1.7 Textures or solitons or ...

$\pi_{q-1}(V)/\pi_1(V)$ classifies codim q defects

what about codim $d+1$?

Medium breaks $G \to H$ living in $\mathbb{R}^d$.

continuous $\phi : \mathbb{R}^d \to V$ "texture" "soliton"

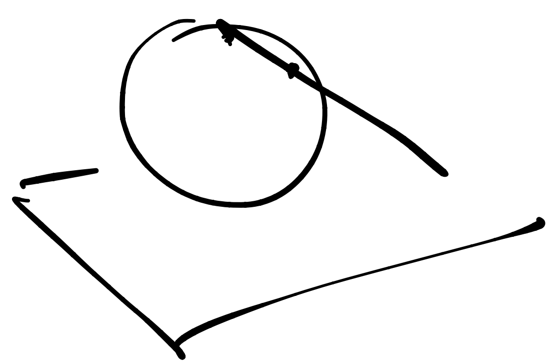
w/ finite $E_{LG}[\phi] < \infty$

$$\Rightarrow \int (\partial\phi)^2$$

$$\Rightarrow \phi \xrightarrow{x\to\infty} \phi_0.$$

$$\phi : (\mathbb{R}^d, \infty) \to (V, \phi_0) \cong \phi : S^d \to V.$$

ie $S^d = \mathbb{R}^d/\infty$



$\Rightarrow [\phi] \in \pi_d(V)$

If $[\phi_1] = [\phi_2] \Leftrightarrow$ can smoothly deform ϕ_1 to ϕ_2

eg $d=2$, $V=S^2$ $\qquad \pi_2(S^2) = \mathbb{Z}$ $\qquad$ skyrmion

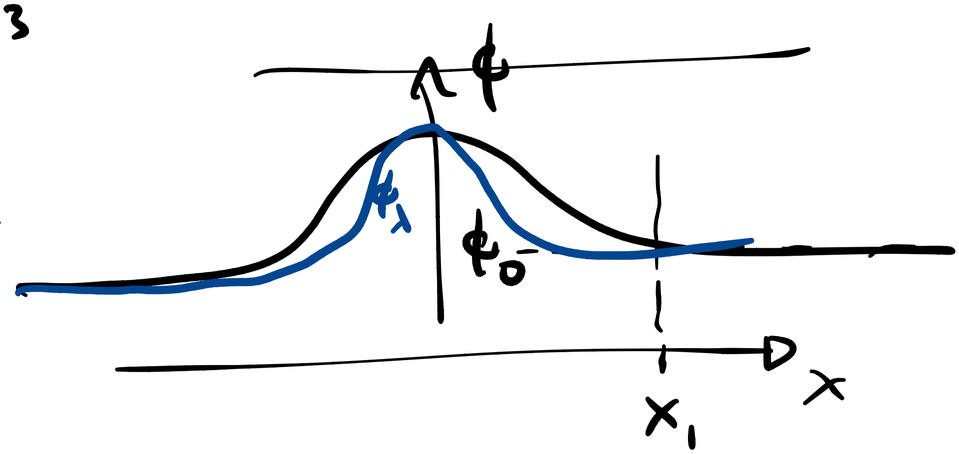
$d=3$, $V=G$ (simple) $\qquad \pi_3(G) = \mathbb{Z}$ $\qquad$ "

"skyrmion"

$$\frac{1}{2^{\#}\pi^{\#}} \int_{S^d} \phi \wedge (d\phi)^d \, .$$

eg: $d=3$, $V = SU(2) \simeq S^3$: $\phi = e^{i f(r) \hat{x} \cdot \vec{\sigma}}$.

Energetic αs :



Given $\phi(x) \xrightarrow[x > x_1]{} \phi_0$

$$\phi_\lambda(x) \equiv \phi(x/\lambda) \xrightarrow[x > x_{1/\lambda}]{} \phi_0 \qquad \lambda < 1.$$

$\lambda = 0$ is singular.

$$F^0_{LG}[\phi_\lambda] \equiv \rho \int d^d x \, (\partial_x \phi_\lambda)^2 \equiv \rho \int d^d \tilde{x} \, \lambda^d \lambda^{-2} (\partial_{\tilde{x}} \phi)^2 \qquad \tilde{x} = x/\lambda$$

$$\sim \lambda^{d-2} F^0_{LG}[\phi]$$

$d>2 \Rightarrow \lambda \to 0$ minimizes $F^0_{LG}[\phi_\lambda]$

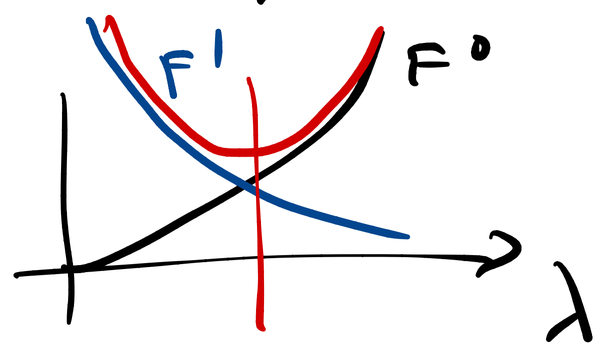
$\Rightarrow$ <u>must</u> consider (+ higher deriv. terms) in F_{LG}

"Derrick's Thm"

<u>eg</u>: $F' \equiv \frac{1}{\Lambda^\Delta} \int d^dx \quad (\partial\phi)^4$

$\phi(x) \to \phi_\lambda(x) = \phi(x/\lambda)$, $\quad F' \to \lambda^{d-4} F'$

so for $2<d<4$

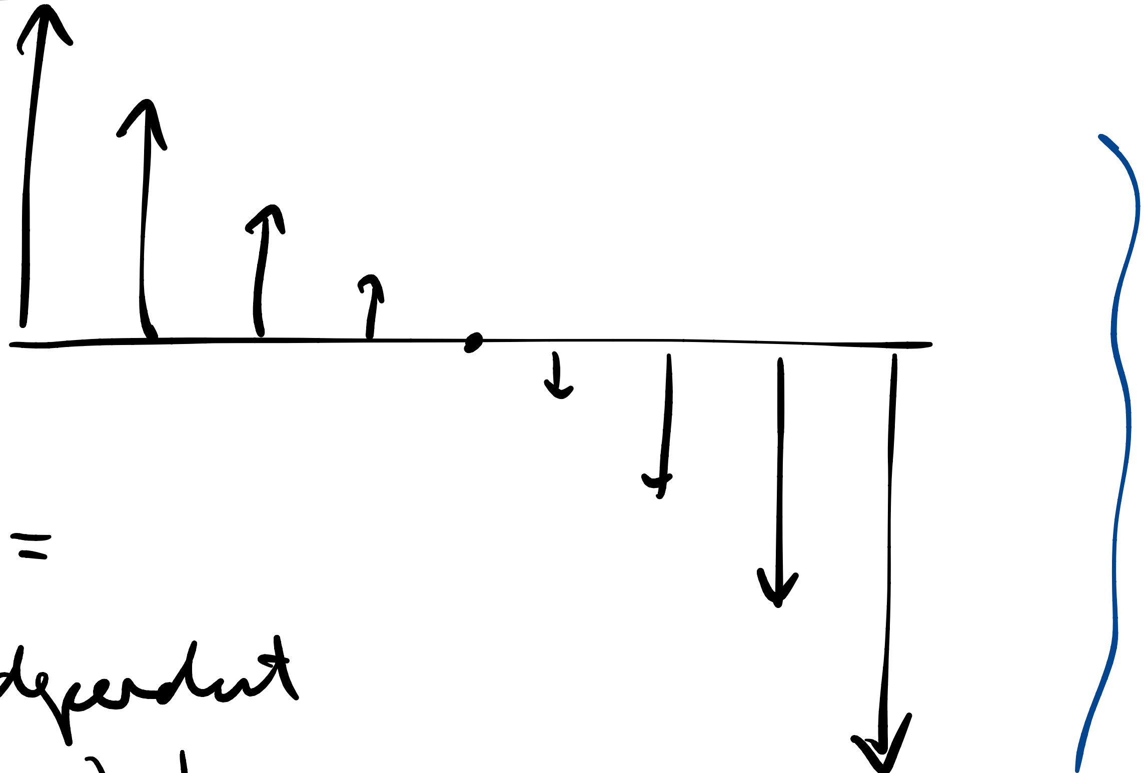


$\underline{\lambda_* \sim \Lambda^\#}$

$\boxed{d=2\,?}$

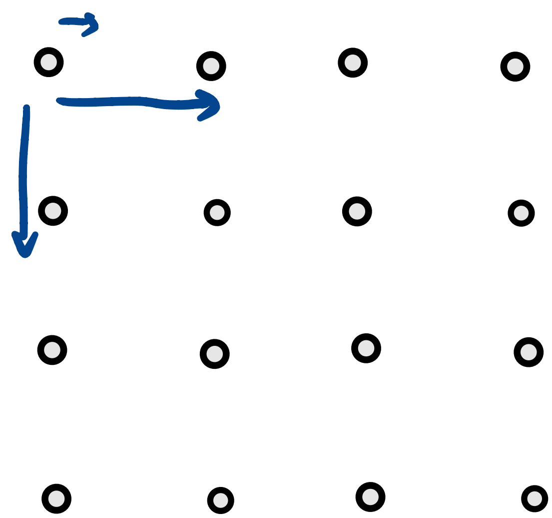
1.8 Defects of broken spatial symmetries

<u>why it's more difficult:</u>



Rotation = position-dependent translation.

<u>Elasticity theory in terms of goldstones:</u>

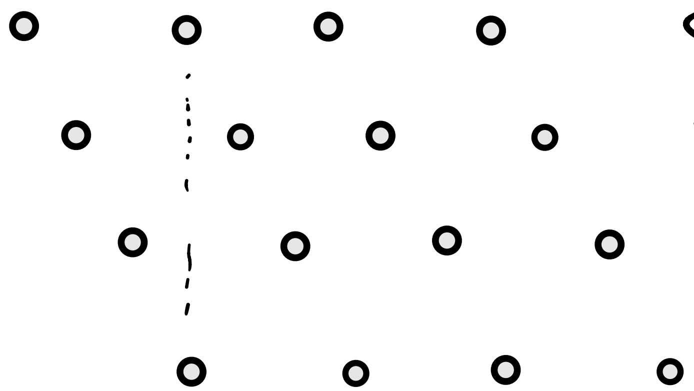


eg $d=1$

$G = \mathbb{R}$ cont. transl.

$H = \mathbb{Z}$ lattice transl

$\mathcal{V} = G/H = \mathbb{R}/\mathbb{Z} = S^1$



$\leftarrow \quad \phi_0$

$\leftarrow \quad \phi_0 + \epsilon$

$\phi_0 + 2\epsilon$

$\phi_0 + a \cong \phi_0$.

In d dims, $V = \mathbb{R}^d / \Gamma = T^d = S^1 \times S^1 \times S^1$

transl in d dims

lattice

$\ni \theta^{I=1\dots d}$

coords periodic

$$F_{LG}[\theta^I] = \int d^d x\, dt \, \underbrace{\kappa^{ijKL}\, \partial_i \theta^K \, \partial_j \theta^L}_{} + \dots$$

elasticity tensor

Quasicrystal $\equiv$ quasiperiodic solid

= project a D-dim'l lattice into $d < D$ dimensions

$\theta^{I=1..D} \in T^D$ extra goldstones $\equiv$ "phasons."

Hohenberg-Coleman Mermin Wagner:

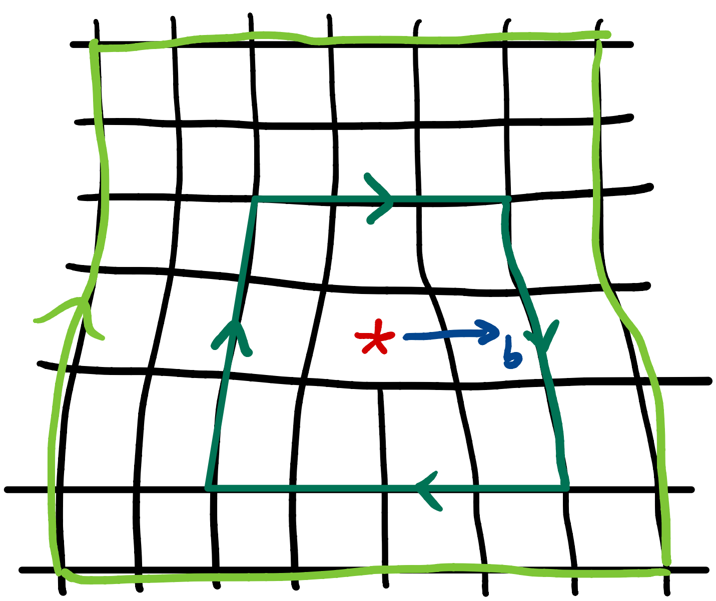
$$\langle \phi(x)\,\phi(y)\rangle \underset{\substack{d\leq 2\\ T>0}}{\sim} f(x-y) \not\to 0 \quad x-y\to\infty.$$

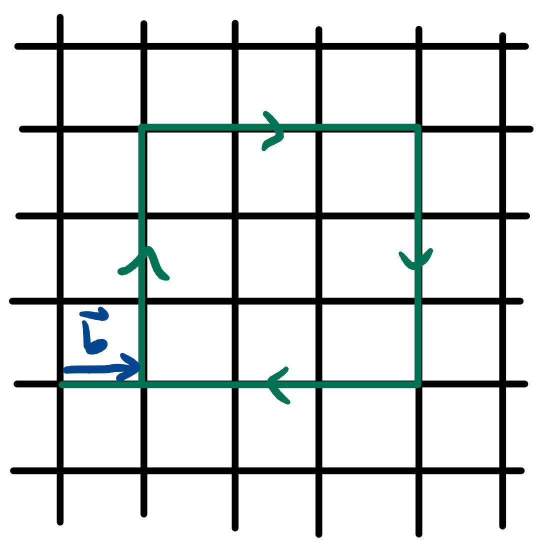
$\Longrightarrow$ no long range order.

DEFECTS.

Dislocations. (codim 2)

$$\pi_1(V) = \pi_1(\mathbb{R}^d/\Gamma) = \Gamma.$$





$\vec{b}$ = extra translation req'd to close path on perfect lattice.

dislocation = vortex of θ^I

$$b^I = \frac{1}{2\pi} \oint_C dx^i \, \partial_{x^i} \theta^I = \frac{1}{2\pi} \int_{\theta(C)} d\theta^I$$

C around dislocation

Burgers' vector. detects dislocation from far away

A first encounter w/ Lieb-Schultz-Mattis-Hastings-Oshikawa (LSMOH) constraints

Suppose atom # $\in \mathbb{Z}$ is conserved. U(1) symmetry.

Not a SF $\Rightarrow$ U(1) unbroken.

$$V = G/H \longrightarrow G \times U(1) / H \times U(1) = V.$$

Manouver: couple to background U(1) gauge field A_μ

A

Q: what is $F_{LG} \equiv S[\theta, A_\mu]$?

$$\left(\text{eg: } \quad \rho(x) = \frac{\delta S}{\delta A_0(x)}. \right)$$

constraint: gauge-invariant:

$$\begin{cases} A_\mu \to A_\mu + g^{-1} \partial_\mu g \\ g: \text{ spacetime} \longrightarrow U(1) \end{cases}$$

idea: $$e^{i S[\theta, A]} = \int D[\text{stuff}]\, e^{i S_{\text{micro}}[\theta, A, \text{stuff}]}$$

if all the stuff is gapped $\Rightarrow S[\theta, A]$ is local.

$$= \int d^d x\, dt\, \mathcal{L}(\theta, A, \partial_\mu \theta \ldots)$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Note: θ is gauge-neutral.

$$S = \int k(\partial\theta)^2 + \int F_{\mu\nu}F^{\mu\nu} + \dots$$

$$+ \underline{\underline{S_\nu}}$$

$$S_\nu[\theta, A] = \frac{\nu}{2\pi}\int A \wedge d\theta = \frac{\nu}{2\pi}\int dx\,dt\, A_\mu \partial_\nu \theta\, \epsilon^{\mu\nu}$$

$\boxed{\text{in } d=1}$

$$\delta_g S_\nu = -\underbrace{\frac{\nu}{2\pi}\int d\theta \wedge g^{-1}dg}$$

Recall: given $g: S^1 \to S^1$

$$\underline{\frac{1}{2\pi}\oint g^{-1}dg \in \mathbb{Z}.}$$

Claim: $\frac{1}{2\pi}\int d\theta \wedge g^{-1}dg \in 2\pi\mathbb{Z}$

for smooth θ, g.

e^{iS} is gauge inv't $\iff$ $\underline{\delta S \in 2\pi\mathbb{Z}}$.

$\Rightarrow S_\nu$ is allowed if $\nu \in \mathbb{Z}$.

Pf of claim: ① $\delta_g S_\nu[\theta, g]$ is topological

$$\frac{\delta}{\delta\theta(x)}(\delta_g S_\nu) = 0 = \frac{\delta}{\delta g(x)}(\delta_g S_\nu)$$

② evaluate for spacetime $= T^2$

$$T^2 \longrightarrow T^2$$
$$(x,t) \longrightarrow (g(x,t), \theta(x,t))$$

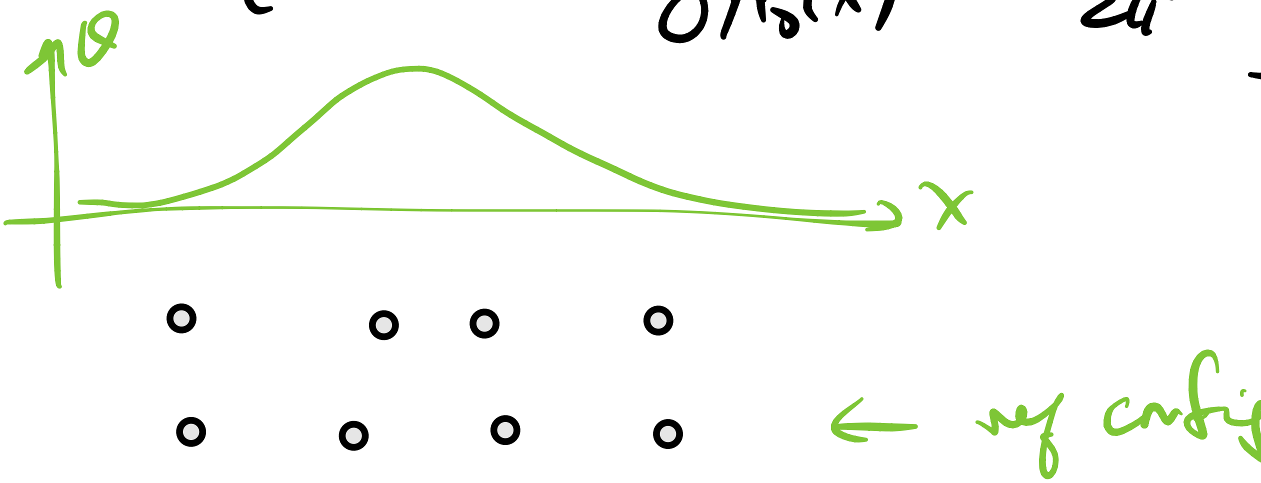
$\delta_g S_\nu$ is the winding # of this map. $\times 2\pi$

eg: $\theta = \theta(t)$
$g = g(x)$.

$$\delta_g S = \frac{1}{2\pi} \oint dt\, \theta(t) \oint dx\, g^{-1} dg \,.$$

What does S_1 do?

$$\rho(x) = \frac{\delta S}{\delta A_0(x)} = \frac{\nu}{2\pi}\partial_x \theta + \dots$$



<u>eg</u>: $S = S + \int A_\mu j_0^\mu$

<u>claim</u>: is <u>gauge inv't</u> if $\partial_\mu j^\mu = 0$.

even under

$$A_\mu \to A_\mu + g^{-1}\partial_\mu g$$

w g <u>not</u> connected to $\mathbb{1}$.

<u>"large gauge transformation".</u>

let $u^i(x,t) \equiv$ displacement of the atom at x from its eqbm position.

claim: $= \frac{1}{2\pi} a^i_I \, \theta^I(x,t) - x^i$

a^i_I are generators of Γ.

$\Gamma = \{ n^I a^i_I , n^I \in \mathbb{Z} \}$

eqbm config:

$\theta^I(x,t) = K_i^{\ I} x^i$

w/ $K^I_{\ i} \frac{a^j_I}{2\pi} = \delta_i^{\ j}$

(cols are reciprocal lattice generators)

In d dims,

$$S_\nu[\theta, A] = \frac{\nu}{(2\pi)^d}\int \underbrace{A\wedge d\theta^1\wedge d\theta^2\wedge\dots\wedge d\theta^d}$$

$\nu\in\mathbb{Z} \Leftrightarrow$ gauge inv.

$$\rho(x) = \frac{\delta S}{\delta A_0(x)} = \frac{\nu}{d!\,(2\pi)^d}\,\epsilon_{I_1\dots I_d}\,\epsilon^{i_1\dots i_d}\,\underline{\underline{\partial_{x_{i_1}}\theta^{I_1}\dots\partial_{x_{i_d}}\theta^{I_d}}}$$

<u>eqbm density:</u> $\partial_{x_i}\theta^I_{\text{eqbm}} = K_i^{\ I}$

$$\rho_0(x) = \frac{\nu}{(2\pi)^d}\,\underbrace{\frac{\epsilon_{I_1\dots I_d}\,\epsilon^{i_1\dots i_d}}{d!}\,K_{i_1}^{\ I_1}\dots K_{i_d}^{\ I_d}}_{=\det K}$$

$$= \frac{\nu\cdot\det K/2\pi}{\text{vol. unit cell}} = \nu\,\frac{1}{\det a} = \frac{\nu}{V}$$

$$\nu = \rho_0 \text{ vol of unit cell}$$

$$= \text{charge}/\text{unit cell} \in \mathbb{Z}.$$

$\underline{d=3}$.

$$\Delta S = \int A \wedge dA \wedge d\theta^I \, \frac{k^I}{4\pi} \qquad k \in \mathbb{Z}.$$

$$\Delta \rho = \frac{k^I}{2\pi} \epsilon^{ijk} \partial_i A_j \partial_k \theta^I$$

$$\Delta j_\mu = \frac{k^I}{2\pi} \epsilon_{\mu\nu\rho\lambda} \partial_\nu A_\rho \partial_\lambda \theta^I$$

$$\underset{\text{IBP}}{=} -\int \frac{k^I}{4\pi} \theta^I \, F \wedge F$$

$$\Delta S(\theta + 2\pi) \overset{!}{=} \Delta S(\theta) + 2\pi \mathbb{Z}.$$

dislocation density:

$$j_0^I = \frac{\epsilon^{ij}\partial_i\partial_j\theta^I}{2\pi}$$

<u>$d=1$</u>: $$S_\nu = \frac{\nu}{2\pi}\int \theta^I \; F$$

$$\int_{\text{spacetime}} F = 2\pi n$$

$$0 = \frac{\delta S}{\delta \theta} = -\partial^2\theta + \frac{\nu}{2\pi}F$$

$$\Rightarrow \theta = \tilde{\theta} + \frac{\nu}{2\pi}F\,X$$

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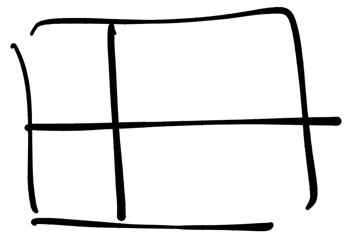
each of θ, g : $T^2 \to S^1$

$\theta : T^2 \to S^1$

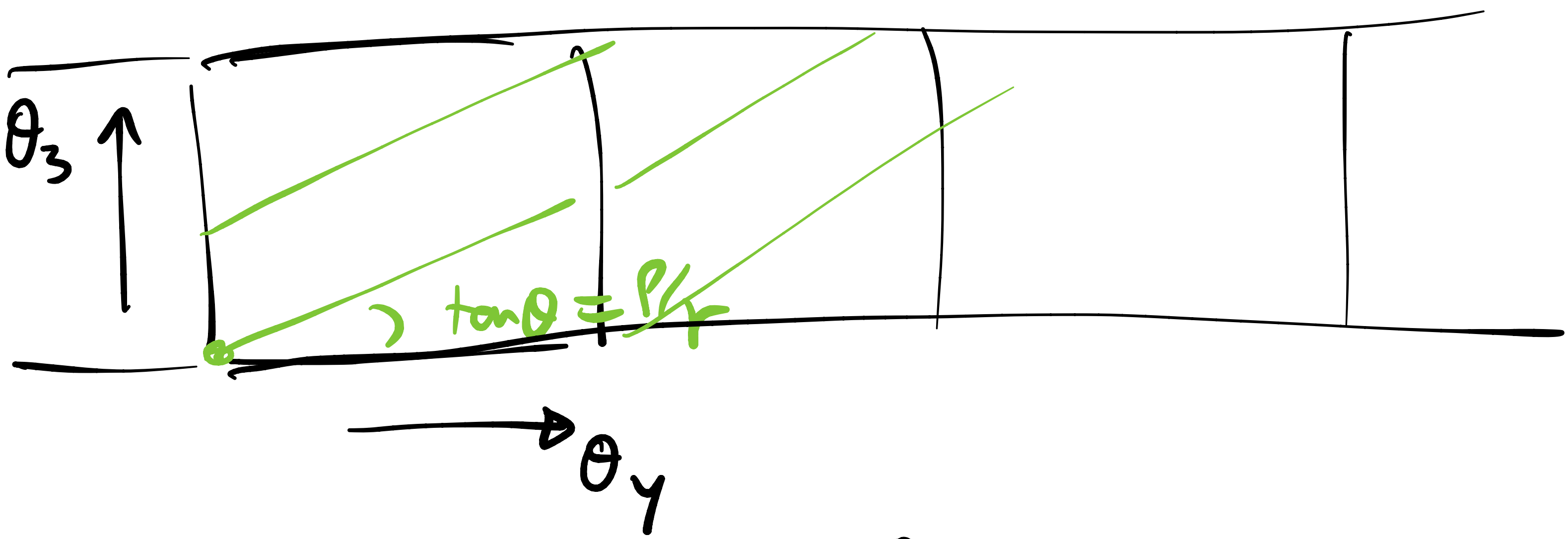


$\leftrightarrow (n_\theta^x, n_\theta^y)$

$$\frac{1}{2\pi}\int d\theta \wedge g^{-1}dg \overset{?}{=} 2\pi\left(n_\theta^x n_g^y - n_\theta^y n_g^x\right).$$

$$SU(2) \times U(1)_Y \,/\, U(1)_Q$$

$$Q = p\,T^3 + r\,Y$$



$$U(1)_Q \subset U(1)_Y \times U(1)_3$$

$$= \left\{ e^{i\lambda(pT^3 + rY)} \right\} \qquad \left\{ e^{i\theta_Y Y} \right\} \qquad \left\{ e^{i\theta_3 T^3} \right\}$$

$$\pi_1(G/H) =: \pi_0\big(\;/\;/\;/\;/\;/\; \big) \;?$$

$$\underline{\underline{SU(2)\times U(1)_Y}} \Big/ \underline{\underline{U(1)_Q}} \simeq \underline{\underline{SU(2)/\mathbb{Z}_r}}$$

$$= \left(g, e^{i\theta}\right) \sim \left(e^{i\theta_Q p T^3} g, e^{i(\theta + r\theta_Q)}\right)$$

$$\Rightarrow \underline{\underline{\theta_Q = -\theta/r + \frac{2\pi}{r}k}}$$

$$\sim \left(e^{i\left(-\frac{\theta}{r} + \frac{2\pi}{r}k\right)pT^3} g, 1\right)$$

if $(p, r) = 1$ this is a residual $\mathbb{Z}_r$ action.

$G = U(1) \longrightarrow$ nothing

couple to a $\overset{BG}{}$ A_μ

$$S[\phi, A] = \int \rho(\partial_\mu \phi + A_\mu)^2 + \dots$$

$$\phi(x,y) = \varphi$$

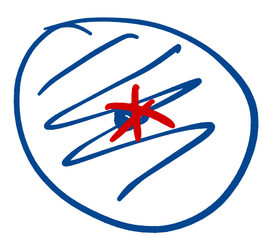
gauge transf:
$$\begin{cases} A_\mu \to A_\mu + i e^{-i\theta} \partial_\mu e^{i\theta} \\ \phi \to \phi + \theta \end{cases}$$

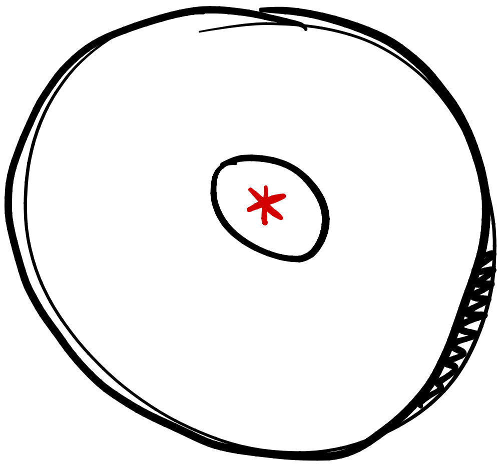
choose $\theta = -\varphi$.

$$0 = \frac{1}{2\pi}\oint_{C_0} A \longrightarrow \overbrace{\frac{1}{2\pi}\oint i e^{-i\theta} \partial_\mu e^{i\theta}}^{-\oint d\varphi/2\pi}$$

$$= 1$$

$$= \int_{R_0} F/2\pi$$





$$\begin{cases}\phi = \varphi \\ \theta = -\varphi\end{cases}$$

$$e^{i\oint_c A} \to e^{i\oint_c A} .$$

$$E[\Phi] = \int d^d x \, \left| (\partial - A)\Phi \right|^2$$

$$\underbrace{(\partial - A)\Phi} \xrightarrow{x \to \infty} 0$$

$$\Rightarrow A_\mu \overset{x\to\infty}{\simeq} \Phi^{-1} \partial_\mu \Phi$$

$$\Rightarrow \int F = \oint_{C_\infty} A = \text{winding \#} \cdot 2\pi$$