

University of California at San Diego – Department of Physics – Prof. John McGreevy
**Physics 211C (239) Phases of Quantum Matter,
Spring 2021
Assignment 8**

Due 12:30pm Monday, June 7, 2021

Thanks for following the submission guidelines on [hw01](#). Please ask me by email if you have any trouble.

1. Helical majorana mode on a surface domain wall of a TI.

Consider a $D = 3 + 1$ fermion topological insulator protected by charge conservation and time-reversal symmetry. If we stick an s -wave superconductor on the surface, we can gap out the surface Dirac cone, but the vortices are interesting because they carry majorana zero modes.

What happens if we (somehow) produce on the surface a domain wall in the phase of the superconducting order? That is, suppose that on the surface ($z = 0$), when $y > 0$ the superconducting pairing field is $\Delta = \Delta_0$, but for $y < 0$, it is $\Delta = e^{i\gamma} \Delta_0$ for some phase γ . If $\gamma = \pi$, what happens on the $D = 1 + 1$ dimensional locus at $y = 0$?

2. Majorana chain.

Consider the majorana chain

$$H = \mathbf{i} \sum_i (t_e \gamma_i \tilde{\gamma}_i + t_o \tilde{\gamma}_i \gamma_{i+1})$$

with $\{\gamma_i, \gamma_j\} = \delta_{ij} = \{\tilde{\gamma}_i, \tilde{\gamma}_j\}$, $\{\gamma_i, \tilde{\gamma}_i\} = 0$.

- (a) Show that the Hamiltonian can be rewritten as a p -wave superconductor of spinless electrons.
- (b) Find the energy spectrum. Check that when $t_e = t_o$, the gap closes.

3. Majorana lasagna.

Consider a layer in the coupled-layer construction to be the critical limit of the Kitaev chain, that is, a massless 1+1d Majorana fermion field. Apply the coupled-layer construction to make a model in $D = 2 + 1$ dimensions with no symmetry. What 2+1d free-fermion SPT state do you make this way?

4. All-fermion toric code.

- (a) Check that $U(1)^4$ CS theory with K -matrix

$$K_{\text{SO}(8)} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

has the same spectrum of anyons as the all-fermion toric code.

- (b) Check that the statistics of the 16 anyons in two copies of the toric code is related to that of two copies of the all-fermion toric code by a relabelling. (At least check the self-statistics.)

- (c) [Bonus] A perhaps better way to address the previous part, using the result of the first part: show that the $U(1)^8$ Chern-Simons theory with K -matrix

$$K_{\text{SO}(8)} \oplus (-K_{\text{SO}(8)})$$

is equivalent to two copies of the toric code, plus trivial theories. (Note that we are not worried about preserving any symmetry here, so there is no notion of charge vector, and the theory with $K = \sigma^x$ is trivial.)

5. **Cluster state from group cohomology.** [Bonus]

- (a) A projective representation of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by π rotations of a spin-half particle. Call the elements of $G = \{e, x, y, z\}$ (with multiplication table

$$\begin{pmatrix} e & x & y & z \\ x & e & z & y \\ y & z & e & x \\ z & y & x & e \end{pmatrix}.$$

Then we can take the representation matrices to be

$$U(e) = 1, U(x) = \mathbf{i}X, U(y) = \mathbf{i}Y, U(z) = \mathbf{i}Z.$$

Check that this is a projective representation of G

$$U(g)U(h) = \nu(e, g, gh)U(gh)$$

and find the 2-cocycle ν . Check that it satisfies the cocycle condition.

- (b) Show that we can regard the single-site Hilbert space $\mathcal{H}_0 = \text{span}\{|e\rangle, |x\rangle, |y\rangle, |z\rangle\}$ as a pair of qubits, and write the state

$$|1\rangle = \sum_{g \in G} |g\rangle$$

in terms of the Pauli operators on this qubit.

- (c) Find the solvable Hamiltonian that results from the construction of Chen-Gu-Wen:

$$H = -U \sum_i |1_i\rangle\langle 1_i| U^\dagger$$

in terms of the Pauli operators.

Compare to the cluster hamiltonian

$$H = - \sum_i Z_{i-1} X_i Z_{i+1} = -U_{CZ} X_i U_{CZ}^\dagger,$$

with $U_{CZ} \equiv \prod_i CZ_{i,i+1}$ where CZ is the control-Z operation.