University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 211C (239) Phases of Quantum Matter, Spring 2021
Assignment 3

Due 12:30pm Friday, April 23, 2021

Thanks for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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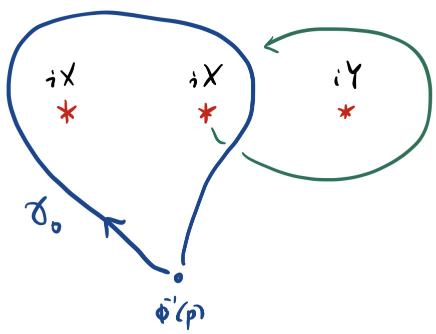
1. Consider a biaxial nematic in 2 spatial dimensions, in the presence of two $\mathbf{i}X$ disclinations and a $\mathbf{i}Y$ disclination. (Here I am choosing a base point and measuring the homotopy class of each of the defects by the image of a path starting at the pre-image of the base point, and going around only that defect.)

   Consider a path γ_0 that goes around the two $\mathbf{i}X$ disclinations, so that the holonomy around the path is $\mathbf{i}X \cdot \mathbf{i}X = -\mathbb{1}$.

   Now deform the configuration to move one of the $\mathbf{i}X$ disclinations in a circle around the $\mathbf{i}Y$ disclination. As you do this, deform the path γ_0 so that it continues to go only around the two $\mathbf{i}X$ disclinations. Now decompose the final path into a sequence of paths going only around one defect at a time (by deforming parts of the path to the base point). What element of Q_8 do you find?

2. Consider the term
$$S_0[A] = \int d^d x dt A_\mu j^\mu(x,t).$$
   Show that this is gauge invariant, *i.e.* invariant under
$$A_\mu \to A_\mu + g^{-1}\partial_\mu g$$
   with an arbitrary smooth map g : spacetime $\to \mathsf{U}(1)$, as long as the current j is conserved, $\partial^\mu j_\mu = 0$.

3. According to our result for its vacuum manifold $V = G/H$, what are the point-like and string-like topological defects of the A-phase of ^{3}He?

   This is a bit of an open-ended question. Here are two concrete parts of it:

   (a) Show that the charge-(-2) superfluid vortex line

   $$\hat{d} = \hat{z}, \quad e^{(1)} + \mathbf{i}e^{(2)} = (\hat{x} + \mathbf{i}\hat{y})e^{-\mathbf{i}2\varphi} = (\hat{\rho} + \mathbf{i}\hat{\varphi})e^{-\mathbf{i}\varphi}$$

   can be homotoped (through axisymmetric configurations) to a smooth configuration (with the same winding far away). (Above I am using cylindrical coordinates ρ, φ, z in $\mathbb{R}^3$.) The final smooth configuration at $\rho = 0$ is

   $$\hat{d} = \hat{z}, \quad e^{(1)} + \mathbf{i}e^{(2)} = (-\hat{x} + \mathbf{i}\hat{y})e^{-\mathbf{i}2\varphi} = (-\hat{\rho} + \mathbf{i}\hat{\varphi})e^{-\mathbf{i}\varphi}.$$

   More generally, it is

   $$\hat{d} = \hat{z}, \quad e^{(1)} + \mathbf{i}e^{(2)} = \left(-\hat{z}\sin\eta(\rho) + \hat{\rho}\cos\eta(\rho) + \mathbf{i}\hat{\varphi}\right)e^{-\mathbf{i}\varphi}$$

   where $\eta(\rho)$ is a function that interpolates between $\eta(\rho = 0) = \pi$ and $\eta(\rho = \infty) = 0$, such as $\cos\eta(\rho) = 1 - 2e^{-\rho/\rho_0}$ for some core size ρ_0. (First check that this configuration is indeed smooth at $\rho = 0$. Hint: write the real and imaginary parts in terms of $\hat{x}$ and $\hat{y}$.)

   (b) Show using the long exact sequence of relative homology that

   $$\pi_2(V_A, V_A^{\text{wall}}) = \mathbb{Z}$$

   where V_A is the vacuum manifold of the A-phase, and V_A^{wall} is the vacuum manifold at a boundary to which $\hat{\ell}$ is restricted to be normal.

4. **Vacancies.** [bonus problem] A vacancy is simply a lattice site in a crystal that is missing its atom. A vacancy is freely mobile, but cannot be destroyed locally. How are these properties reflected in the effective action $S[A, \theta]$?