Topological Defects, cont'd:

Given $F_{LG}[\Phi]$, why not solve EoM

$$\frac{\delta}{\delta\Phi(x)} F_{LG}[\Phi] \overset{!}{=} 0 \quad ?$$

- harder
- less conclusive.

"boojum"

Had shown: If $\pi_{q-1}(V) = 0$

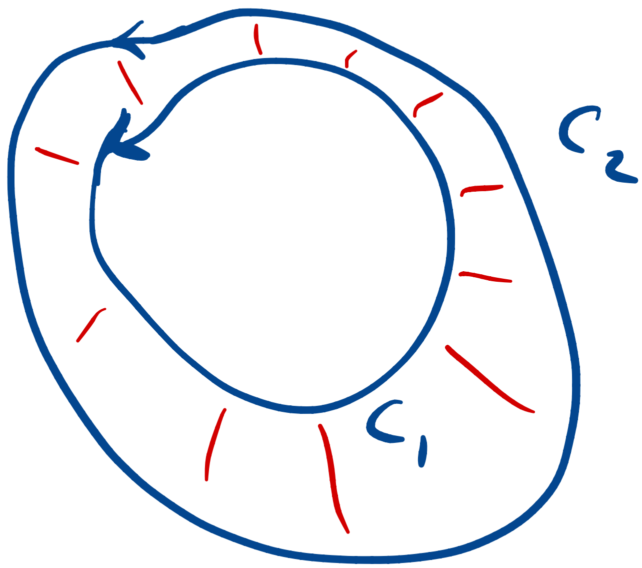
then no stable codim-q defects.

To show: $\left\{ \begin{matrix} \text{codim-}q \\ \text{defects} \end{matrix} \right\} / \simeq \;\longleftrightarrow\; \underbrace{\pi_{q-1}(V,p) / \pi_1(V,p)}$

(for $q \geq 2$)

Free homotopy

$q=2$ for a moment.



$\phi|_{c_1} : S^1 \to S^1$

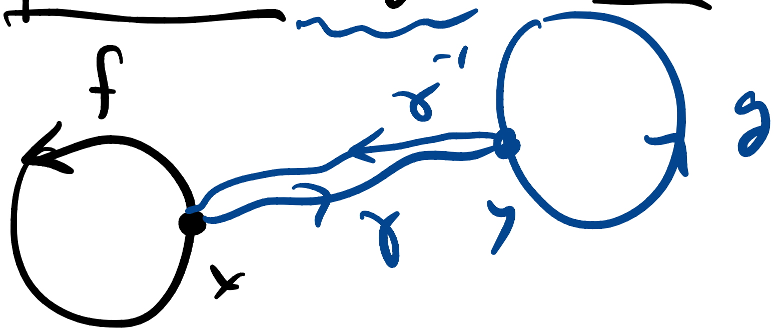
$\phi|_{c_2} : S^1 \to S^1$

are freely homotopic (no base point)

claim: f, g (loops in V) are freely homotopic

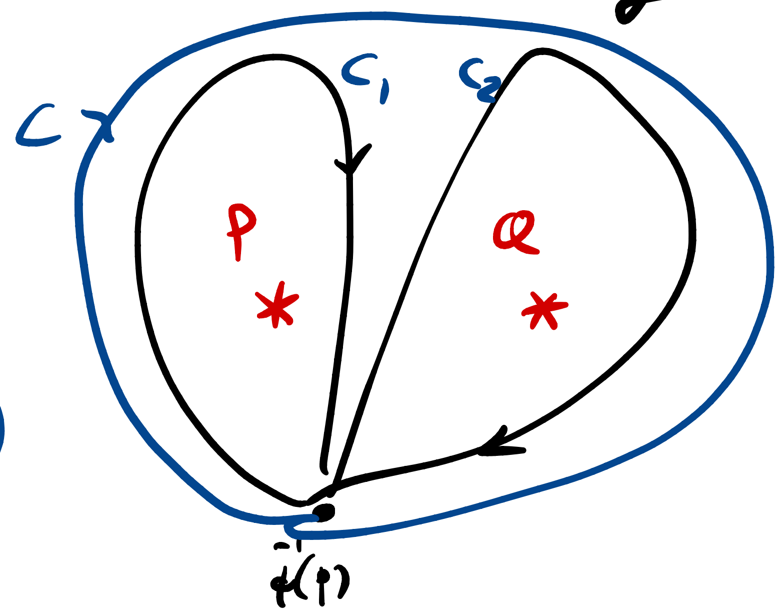
$\iff$ $\exists$ path γ connects $x \in f$ to $y \in g$

with $\boxed{f \simeq \gamma g \gamma^{-1}} \simeq g$



$t=0$ $\qquad\qquad$ $t=1$

$$\gamma g \gamma^{-1} \simeq g$$



<u>Composite codim 2 defects</u> :

$$\phi(c) \simeq \phi(c_2)\circ\phi(c_1) = Q\cdot P$$

$Q, P \in \pi_1(V, p)$

<u>basic fact</u> : $a, b \in G$

$$ab = a(ba)a^{-1}$$

$[ab] = [ba]$ are in the same conjugacy class!

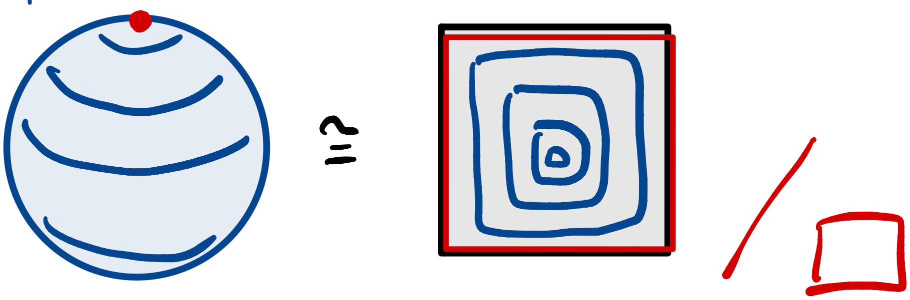
why (if V is path-connected) is

$\pi_k(V, x) \cong \pi_k(V, y)$?

Given

$f : (I^k, \partial I^k) \to (V, x)$

a rep. of $\pi_k(V, x)$



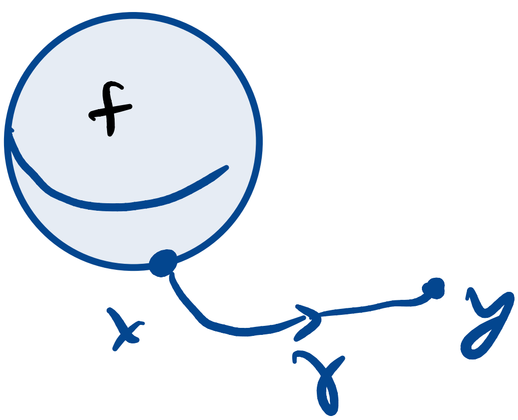
$S^k \cong I^k / \partial I^k$

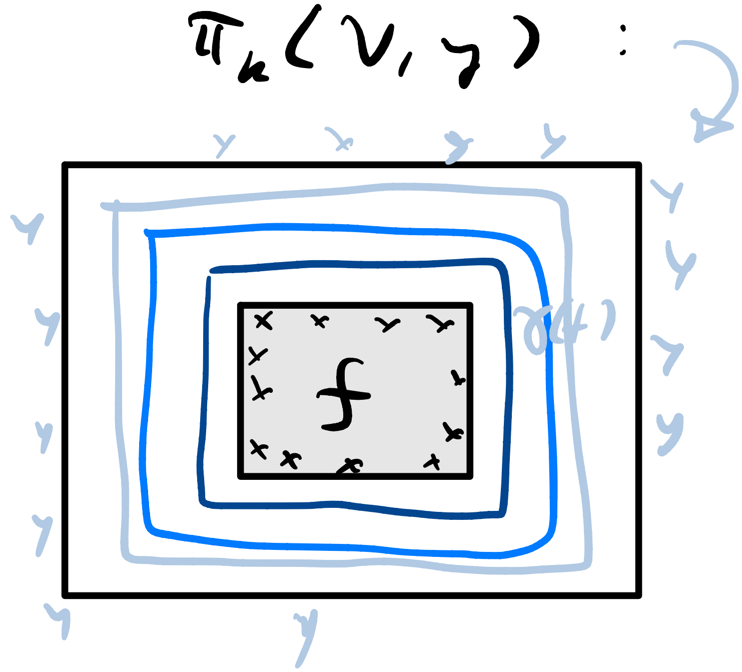
and a path γ from x to y

using f, γ

Construct a rep. of $\pi_k(V, y)$:

$k=2$:





I^k

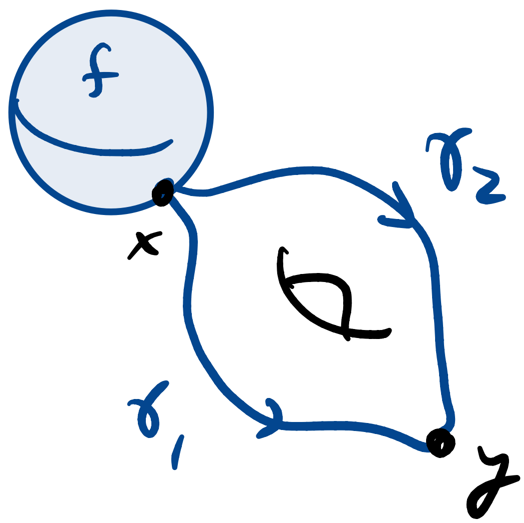
$\underline{k=1}$:

$$\gamma^{-1} \times f \times \gamma$$

$$\underline{f \longrightarrow \gamma^{-1} f \gamma}.$$

is an isomorphism.

But : can depend on choice of γ.



differ by

$[\gamma_1 \gamma_2^{-1}] \in \pi_1(V,x)$

"loop automorphism"

action of $\pi_1(V,x)$

on $\pi_k(V,x)$

$\underline{k \geq 1}$.

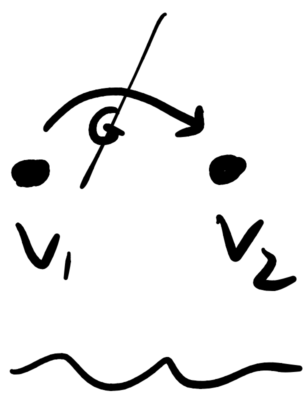
1.3 Who is the vac. mfld V?

- G acts transitively on V.

  i.e. $\forall\, \phi_1, \phi_2 \in V \quad \exists\, g \in G$

  s.t. $\quad \phi_2 = g\,\phi_1$.

Pf: If not



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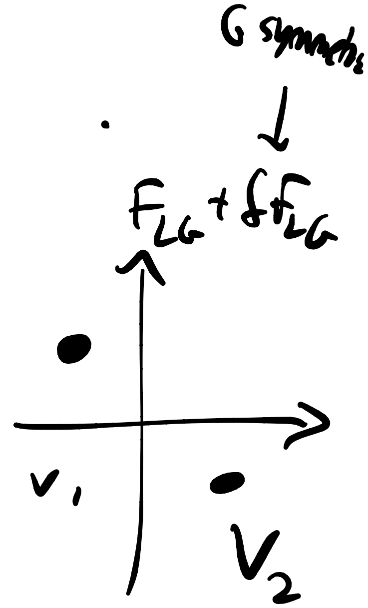


$\Rightarrow$ either ① fine-tuning of parameters in F_{LG}

② $\exists$ larger $\hat{G} \supset G$ which does act transitively.

$\leftarrow$ ③ supersymmetry.

Given $\phi \in V$, Let $H_\phi \equiv \{ g \in G \mid g\phi = \phi \} \subset G$

'stabilizer' of ϕ.

if $\phi_2 = g\phi_1$ then $H_{\phi_2} = g H_{\phi_1} g^{-1}$

$= \{ g h g^{-1}, h \in H_{\phi_1} \}$

$\neq H_{\phi_1}$

$h \in H_{\phi_2}$

$h\phi_2 = \phi_2$

$= h g \phi_1 = g\phi_1$

$\Rightarrow (g^{-1} h g)\phi_1 = \phi_1$

$\Rightarrow H_{\phi_1} = g^{-1} H_{\phi_2} g$

$\Rightarrow H_\phi$ is ind of ϕ as a group

Let $H_{\phi_0} \equiv H$.

if $H = \{1\} \xRightarrow{\text{transitivity}} V = G$.

More generally : $\boxed{V = G/H}$

$$\boxed{V = G/H} = G/\{g \sim gh,\ h \in H\}$$

$$= \{\text{left cosets of } H \text{ in } G\}$$

$$= \{gH,\ g \in G\}$$

$$V \ni \phi_1 = g\phi_0 \longleftrightarrow gH.$$

check: given $g_1\phi_0 = g_2\phi_0 \in V$.

$$\Rightarrow \phi_0 = g_1^{-1}g_2\phi \Rightarrow g_1^{-1}g_2 \in H = H_{\phi_0}$$

$$\Rightarrow g_2 = g_1\underbrace{(g_1^{-1}g_2)}_{H} \in g_1H$$

$$g_1 = g_1\underset{\substack{\uparrow \\ \in H}}{e} \in g_1H.$$

given $\underline{gH} \in G/H \longrightarrow \phi = g\phi_0 = gh\phi_0 \ \forall h \in H$

$\blacksquare$

<u>G is very ambiguous!</u> But V is not.

1.4 examples

1. Planar spins or superfluids

$$V = \{ (S_x, S_y),\ S_x^2 + S_y^2 = 1 \} = S^1$$

if $G = SO(2) = U(1) \Rightarrow H = \{e\}$

$\Rightarrow V = G/H = S^1.$

if $G = O(2)$ $\quad \begin{pmatrix} S_x \\ S_y \end{pmatrix} \to \begin{pmatrix} +S_x \\ -S_y \end{pmatrix}$

$= \underbrace{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}_{=P} \begin{pmatrix} S_x \\ S_y \end{pmatrix}$

$H = \langle P \rangle \cong \mathbb{Z}_2.$

$$V = G/H = O(2)/\mathbb{Z}_2 = SO(2)$$

$$= S^1.$$

$$= \{ \{R, P \circ R\}, R \in SO(2) \}.$$

$\pi_q(S^1) = \mathbb{Z}\, \delta_{q,1} \Rightarrow$ only codim 2 defects

2. Heisenberg spins : $V = \{ \vec{S} \mid S^2 = 1 \} = S^2$ ✓

If $G = SO(3)$ take $\phi_0 = (0,0,1) = \hat{z}$

$H = SO(2)$ rots abt $\hat{z}$

$$V = G/H = SO(3)/SO(2) = S^2 \checkmark$$

If $G = SU(2)$

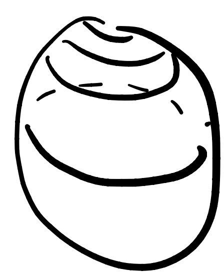
$$H = \left\{ e^{i\frac{\theta}{2}\sigma^3} = \begin{pmatrix} e^{i\theta/2} & \\ & e^{-i\theta/2} \end{pmatrix} \right\}$$

$\pi_1(S^2) = 0.$

$\pi_2(S^2) = \pi_3(S^2) = \mathbb{Z}$

$\pi_4(S^2) = \pi_5(S^2) = \mathbb{Z}_2.$

$\vdots$



particle defects in $d=3$.

3. Nematic: headless vector $\in \dfrac{\{\vec{n}, n^2=1\}}{\{\vec{n} \sim -\vec{n}\}}$

$$= S^2/\mathbb{Z}_2 = \mathbb{RP}^2.$$

OR $M_{ij} = M_{ji}$, $M_{ii} = 0$

$M_{ij} = n_i n_j - \delta_{ij}$ (inv't under $n \to -n$)

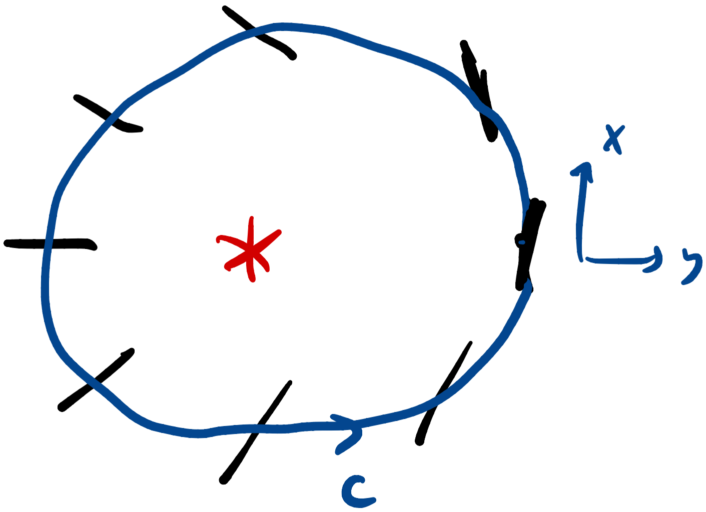
w/ 2 degenerate evals.

if $G = SO(3)$ $\quad H = \langle \underbrace{\text{all rots about } \vec{n}}_{SO(2)}, \underbrace{\pi \text{ rots about a } \perp \text{ axis}}_{\mathbb{Z}_2} \rangle$

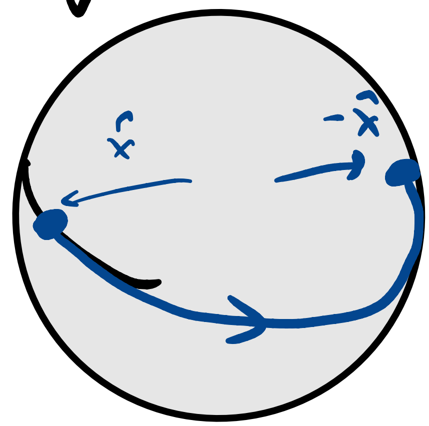
$$= D_\infty$$

$$V = SO(3)/D_\infty = S^2/\mathbb{Z}_2 = \underline{\mathbb{RP}^2}.$$

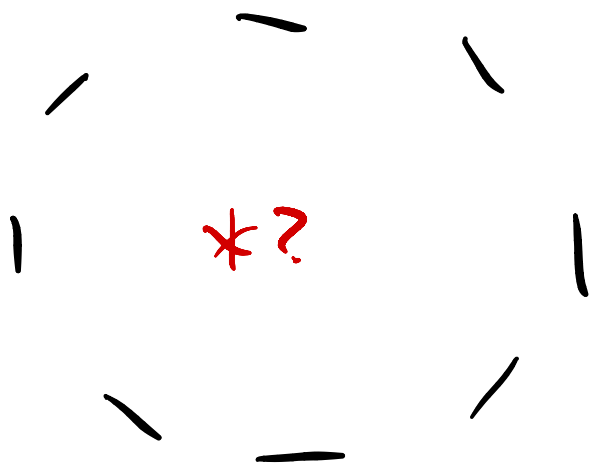
$\underline{\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2}$. $\longrightarrow$ ordinary defects "π disclinations"



$V = S^2/\mathbb{Z}_2$

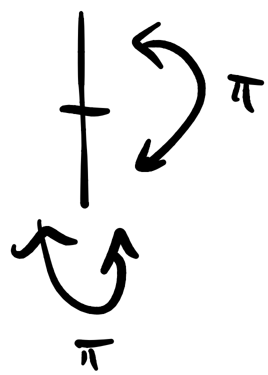


$\phi(C)$



see movie.

4. Biaxial nematic



If $G = SO(3) \Rightarrow H = D_2 = \langle \pi$ rots about $3 \perp$ axes $\rangle$

$V = SO(3)/D_2$ what's $\pi_1(V)$?

univ. cover of $SO(3)$ is $SU(2)$: $SO(3) = SU(2)/\mathbb{Z}_2$

$\underline{\underline{V}} = SO(3)/D_2 = SU(2)/(\text{lift of } D_2 \text{ to } SU(2))$

$\pi_1\left(\text{simply connected space} / \text{free action of } \Gamma\right) = \Gamma.$

π rot about $\hat{x}$ on 2 $= e^{i\frac{\theta \hat{x}\cdot\vec{\sigma}}{2}}\Big|_{\theta=\pi} = \cos\frac{\theta}{2} + i\hat{x}\cdot\vec{\sigma}\sin\frac{\theta}{2}$

$= i\sigma^x.$

lift of D_2 to $SU(2)$ $= \{\pm 1\!\!1, \pm iX, \pm iY, \pm iZ\}$

$= Q_8.$ **NON-ABELIAN !**