University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 211C (239) Phases of Quantum Matter, Spring 2021
Assignment 7

Due 12:30pm Friday, May 28, 2021

Thanks for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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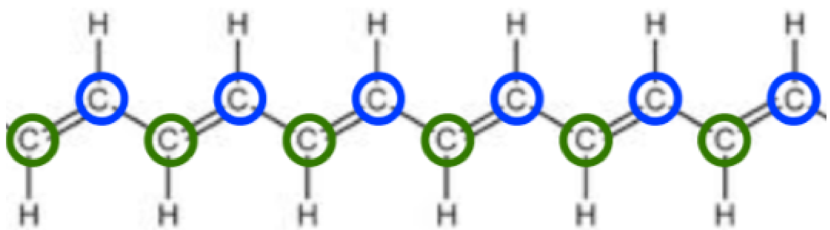
1. **Su-Schrieffer-Heeger model as an SPT.** Consider the following system, which we can regard as protected by $G = \mathsf{U}(1) \times \mathbb{Z}_2$ in $D = 1 + 1$. It is a sort of model of polyacetylene, which looks something like this:



The Hilbert space is a representation of two complex fermion modes per unit cell (c_a and c_b – we'll ignore spin), with Hamiltonian

$$H = -\sum_j \left(t_1 c_{ja}^\dagger c_{jb} + h.c. \right) - \sum_j \left(t_2 c_{ja}^\dagger c_{(j+1)b} + h.c. \right). \quad (1)$$

(The strengths of the hopping amplitudes t_1 and t_2 model the lengths of the two kinds of bonds in the figure.)

(a) Describe the physics of an open chain at half-filling (one particle per unit cell) in the limits $(t_1, t_2) = (1, 0)$ and $(t_1, t_2) = (0, 1)$.

(b) Diagonalize H in the momentum basis and draw the spectrum. What happens when $t_1 = t_2$?

(c) Check that if $t_{1,2}$ are real, then this model has an antiunitary $\mathbb{Z}_2$ particle-hole symmetry acting by

$$\mathcal{C} : c_a \leftrightarrow c_a^\dagger, c_b \leftrightarrow -c_b^\dagger, \mathbf{i} \leftrightarrow -\mathbf{i}.$$

What does this symmetry do to the single-particle hamiltonian $\mathcal{H}(k)$ defined by $H = \oint đk \left(c_{ka}^\dagger, c_{kb}^\dagger \right) \mathcal{H}(k) \begin{pmatrix} c_{ka} \\ c_{kb} \end{pmatrix}$? What does this imply for the single-particle spectrum?

(d) Compute the polarization

$$P = \frac{1}{2\pi} \oint dk \, \langle \psi_k | \, \mathbf{i}\partial_k \, | \psi_k \rangle$$

at half-filling (one particle per unit cell) as a function of t_1/t_2. (In this expression $|\psi_k\rangle$ is the occupied state.) Relate the resulting surface charge to your answer in part 1a.

(e) What happens to the polarization if we allow imaginary hoppings? Can you use this to design a Thouless pump?

(f) [Bonus] What happens at a domain wall between a region with $t_1/t_2 > 1$ and one with $t_1/t_2 < 1$?

(g) Actually, the model (1) has many symmetries. Check that the *unitary* particle-hole symmetry

$$\mathcal{S} : c_a \leftrightarrow c_a^\dagger, c_b \leftrightarrow -c_b^\dagger, \mathbf{i} \leftrightarrow \mathbf{i}$$

also preserves H. (Note that $\mathcal{C} = \mathcal{S}\mathcal{T}$ where $\mathcal{T} : c \to c, \mathbf{i} \to -\mathbf{i}$ is ordinary time reversal symmetry.) What does $\mathcal{S}$ do to $\mathcal{H}(k)$?

(h) Show that the single-particle hamiltonian $\mathcal{H}(k)$ has the form

$$\mathcal{H}(k) = h_x(k)\sigma^x + h_y(k)\sigma^y = \vec{h}(k) \cdot \vec{\sigma} \tag{2}$$

with $|h(k)|^2$ nonzero for all k (where $h(k) \equiv h_x(k) + \mathbf{i}h_y(k)$). It therefore defines a map from the Brillouin zone to $\mathbb{R}^2 \setminus \{0\} \simeq S^1$

$$\mathcal{H}(k) : S^1 \to \mathbb{R}^2 \setminus \{0\}$$

which has a winding number $\nu \in \mathbb{Z}$. Respecting the symmetry $\mathcal{S}$ thus produces an integer classification of such states.

What is the physical interpretation of $\nu > 1$? Can you find an $\mathcal{S}$-invariant Hamiltonian that has $\nu > 1$?

Relate this winding number (mod two) to the polarization.

(i) What terms can you add to H that respect $\mathcal{C}$ but break $\mathbf{S}$? What terms respect $\mathcal{S}$ but break $\mathcal{C}$? What do these do to the $\mathbb{Z}$-valued invariant?

(j) [Bonus] This has been a long problem, and I've still left out a crucial part of the story about polyacetylene. This is that the dimerization pattern, t_1/t_2 is a dynamical variable, a mode of the lattice. Because the energy of the electrons is lowered when $t_1/t_2 \neq 1$, the system prefers to be dimerized. But the potential for t_1/t_2 is symmetric about 1, so the dimerization pattern spontaneously breaks a $\mathbb{Z}_2$ symmetry. The domain walls of this broken symmetry are the ones in part 1f. They carry charge but not spin, unlike the electron. (Pretend that we included the spin degree of freedom of the electron in all of the above.) Use this to explain the observation that upon doping in extra charge, the conductivity of polyacetylene increases rapidly, but the magnetic susceptibility does not.

(k) [Bonus] Find the spectrum of an open chain (of, say, 10 or 15 unit cells), and plot it as a function of t_1/t_2. Watch the edge states get absorbed into the continuum near the phase transition. Plot the wavefunctions of the edge states and compare them to generic states.

2. **Chern-number changing transition.**

Consider the following approximate single-particle Hamiltonian for a particle in $D = 2 + 1$:

$$\mathcal{H}(k) = v(k_x\sigma_x + k_y\sigma_y) + m\sigma_z$$

(valid near $\vec{k} = 0$), describing a single Dirac cone as $m \to 0$.

(a) [Bonus] Argue that, up to relabellings and rescalings, this is the generic form for the single-particle Hamiltonian near a point in parameter space where two bands are colliding.

(b) Compute the Chern number of the bands as a function of m.

(c) Argue that such an $\mathcal{H}$ cannot arise from a local lattice model, without additional contributions to the Berry curvature elsewhere in the Brillouin zone.