University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 211C (239) Phases of Quantum Matter, Spring 2021
Assignment 4 – Solutions

Due 12:30pm Monday, May 3, 2021

Thanks for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

Abelian Chern-Simons problems.

1. Under a large $\mathsf{U}(1)$ gauge transformation,

$$a \to a - \mathbf{i} g^{-1} dg$$

find the variation of the $\mathsf{U}(1)$ Chern-Simons action on a closed 3-manifold M

$$S_0[a] = \int_M \frac{k}{4\pi} a \wedge da \ .$$

Conclude that in the absence of *other* interestingness (such as degenerate groundstates not coming from the dynamics of a), the level k must be an even integer.

Let's write $\omega = d\phi = -\mathbf{i} g^{-1} dg$. This is a closed form, $d\omega = 0$, but it is not exact, since ϕ is not necessarily a globally well-defined function (it can jump by 2π anywhere).

The variation is $\delta S_0 = \frac{k}{4\pi} \int_M \omega \wedge da$. You might be tempted to integrate by parts and say $d\omega = 0$ and therefore this vanishes. But a is not globally well-defined, so it's not true that d of something involving a has to vanish on a closed manifold. A familiar example is $\int_{S^2} F = 2\pi$ for the sphere surrounding a magnetic monopole.

We can argue that $\frac{1}{2\pi} \int \omega \wedge da$ is 2π times an integer by following the logic we used earlier when we studied the variation of $S_\nu[\theta, A]$: first show that it's topological, in the sense of independent of local variations of its arguments, then evaluate it on nice configurations where we can do the integral.

The first step follows because both ω and da are closed. For the second step, we can choose a nice 3-manifold, such as $S^1 \times S^2$, where the period of the circle is L and the coordinate is t ($t \equiv t + L$). Consider a field configuration where the gauge flux is constant in t. If we take $g = e^{\frac{2\pi \mathbf{i} t}{L}}$, then $\omega = \frac{2\pi}{L} dt$, we find

$$\delta S_0 = -\frac{k}{4\pi} \int_0^L \frac{2\pi}{L} dt \underbrace{\int_{S^2} f}_{\in 2\pi\mathbb{Z}} \in \pi k \mathbb{Z}.$$

Therefore, k must be an even integer, if there is nothing else around to make the amplitude gauge invariant. But, you say, we've been talking about the case $k = 1$ all the time as a description of the integer QHE! The answer is that the theory with odd k does make sense, but only if the system is fermionic. We'll come back to this later.

I apologize for the misleading problem statement.

2. For the abelian Chern-Simons theory with gauge group $\mathsf{U}(1)$ at level k,

$$S[a, \mathcal{A}] = \int \left(\frac{k}{4\pi} a \wedge da + \mathcal{A} \wedge \frac{da}{2\pi} \right).$$

do the (gaussian!) path integral over a to find the effective action for the background field $\mathcal{A}$. Find the Hall conductivity.

See the next problem.

3. Now do it for the general K matrix and general charge vector t^I, with

$$S[a^I, \mathcal{A}] = \int \left(\frac{K_{IJ}}{4\pi} a^I \wedge da^J + \mathcal{A} \wedge t_I \frac{da^I}{2\pi} \right).$$

Let's just do it all at once. The path integral is

$$\int [Da] e^{\mathbf{i}S[a,\mathcal{A}]} = e^{\mathbf{i}S_{\text{eff}}[\mathcal{A}]}.$$

Since the Hall conductivity is a local quantity, let's just put the system on the plane or the sphere, where there is no opportunity for a to create any topological mischief, and we can just do the integral. Complete the square in the exponent:

$$\mathbf{i} \int \left(\frac{K_{IJ}}{4\pi} a^I \wedge da^J + \mathcal{A} \wedge t_I \frac{da^I}{2\pi} \right)$$
$$= \mathbf{i} \int \frac{K_{IJ}}{4\pi} \left(a^I + \left(K^{-1}\right)^{IK} t_K \mathcal{A} \right) d \left(a^J + \left(K^{-1}\right)^{JL} t_L \mathcal{A} \right) - \frac{K_{IJ}}{4\pi} \left(\left(K^{-1}\right)^{IK} t_K \mathcal{A} \right) d \left(\left(K^{-1}\right)^{JL} t_L \mathcal{A} \right).$$

Now change variables in the integral $a^I \to a^I + (K^{-1})^{IK} t_K \mathcal{A}$. On the plane this is fine, and the integral is just a constant. All that is left is

$$S_{\text{eff}} = -t_I \left(K^{-1}\right)^{IJ} t_J \int \frac{\mathcal{A} \wedge d\mathcal{A}}{4\pi}.$$

We conclude that the Hall conductivity is

$$\sigma^{xy} = \frac{e^2}{h} t_I \left(K^{-1}\right)^{IJ} t_J.$$

4. **Flux attachment.** Now consider

$$S_j[A] = \int \left(\frac{k}{4\pi} a \wedge da + a \wedge \star j \right).$$

   Find the equations of motion. Show that the Chern-Simons term *attaches k units of flux* to the particles: $F_{12} \propto \rho$.

5. **Anyons.** Show using the Bohm-Aharonov effect that the particles whose current density is j^μ have anyonic statistics with exchange angle $\frac{\pi}{k}$ (supposing they were bosons before we coupled them to A).

   One way to do this is to consider a configuration of j which describes one particle adiabatically encircling another. Show that its wavefunction acquires a phase $e^{\mathbf{i}2\pi/k}$. This is twice the phase obtained by going halfway around, which (when followed by an innocuous translation) would exchange the particles.

   See the next problem.

6. Describe the statistics of the anyonic quasiparticles in the case with general K matrix.

   The EoM are

$$\frac{K_{IJ}}{2\pi} da^J = \star j_I$$

   which means $da^I = 2\pi \left(K^{-1}\right)^{IJ} \star j_J$. Bringing anyon one with charge l_1 all the way around anyon two with charge l_2 gives the phase

$$\Phi_{2\pi} = (l_1)_I \oint_C a^I = (l_1)_I \int_{R, \partial R = C} 2\pi \left(K^{-1}\right)^{IJ} (\rho_2)_J = 2\pi (l_1)_I \left(K^{-1}\right)^{IJ} (l_2)_J .$$

   The exchange phase is half of this.