   Consider a path γ_0 that goes around the two $\mathbf{i}X$ disclinations, so that the holonomy around the path is $\mathbf{i}X \cdot \mathbf{i}X = -\mathbb{1}$.



   Now deform the configuration to move one of the $\mathbf{i}X$ disclinations in a circle around the $\mathbf{i}Y$ disclination. As you do this, deform the path γ_0 so that it continues to go only around the two $\mathbf{i}X$ disclinations. Now decompose the final path into a sequence of paths going only around one defect at a time (by deforming parts of the path to the base point). What element of Q_8 do you find?



To me, this looks like

$$\mathbf{i}X(\mathbf{i}Y)^{-1}\mathbf{i}X\mathbf{i}Y = \mathbf{i}X\,(-\mathbf{i}Y\,\mathbf{i}X\,\mathbf{i}Y) = \mathbf{i}X\,(\mathbf{i}Y\,XY) = \mathbf{i}X\,(-\mathbf{i}X) = \mathbb{1}.$$

2. Consider the term

$$S_0[A] = \int d^d x dt A_\mu j^\mu(x,t).$$

Show that this is gauge invariant, *i.e.* invariant under

$$A_\mu \to A_\mu + g^{-1}\partial_\mu g$$

with an arbitrary smooth map g : spacetime $\to \mathsf{U}(1)$, as long as the current j is conserved, $\partial^\mu j_\mu = 0$.

The variation is

$$\delta S_0 = \int j^\mu g^{-1}\partial_\mu g = \int j^\mu e^{-\mathbf{i}\theta}\partial_\mu e^{\mathbf{i}\theta} \tag{1}$$

$$\overset{\text{IBP}}{=} -\int \partial_\mu\left(j^\mu e^{-\mathbf{i}\theta}\right)e^{\mathbf{i}\theta} \tag{2}$$

$$\overset{\partial^\mu j_\mu=0}{=} -\int j^\mu\left(\partial_\mu e^{-\mathbf{i}\theta}\right)e^{\mathbf{i}\theta} \tag{3}$$

$$= \frac{1}{2}\int j^\mu\left(e^{-\mathbf{i}\theta}\partial_\mu e^{\mathbf{i}\theta} - \left(\partial_\mu e^{-\mathbf{i}\theta}\right)e^{\mathbf{i}\theta}\right) \tag{4}$$

$$= \frac{1}{2}\int j^\mu \partial_\mu\left(e^{-\mathbf{i}\theta}e^{\mathbf{i}\theta}\right) = \frac{1}{2}\int j^\mu\partial_\mu(1) = 0. \tag{5}$$

Alternatively, we can write $g = e^{i\alpha}$, in which case $-\mathbf{i}g^{-1}dg = d\alpha$. The only catch is that α is not single-valued. Its derivative is perfectly well-defined, though. Then

$$\delta S_0 \propto \int \partial_\mu \alpha j^\mu \overset{\text{IBP}}{=} \int \alpha\partial_\mu j^\mu = 0.$$

3. According to our result for its vacuum manifold $V = G/H$, what are the point-like and string-like topological defects of the A-phase of ^{3}He?

This is a bit of an open-ended question. Here are two concrete parts of it:

(a) Show that the charge-(-2) superfluid vortex line

$$\hat{d} = \hat{z}, \quad e^{(1)} + \mathbf{i}e^{(2)} = (\hat{x} + \mathbf{i}\hat{y})e^{-\mathbf{i}2\varphi} = (\hat{\rho} + \mathbf{i}\hat{\varphi})e^{-\mathbf{i}\varphi}$$

can be homotoped (through axisymmetric configurations) to a smooth configuration (with the same winding far away). (Above I am using cylindrical coordinates ρ, φ, z in $\mathbb{R}^3$.) The final smooth configuration at $\rho = 0$ is

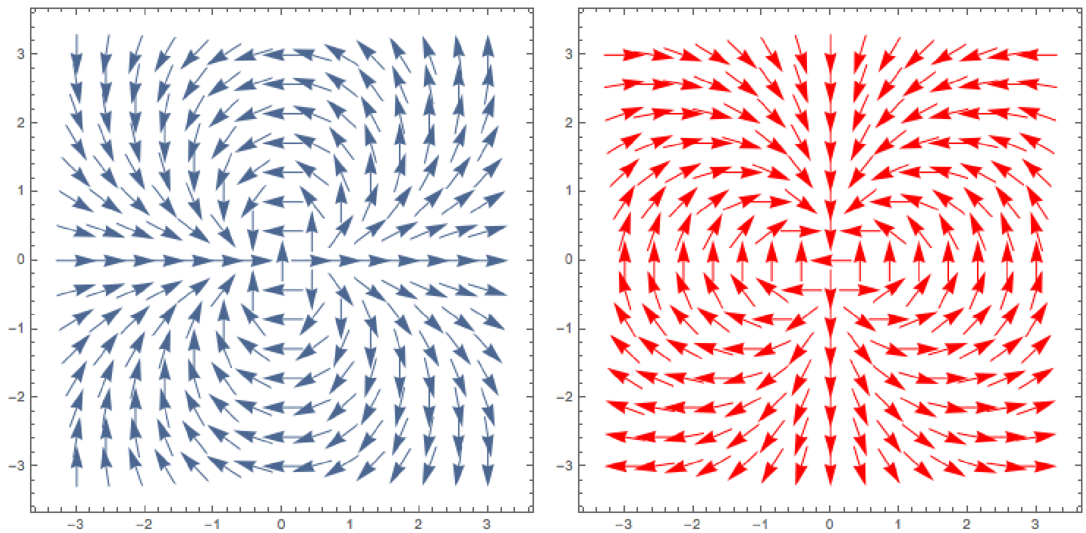
$$\hat{d} = \hat{z}, \quad e^{(1)} + \mathbf{i}e^{(2)} = (-\hat{x} + \mathbf{i}\hat{y})e^{-\mathbf{i}2\varphi} = (-\hat{\rho} + \mathbf{i}\hat{\varphi})e^{-\mathbf{i}\varphi}.$$

More generally, it is

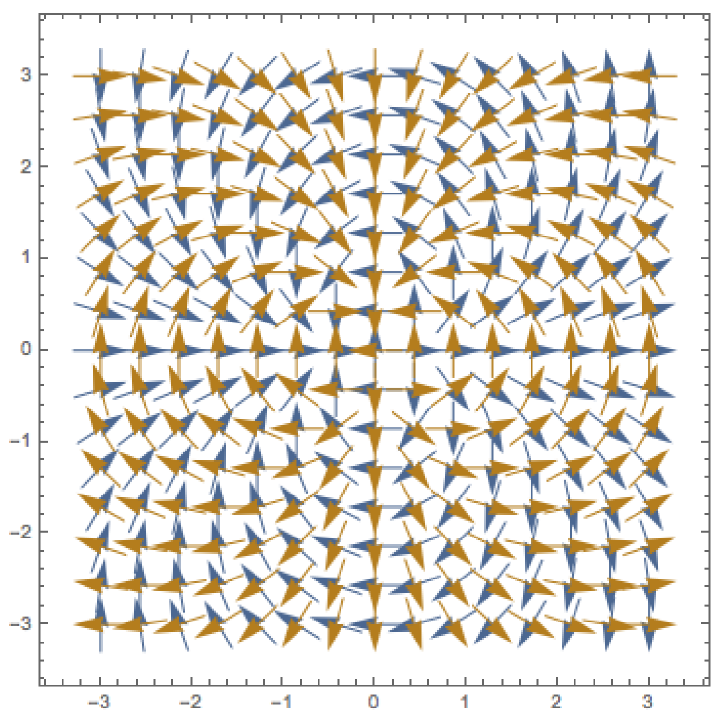
$$\hat{d} = \hat{z}, \quad e^{(1)} + \mathbf{i}e^{(2)} = \left(-\hat{z}\sin\eta(\rho) + \hat{\rho}\cos\eta(\rho) + \mathbf{i}\hat{\varphi}\right)e^{-\mathbf{i}\varphi}$$

where $\eta(\rho)$ is a function that interpolates between $\eta(\rho = 0) = \pi$ and $\eta(\rho = \infty) = 0$, such as $\cos\eta(\rho) = 1 - 2e^{-\rho/\rho_0}$ for some core size ρ_0. (First check that this configuration is indeed smooth at $\rho = 0$. Hint: write the real and imaginary parts in terms of $\hat{x}$ and $\hat{y}$.)

The real and imaginary parts of the original configuration look like:



or both together:



You can see the singularity in the core.

To check that the final configuration is smooth we can write it in terms of $\hat{x}$ and $\hat{y}$. We can do this using the relations

$$\hat{\rho} = \cos\varphi\hat{x} + \sin\varphi\hat{y}, \quad \hat{\varphi} = -\sin\varphi\hat{x} + \cos\varphi\hat{y}.$$

We have

$$\text{Re}\left(e^{(1)} + \mathbf{i}e^{(2)}\right)|_{t=1,\rho=0} = -\cos\varphi\hat{\rho} + \sin\varphi\hat{\varphi} = -\hat{x} \tag{6}$$

$$\text{Im}\left(e^{(1)} + \mathbf{i}e^{(2)}\right)|_{t=1,\rho=0} = \sin\varphi\hat{\rho} + \cos\varphi\hat{\varphi} = \hat{y}. \tag{7}$$

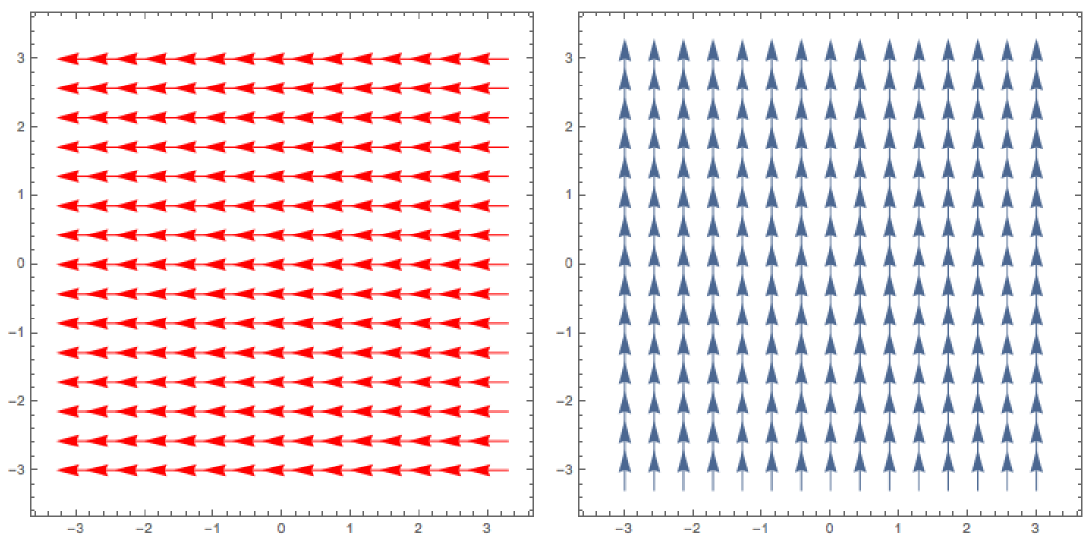
Therefore

$$\left(e^{(1)} + \mathbf{i}e^{(2)}\right)|_{t=1,\rho=0} = -\hat{x} + \mathbf{i}\hat{y}$$

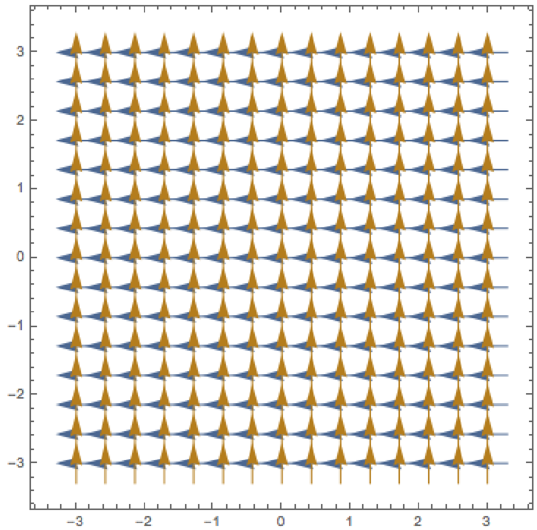
is perfectly well-defined at $\rho = 0$.

Another nice way (I learned from Zipei Zhang) is to use the relations $\partial_\varphi\hat{\rho} = \hat{\varphi}, \partial_\varphi\hat{\varphi} = -\hat{\rho}$.

Its real and imaginary parts (just near the core) look like this:



Or we can plot them both at the same time:



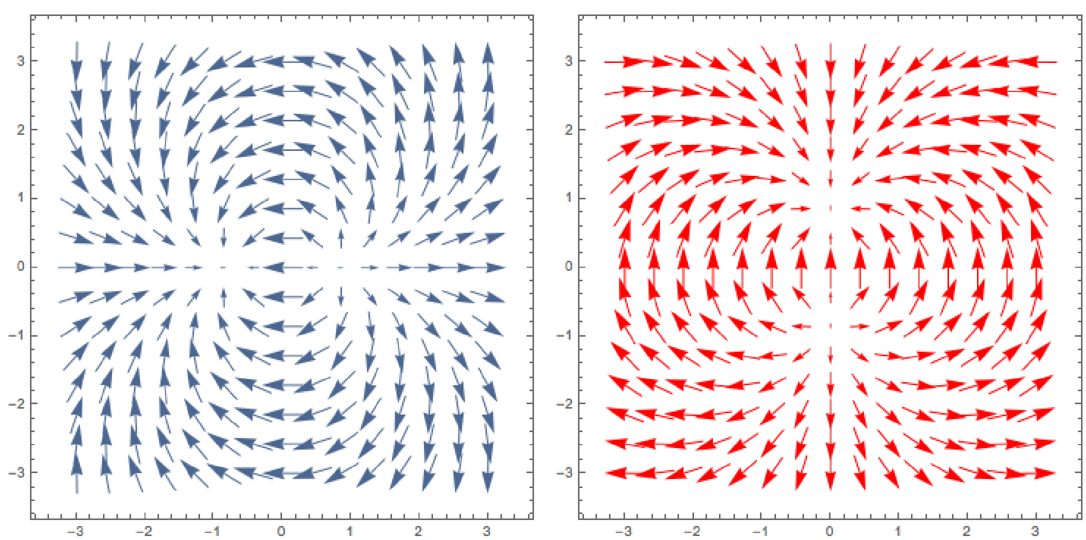
We can make a homotopy between the initial and final configurations by writing

$$\hat{d} = \hat{z}, \quad e^{(1)} + \mathbf{i}e^{(2)} = \left(-\hat{z}\sin\eta(\rho,t) + \hat{\rho}\cos\eta(\rho,t) + \mathbf{i}\hat{\varphi}\right)e^{-\mathbf{i}\varphi}$$

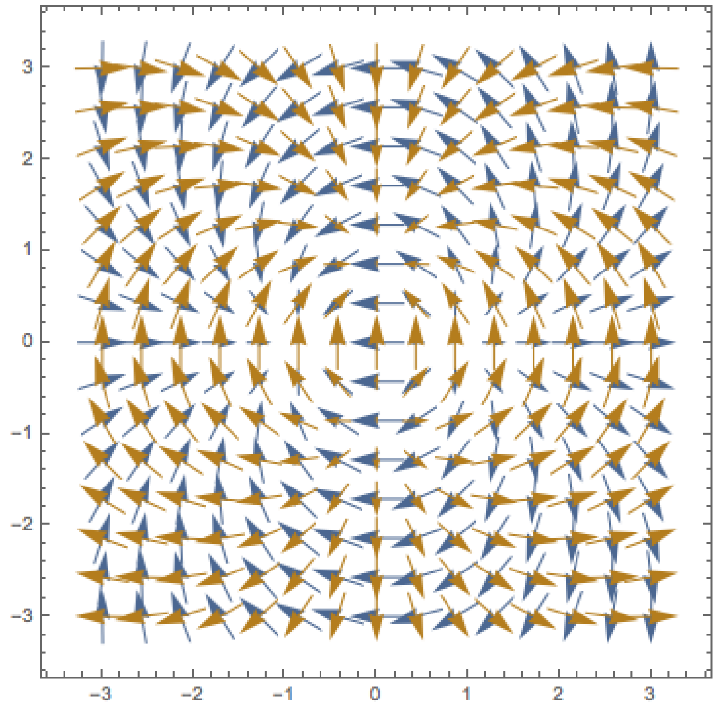
where $\eta(\rho,t)$ homotopes between $\eta(\rho,0) = 0$ and $\eta(\rho,1) = \eta(\rho)$ above. An example is

$$\cos\eta(\rho) = 1 - 2te^{-\rho/\rho_0}.$$

The real and imaginary parts of the full final configuration look like:

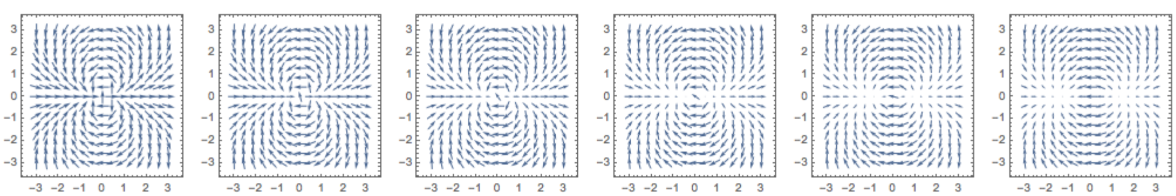


or both together:



They are visibly smooth.

Here's the homotopy on the real part (with $\rho_0 = 2$):



(b) Show using the long exact sequence of relative homology that

$$\pi_2(V_A, V_A^{\text{wall}}) = \mathbb{Z}$$

where V_A is the vacuum manifold of the A-phase, and V_A^{wall} is the vacuum manifold at a boundary to which $\hat{\ell}$ is restricted to be normal.

Let's study the simpler case of $\pi_2(G, H)$ with $G = \mathsf{SU}(2)$ and $H = \mathsf{U}(1)$. Then the long exact sequence is

$$\cdots \to \pi_2(\mathsf{U}(1)) \to \pi_2(\mathsf{SU}(2)) \to \pi_2(\mathsf{SU}(2), \mathsf{U}(1)) \to \pi_1(\mathsf{U}(1)) \to \pi_1(\mathsf{SU}(2)) \tag{8}$$

$$\cdots \to 0 \to 0 \to \pi_2(\mathsf{SU}(2), \mathsf{U}(1)) \to \pi_1(\mathsf{U}(1)) \to 0 \tag{9}$$

from which we learn $\pi_2(\mathsf{SU}(2), \mathsf{U}(1)) \cong \pi_1(\mathsf{U}(1)) = \mathbb{Z}$. The problem in the question is essentially the same,

$$\cdots \to \pi_2(H) \xrightarrow{i_\star} \pi_2(G) \to \pi_2(G, H) \to \pi_1(H) \to \pi_1(G)$$

because although the S^2 factor in V_A has a nontrivial π_2, but this is also there in $\pi_2(V)$, and so is in the image of the inclusion map $i_\star$. The only subtlety is in the $\mathbb{Z}_2$ action on H:

$$\pi_1(H) = \pi_1((S^2 \times S^1)/\mathbb{Z}_2 \times \mathbb{Z}_2) = \pi_1((S^2 \times S^1)/\mathbb{Z}_2)$$

but since it also acts on the simply-connected S^2 I believe it doesn't change the conclusion that $\pi_1(H) = \mathbb{Z}$, and therefore $\pi_2(G, H) \cong \pi_1(H) \cong \mathbb{Z}$.

4. **Vacancies.** [bonus problem] A vacancy is simply a lattice site in a crystal that is missing its atom. A vacancy is freely mobile, but cannot be destroyed locally. How are these properties reflected in the effective action $S[A, \theta]$?