

**Physics 211C (239) Phases of Quantum Matter,
Spring 2021
Assignment 6 – Solutions**

Due 12:30pm Wednesday, May 19, 2021

Thanks for following the submission guidelines on [hw01](#). Please ask me by email if you have any trouble.

1. Boson Integer Quantum Hall State from Partons.

Consider a system made from two species of bosons, b_\uparrow, b_\downarrow . They could be two layers. We'll assume that only the total boson number, acting by $(b_\uparrow, b_\downarrow) \rightarrow e^{i\alpha}(b_\uparrow, b_\downarrow)$ is conserved, and couple to a background field \mathcal{A} for that symmetry.

(a) Consider the parton ansatz:

$$b_\uparrow = f_0 f_\uparrow, \quad b_\downarrow = f_0 f_\downarrow f_1 f_2$$

where all the f s are fermionic partons. There are three U(1) gauge fields that glue these partons back together, and the charge assignments are as follows:

	a_1	a_2	a_3	\mathcal{A}	Chern # in Phase 1	Chern # in Phase 2	Chern # in Phase 3
f_\uparrow	1	0	0	1	1	1	1
f_\downarrow	1	1	0	1	1	1	1
f_0	-1	0	0	0	-1	-1	-1
f_1	0	-1	1	0	-1	-1	-1
f_2	0	0	-1	0	-1	0	1

Also in the table are the Chern numbers of the bands filled by each of the partons in three distinct phases. (Only the Chern number of f_2 changes.) Identify the three phases, and describe the critical theories separating them.

Hint: I recommend describing the parton currents in terms of dynamical gauge fields $j_\mu^{(\alpha)} = \frac{1}{2\pi} \epsilon_{\mu\nu\rho} \partial_\nu b_\rho^{(\alpha)}$, where $\alpha = \uparrow, \downarrow, 0, 1, 2$.

Following the advice of the hint, the effective action is

$$4\pi L = 2((b_\uparrow + b_\downarrow - b_0)da_1 + (b_\downarrow - b_1)da_2 + (b_1 - b_2)da_3) \quad (1)$$

$$+ b_\uparrow db_\uparrow + b_\downarrow db_\downarrow - b_0 db_0 - b_1 db_1 + b_2 db_2 + 2\mathcal{A}d(b_\uparrow + b_\downarrow). \quad (2)$$

The $a_{1,2,3}$ are lagrange multipliers gluing the partons back together. Solving the constraints as $b_1 = b_2 = b_\downarrow, b_0 = b_\uparrow + b_\downarrow$, we have

$$4\pi L = b_\uparrow db_\uparrow + cb_\downarrow db_\downarrow - (b_\uparrow + b_\downarrow)d(b_\uparrow + b_\downarrow) + 2(b_\uparrow + b_\downarrow)d\mathcal{A}.$$

Here $c = -1, 0, 1$ in Phases 1-3. In phase 3, we find

$$4\pi L = Kbdb + 2tbd\mathcal{A}, \quad K = \sigma^x, t = (1, 1)$$

which is our K-matrix description of the boson IQHE. In phase 1 where $c = -1$, we find $K = \begin{pmatrix} 0 & 1 \\ 1 & 1 - c \end{pmatrix}$ which when $c = -1$ gives $\sigma^{xy} = 0$.

Finally, phase 2 is trickier. It is a superfluid. To see this, integrate out all the b_α leaving the a_i instead.

This result, and the tex for the table, are from [this paper](#). [This paper](#) came out on the same day (from the same department) and is very similar.

- (b) For this part of the problem, let's retreat to the continuum. Consider the simpler parton ansatz:

$$b_\uparrow = f_0 f_\uparrow, \quad b_\downarrow = f_0 f_\downarrow$$

where all the f s are fermionic partons. Choose the $U(1)_{\mathcal{A}}$ to be charges $q_0 = 2, q_\uparrow = -1, q_\downarrow = -1$.

Consider an equal number N of b_\uparrow and b_\downarrow particles, so that the total filling fraction is $\nu = 2$. How many f_0 particles are there, and how many f_\downarrow, f_\uparrow particles are there?

There are $2N$ f_0 s and N of each of the other two species.

Write a candidate groundstate wavefunction $\Psi(r_i^\uparrow, r_i^\downarrow)$ for the bosons.

There are twice as many f_0 particles, but they see a magnetic field that's twice as big, so they can fill their Landau level: $\psi_0 = \prod_{i<j} z_{ij}^\uparrow z_{ij}^\downarrow \prod_{ij} (z_i^\uparrow - z_j^\downarrow)$. Similarly the f_\uparrow and f_\downarrow fill their Landau levels, but they see a magnetic field of the opposite sign, so $\psi_{\uparrow/\downarrow} = \prod_{i<j} \bar{z}_{ij}^{\uparrow/\downarrow}$. The gauge projection glues the coordinates of the f_0 to both the coordinates of f_\downarrow and f_\uparrow . Therefore

$$\Psi(r_i^\uparrow, r_i^\downarrow) = \prod_{i<j} |z_{ij}^\uparrow|^2 |z_{ij}^\downarrow|^2 \prod_{ij} (z_i^\uparrow - z_j^\downarrow).$$

- (c) Bonus question: why does the simpler ansatz of the previous part produce a wavefunction in the same phase as one of the phases of the first part?

The change in the wavefunction from including f_1, f_2 with the assigned Chern numbers is just a factor of a real number $\prod |z_{ij}^\downarrow|^2$. This doesn't change

the statistics of the quasiparticles. The factors $|z_{ij}^{\uparrow/\downarrow}|^2$ could similarly be argued not to affect things. However, they matter for producing a uniform particle density in each layer and make a big difference for the quality of the variational ansatz.

- (d) Actually, here is a simpler description of the same phase diagram, closer to what I said in lecture. Consider a single species of boson, with the simple parton ansatz with $b = d_1 d_2$ in terms of two fermions. Let d_1 and d_2 fill Chern bands with total Chern number c_1 and c_2 . Fix $c_1 = -1$. Consider what happens when $c_2 = 2$.

Describe the effective field theory of d_2 filling two bands with chern number 1 by introducing two gauge fields each with CS term $\frac{1}{4\pi} b_a db_a$.

See Appendix B of [this paper](#).