University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 211C (239) Phases of Quantum Matter, Spring 2021
Assignment 5 – Solutions

Due 12:30pm Monday, May 10, 2021

Thanks for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Quasiparticle wavefunctions.**

   (a) Use the parton construction of the Laughlin $\nu = \frac{1}{m}$ state to construct wavefunctions for the quasihole and quasiparticle.

   (Hint: add or remove a single parton. Don't forget to project onto the lowest Landau level.)

   A simple (parton-independent) way to motivate the quasihole wavefunction is to find the wavefunction that results by threading 2π flux at the point w in the complex plane. (We saw earlier that on general grounds, if the state is gapped, this produces an excitation with statistics $\pi\sigma^{xy}$.) Threading 2π flux at w means that the wavefunction should acquire the phase $e^{\mathbf{i}\theta}$ when we move *any* of the electrons around the point w: $z_i - w \to e^{\mathbf{i}\theta}(z_i - w)$, $\forall i = 1..N$. A very easy way to accomplish this is to multiply the wavefunction by the factor

   $$\prod_{i=1}^{N}(z_i - w) \ .$$

   That's it. No need for an LLL projection, since it's still holomorphic. The full wavefunction for a quasihole at w is then

   $$\tilde{\Psi}_w(z) = \prod_{i=1}^{N}(z_i - w)\prod_{i<j}^{N}(z_i - z_j)^m e^{-\sum_i \frac{|z_i|^2}{4\ell_B^2}} \ .$$

   Notice that this is still a wavefunction for N electrons.

   The quasiparticle wavefunction should acquire the opposite phase, so we'd like to multiply by

   $$\prod_{i=1}^{N}(\bar{z}_i - \bar{w})$$

but that's not an LLL wavefunction. The projection of this to the LLL is the quasiparticle wavefunction.

$$\tilde{\Psi}_{\bar{w}}(z) = \mathcal{P}_{LLL} \prod_{i=1}^{N} (\bar{z}_i - \bar{w}) \prod_{i<j}^{N} (z_i - z_j)^m = \prod_i \left(2\ell_B^2 \partial_{z_i} - \bar{w}\right) \prod_{i<j} (z_i - z_j)^m.$$

Let's try acting on the parton groundstate with a single parton annihilation operator, $f_\alpha(w)$, where α is a species label on the partons: $c = \prod_\alpha f_\alpha$. The problem with this idea is that it removes a parton. But the projection to the gauge invariant Hilbert space requires that there be the same number of electrons as each type of parton, so that the state has a nonzero overlap $\Psi(r) = \langle 0| \prod_i c(r_i) |\text{parton state}\rangle$.

If we act with a single creation operator $f^\dagger(\eta)$, we must start occupying the second parton Landau level, so that's a good sign that we'll need an LLL projection.

But I conclude that at the moment I don't know how to motivate the Laughlin quasihole and quasiparticle wavefunctions from partons. Please let me know if you do.

(b) Construct a wavefunction with *two* quasiholes and use it to verify their statistics.

This calculation was first done here.

The state is

$$\tilde{\Psi}_{12}(z) = \prod_{i=1} (z_i - w_1) \prod_{i=1} (z_i - w_2) \prod_{i<j} (z_i - z_j)^m.$$

Let's compute the Berry connection for varying w_1:

$$\mathcal{A}_{w_1} = \langle \Psi_{12}| \mathbf{i}\partial_{w_1} |\Psi_{12}\rangle = \langle \Psi_{12}| \sum_i \frac{\mathbf{i}}{w_1 - \hat{z}_i} |\Psi_{12}\rangle .$$

The Berry phase accumulated by moving w_1 in a circle (of radius, say R)around w_2 is then

$$\gamma_{12} \equiv \oint_{C_{w_2}} dw_1 \mathcal{A}_{w_1} = \langle \Psi_{12}| \mathbf{i} \oint_{C_{w_2}} dw_1 \sum_i \frac{1}{w_1 - \hat{z}_i} |\Psi_{12}\rangle = \langle \Psi_{12}| (-2\pi) \sum_i \Theta(\hat{z}_i \in C_{w_2}) |\Psi_{12}\rangle$$

where we used Cauchy's theorem, and

$$\Theta(s) \equiv \begin{cases} 1, & \text{if the statement } s \text{ is true} \\ 0, & \text{else} \end{cases} .$$

This last expression is the average number of electrons inside the circle of radius R about w_2 (times -2π). If there were no quasihole at w_2, this would be (for large enough R) just $-2\pi\nu\frac{\Phi}{\Phi_0}$, where $\Phi = \int_{C_{w_2}} \vec{B}\cdot d\vec{a}$ is the flux through the circle. This contribution is not necessarily 2π times an integer and I think should be regarded as some background noise.

The presence of the quasihole at $z = w_2$ decreases the electron density. It decreases the expected number of electrons in the neighboring region by $\frac{1}{m}$[1], and therefore the contribution from w_2 to the Berry phase is $\gamma_{12} = -2\pi\frac{1}{m}$. The quasihole exchange phase is then

$$\theta_{12} = \frac{\gamma_{12}}{2} = \frac{\pi}{m} = \pi\nu.$$

2. **Hall plateaux as a crazy manifestation of quantum oscillations.** Check the claim that the hierarchy states at fillings $\nu = \frac{\nu^\star}{2\nu^\star \pm 1}$ for $\nu^\star \in \mathbb{Z}$ can be regarded as an extreme version of quantum oscillations in the HLR state at $\nu = \frac{1}{2}$.

   We work at fixed electron density ρ throughout, so B and ν are related by $\rho = \nu B/\Phi_0$. Write $B = B_{\nu=\frac{1}{2}} + \delta B = 2\Phi_0\rho + \delta B$, so

$$\delta B = \Phi_0\rho\left(\frac{1}{\nu} - 2\right).$$

   If $\frac{1}{\delta B} = \pm\nu^\star\frac{1}{\rho\Phi_0}$, we find

$$\pm\frac{1}{\nu^\star} = \frac{1}{\nu} - 2$$

   which indeed gives the relation for the states at the first level of the hierarchy.

3. **Charges of quasiparticles in abelian CS EFT.**

[1] Here I am appealing to a result from the plasma analogy. The charge density

$$\rho(z,\bar{z}) = \int \prod_{i=2}^{N} d^2 z_i |\Psi_w(z, z_2 \cdots z_N)|^2 = \int \prod_{i=2}^{N} d^2 z_i e^{\sum_{1<i<j} \log|z_i - z_j|^2 + \sum_{1<i} \log|z-w|^2 - \sum_i \frac{|z_i|^2}{2\ell_B^2}}$$

in the quasihole wavefunction is the density of a one-component plasma of charge-m objects (with logarithmic mutual interactions) that see a neutralizing background (that's the quadratic term) plus an extra potential from a fixed impurity of positive unit charge at $z = w$. As Girvin and Yang say (page 447), 'the chief desire of the plasma is to maintain charge neutrality'. This is accomplished by forming a screening cloud near $z = w$ to screen the impurity. Screening the cloud requires a deficit of $1/m$th of a charge-m particle. Those particles sit at the electron positions, so this is $1/m$th of an electron missing.

In an abelian CS theory with K-matrix K, show that a quasiparticle with charge ℓ^I under CS gauge field a^I has electric charge

$$q_l = tK^{-1}l.$$

The EFT for a charge at the origin is

$$L = \frac{1}{4\pi}K_{IJ}a^I da^J + \frac{1}{2\pi}At_I da^I + \ell_I a_0^I \delta^2(x).$$

The EOM for a_0^I is

$$0 = \frac{\delta S}{\delta a_0} = \frac{1}{2\pi}Kda + \ell\delta^2$$

so

$$da = 2\pi K^{-1}\ell\delta.$$

The source for A_0 is then

$$\frac{1}{2\pi}t_I da^I = tK^{-1}\ell\delta^2(x).$$

4. **Excitations of hierarchy states.** Find the torus groundstate degeneracy, and the charges and statistics of the quasiparticle excitations of the abelian incompressible FQH state at $\nu = \frac{2}{5}$.

   This state is described by the hierarchy construction with $m = 3$ and $\nu^\star = 2$. The EFT is

$$4\pi L = 3ada + 2Ada + 2ad\tilde{a} + 2\tilde{a}d\tilde{a},$$

   that is the K-matrix is

$$K = \begin{pmatrix} k & 1 \\ 1 & \tilde{k} \end{pmatrix}$$

   and the charge vector is $t = (1,0)$. You can check that indeed the Hall conductivity is $tK^{-1}t = \frac{2}{5}$.

$$\det K = 5$$

   so the torus GSD is 5-fold.

   A single qp with charge ℓ^I under $a^I = (a, \tilde{a})$ has electric charge

$$tK^{-1}\ell.$$

   For $\ell = (1,0)$, this gives $q = \frac{2}{5}$ and for $\ell = (0,1)$, this gives $q = -\frac{1}{5}$.