University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 211C (239) Phases of Quantum Matter, Spring 2021
Assignment 8 – Solutions

Due 12:30pm Monday, June 7, 2021

Thanks for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Helical majorana mode on a surface domain wall of a TI.**

   Consider a $D = 3 + 1$ fermion topological insulator protected by charge conservation and time-reversal symmetry. If we stick an s-wave superconductor on the surface, we can gap out the surface Dirac cone, but the vortices are interesting because they carry majorana zeromodes.

   What happens if we (somehow) produce on the surface a domain wall in the phase of the superconducting order? That is, suppose that on the surface ($z = 0$), when $y > 0$ the superconducting pairing field is $\Delta = \Delta_0$, but for $y < 0$, it is $\Delta = e^{\mathbf{i}\gamma}\Delta_0$ for some phase γ. If $\gamma = \pi$, what happens on the $D = 1 + 1$ dimensional locus at $y = 0$?

   The Dirac equation on the surface is

   $$0 = \mathbf{i}\gamma^y \partial_y \eta + \Delta^\star(y)\sigma^2\eta^\star + \mathbf{i}\gamma^a\partial_a\eta.$$

   We choose $\gamma^y = -\mathbf{i}\sigma^2$ and make the ansatz $\eta = e^{\mathbf{i}\pi/2}e^{A(y)}\eta_0$ (with η_0 a $D = 1 + 1$ majorana spinor) and find that we need

   $$\partial_y A = -\Delta^\star(y).$$

   In the special case where $\gamma = \pi$, the solution is $A(y) = -\int_0^y \Delta^\star(y)$. This results in zeromode with a profile that decays exponentially away from the wall. The zeromode satisfies the equation

   $$0 = \mathbf{i}\gamma^a\partial_a\eta_0$$

   for a $D = 1 + 1$ majorana spinor, a helical majorana mode.

   Maybe this phenomenon has actually been observed in this paper. In this material both the superconductivity and the π domain walls in the superconducting phase seem to come for free. It's hard to tell, but it seems like they see some gapless degrees freedom in the domain wall.

2. **Majorana chain.**

   Consider the majorana chain

$$H = \mathbf{i} \sum_i \left(t_e \gamma_i \tilde{\gamma}_i + t_o \tilde{\gamma}_i \gamma_{i+1} \right)$$

   with $\{\gamma_i, \gamma_j\} = \delta_{ij} = \{\tilde{\gamma}_i, \tilde{\gamma}_j\}, \{\gamma_i, \tilde{\gamma}_i\} = 0$.

   (a) Show that the Hamiltonian can be rewritten as a p-wave superconductor of spinless electrons.

   (b) Find the energy spectrum. Check that when $t_e = t_o$, the gap closes.

   See the calculation here.

3. **Majorana lasagna.**

   Consider a layer in the coupled-layer construction to be the critical limit of the Kitaev chain, that is, a massless 1+1d Majorana fermion field. Apply the coupled-layer construction to make a model in $D = 2 + 1$ dimensions with no symmetry. What 2+1d free-fermion SPT state do you make this way?

4. **All-fermion toric code.**

   (a) Check that $\mathsf{U}(1)^4$ CS theory with K-matrix

$$K_{\mathsf{SO}(8)} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

   has the same spectrum of anyons as the all-fermion toric code.

   The matrix of braiding statistics is

$$\frac{\Theta^{IJ}}{\pi} = \left(K^{-1}\right)^{IJ} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} .$$

   So you see that the first particle is a self-boson with trivial statistics with everyone – the identity operator. The other three are self-fermions (e, m, ϵ) that are mutual semions with each other. This is the all-fermion toric code.

(b) Check that the statistics of the 16 anyons in two copies of the toric code is related to that of two copies of the all-fermion toric code by a relabelling. (At least check the self-statistics.)

(c) [Bonus] A perhaps better way to address the previous part, using the result of the first part: show that the $\mathsf{U}(1)^8$ Chern-Simons theory with K-matrix

$$K_{\mathsf{SO}(8)} \oplus \left(-K_{\mathsf{SO}(8)}\right)$$

is equivalent to two copies of the toric code, plus trivial theories. (Note that we are not worried about preserving any symmetry here, so there is no notion of charge vector, and the theory with $K = \sigma^x$ is trivial.)

5. **Cluster state from group cohomology.** [Bonus]

(a) A projective representation of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by π rotations of a spin-half particle. Call the elements of $G = \{e, x, y, z\}$ (with multiplication table

$$\begin{pmatrix} e & x & y & z \\ x & e & z & y \\ y & z & e & x \\ z & y & x & e \end{pmatrix} .$$

Then we can take the representation matrices to be

$$U(e) = 1, U(x) = \mathbf{i}X, U(y) = \mathbf{i}Y, U(z) = \mathbf{i}Z.$$

Check that this is a projective representation of G

$$U(g)U(h) = \nu(e, g, gh)U(gh)$$

and find the 2-cocycle ν. Check that it satisfies the cocycle condition.

(b) Show that we can regard the single-site Hilbert space $\mathcal{H}_0 = \text{span}\{|e\rangle, |x\rangle, |y\rangle, |z\rangle\}$ as a pair of qubits, and write the state

$$|1\rangle = \sum_{g \in G} |g\rangle$$

in terms of the Pauli operators on this qubit.

It is just $H_0 = -\sum_i X_i$.

(c) Find the solvable Hamiltonian that results from the construction of Chen-Gu-Wen:

$$H = -U \sum_i |1_i\rangle\langle 1_i| U^\dagger$$

in terms of the Pauli operators.

Compare to the cluster hamiltonian

$$H = -\sum_i Z_{i-1} X_i Z_{i+1} = -U_{CZ} X_i U_{CZ}^\dagger,$$

with $U_{CZ} \equiv \prod_i \mathsf{CZ}_{i,i+1}$ where CZ is the control-Z operation.