LSMOH thm for a crystal w/ U(1) sym.

$$\left[\begin{array}{l} S[A,\theta] = \ldots \quad S_\nu[A,\theta] + \ldots \\ S_\nu[A,\theta] = \frac{\nu}{(2\pi)^d} \int A \wedge d\theta^1 \ldots \wedge d\theta^d \end{array}\right.$$

is inv't under large gauge transfs

$$A \to A + i g^{-1} dg$$

g: space $\to U(1)$

only if $\nu \in \mathbb{Z}$.

But $\rho = \frac{\delta S}{\delta A_0} = \nu = \frac{\#\text{ particles}}{\text{unit cell}}$

Q: suppose $\rho = 1/2$.
How to make a state w/ a gap?
(at fixed θ)

eg: spontaneously break $\mathbb{Z} \to 2\mathbb{Z}$

$$|\bullet \;\; \circ\rangle \xrightarrow[\text{sym}]{\text{broken}} |\circ \;\; \bullet\rangle$$

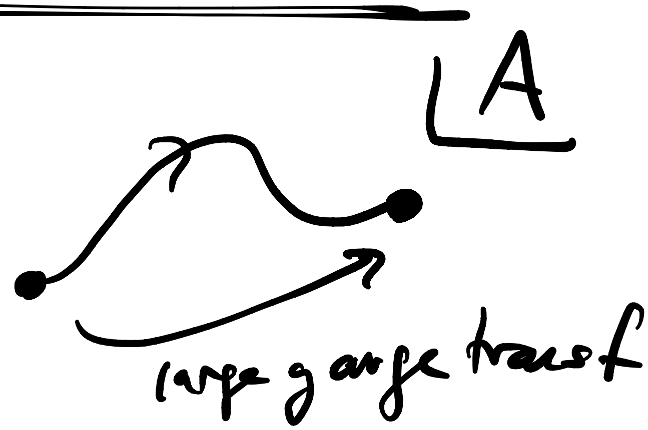
$\rightarrow| a |\leftarrow$

lattice momentum: $P \simeq P + \frac{2\pi}{a} = P + 2\pi$

$$|P=0\rangle = |\bullet \;\; \circ\rangle + |\circ \;\; \bullet\rangle$$

$$|P=\pi\rangle = |\bullet \;\; \circ\rangle - |\circ \;\; \bullet\rangle$$

FLUX-THREADING.



Put the system on a BIG circle w/ $L \in 2\mathbb{Z}$ sites.

$(x \simeq x + L)$

large gauge transf $g = g_n = e^{-i 2\pi n x / L}$

is the endpoint of a process:

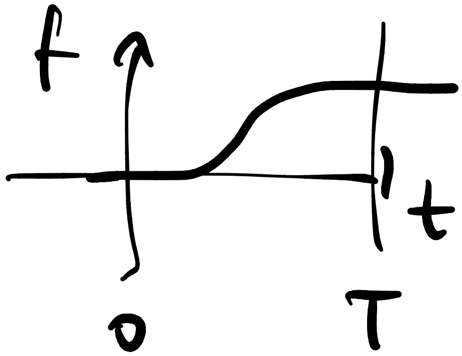
$$\Phi \equiv \oint_x A = \int_0^L dx\, A_x$$

only appears as $e^{i\Phi} = e^{i(\Phi + 2\pi)}$.

Under $A \to A + i\, g^{-1} dg$ w/ $g = g_n$

$$\Phi \to \Phi + \oint i\, g_n^{-1} dg_n$$
$$= \Phi + 2\pi n.$$

Process: $A(t) = \frac{2\pi}{L} f(t)\, dx$



$f(0) = 0, \quad f(T) = 1.$

$T^{-1} \ll$ gap to non phonon excitations

$\boxed{\text{At fixed } \theta}$

$$\Phi(t) = \oint A$$
$$= 2\pi f(t).$$

Step 1:

Claim: start in a g.s. $\xrightarrow[\text{threading at fixed } \theta]{\text{flux}}$ end in a g.s.

Pf: $\dfrac{\partial E}{\partial \Phi} = \langle I \rangle$

current in x direction

1st law of thermo

$dE = TdS + PdN + Id\Phi$

In an insulator $\langle I \rangle \to 0$ as $L \to \infty$.

$\Longleftrightarrow$ • $\dfrac{\delta S}{\delta A_\mu(x)} = \langle j^\mu(x) \rangle$

• $e^{iS} = e^{-iET}$

step 2: apply Newton to the C.O.M.

$$E_x = \frac{\partial A_x}{\partial t} = \frac{2\pi}{L} f'(t)$$

$$\Delta P = \int_0^T dt \sum_i^N E_x(t) = N \frac{2\pi}{L} \int_0^T f'(t)\,dt$$

total change in lattice momentum

particles

$$= N \frac{2\pi}{L} = 2\pi \nu$$

$\cong \Delta P + 2\pi$

Conclusion: start in $|P=0\rangle$

suppose $\nu = 1/2$. end in $|P = 2\pi\nu\rangle$

$$= |P = \pi\rangle.$$

$$|P=\pi\rangle \perp |P=0\rangle$$

$\Rightarrow$ gs is degenerate.

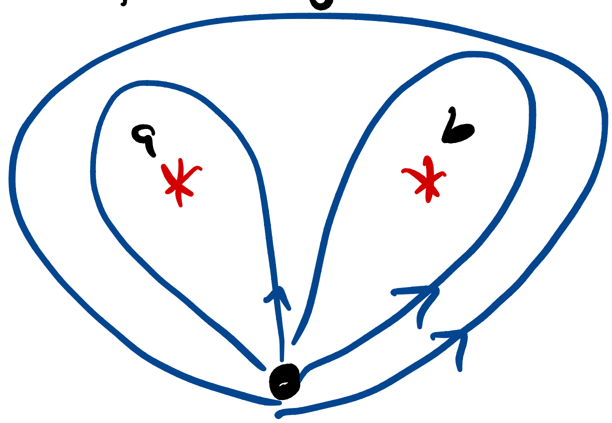
$|P=\pi\rangle$ is the image of $|P=0\rangle$ under the large gauge transf g_1.

- only g_n w/ n even need preserve e^{iS_ν}.

QH

Recall: Top. order means

1) fractionalization of quantum #s
2) G.S.D. depends on topology
3) Long-range entanglement.
   (nonzero top. entanglement entropy)



most data. "UMTC"

$a \times b = c_1 + c_2 + \ldots$

If $a \times b = c$ w/ unique c $\forall\, a, b$

then the T.O. **abelian**, else **non-abelian**.

2.1 EM response of gapped states in D=2+1.

Assume: • U(1) symmetry, j^μ

• Gap.

$$S_{micro}[\text{the stuff}, A] = S_{micro}[\text{the stuff}] + \int j^\mu A_\mu$$

$+ \ldots$

gauge invariant.

A is a background field.

$$e^{iS_{eff}[A]} = \int [D\text{stuff}]\, e^{iS_{micro}[\text{stuff}, A]}$$

$$\text{gap} \Rightarrow S_{eff}[A] = \int d^dx\,dt\, \mathcal{L}(A, F \ldots)$$

gauge inv't.

eg: $$\langle j^\mu_{(x)} \rangle = \frac{\delta S_{eff}}{\delta A_\mu(x)}$$

$$\langle j^\mu(x)\, j^\nu(y) \rangle = \frac{\delta S}{\delta A_\mu(x)\, \delta A_\nu(y)} \xrightarrow{\text{Kubo}} \text{conductivity.}$$

Landau-Ginzburg-Wilson logic:

$D_\mu = \partial_\mu + A_\mu$

$F \equiv dA$. $\Rightarrow [A] = [\partial] = 1$

$$S_{\text{eff}}[A] = \int_3 \Big[0 \cdot A^2 + \underbrace{\frac{\nu}{4\pi} A \wedge F}_{\text{CS term}} + \underbrace{\epsilon \mathcal{E}^2 - \frac{1}{\mu} B^2}_{} + \dots \Big]$$

· gap $\Rightarrow$ no U(1) goldstone ϕ $\Rightarrow$ $\rho(\partial\phi + A)^2$

· time-reversal sym or parity $\underline{\nu \to -\nu}$.

$$\sigma^{xy} = \lim_{\omega \to 0} \frac{1}{i\omega} \underbrace{\langle j^x \, j^y \rangle}_{\frac{\delta}{\delta A_x} \underbrace{\frac{\delta}{\delta A_y} S_{\text{eff}}[A]}_{\frac{\nu}{2\pi} F_{0x}} \Big|_{A=0}} \Big|_{k=0} = \frac{\nu}{2\pi} \Bigg| = \nu \frac{e^2}{h}$$

$\underbrace{}_{\frac{\nu}{2\pi} \cancel{i\omega}}$

Hall conductivity

$e^{i \frac{\nu}{4\pi} \int A \wedge F}$ is inv't under

large gauge transformation

$\iff$ $\nu \in \mathbb{Z}$.

claim: If gap & no fractionalization

then $\nu \in \mathbb{Z}$.

(If, in addition microscopically bosons, $\nu \in 2\mathbb{Z}$.)

$\implies$ ν is a label on phases

w/ $U(1)$ symmetry.

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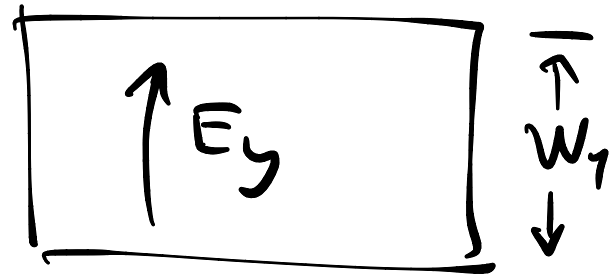
quantum Hall

$\sigma^{xx} \overset{\text{Kubo}}{=} 0$. Insulator

(But: $\rho_{ij} = (\sigma^{-1})_{ij}$ $\rho_{xx} = 0$.)

$$R_{xy} = V_y / I_x = E_y W_y / I_x$$

measure



$$= E_y / (I_x / W_y)$$

$$I_x = j_x W_y$$

$$= E_y / J_x = \frac{1}{\sigma_{xy}}.$$

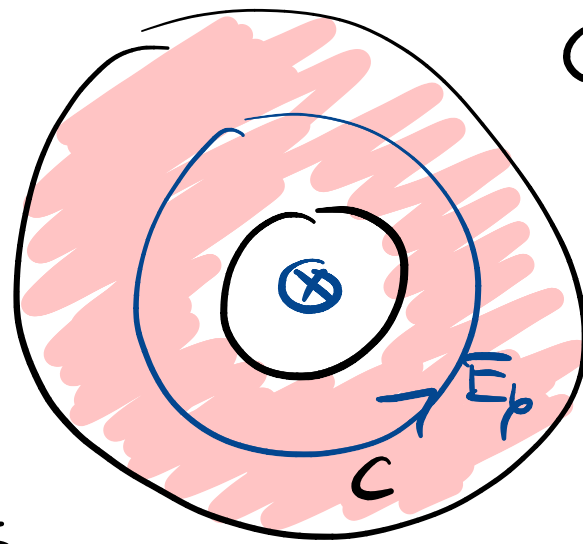
can be quantized (in units of e^2/h).

Flux-threading:

Vary $\Phi(t) = \oint A$ from 0 to 2π.

$$= \Phi_0 = \frac{hc}{e}.$$

Corbino geometry



$$\Phi_0 = \Delta\Phi = \int_0^T dt \, \frac{\partial}{\partial t}\left(\int_{\text{hole}} d\vec{a} \cdot \vec{B}\right) \overset{\text{Faraday}}{=} -c \int dt \oint_C \vec{E} \cdot d\vec{\ell}$$

Stokes

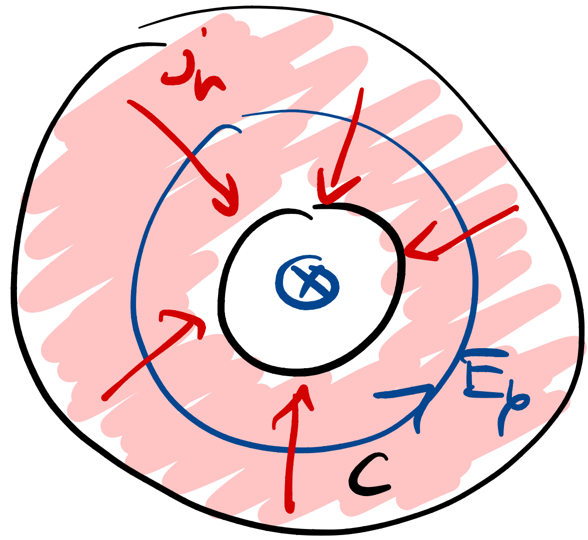
$$j_r = \sigma_{xy} E_\varphi$$

$$\Phi_0 = \ldots = -\frac{c}{\sigma_{xy}} \underbrace{\int_0^T dt\, j_r}_{= \Delta Q}$$

An amount of charge

$$\boxed{\Delta Q = \frac{\sigma_{xy}\Phi_0}{c} = \nu e}$$

is transferred from one edge to the other



$$\sigma_{xy} = \nu \frac{e^2}{hc}$$

$$\Phi_0 = \frac{hc}{e}$$

<u>other step</u>: $H(\Phi = 0) \cong H(\Phi = 2\pi)$

<u>moreover</u>: Work done $\Delta E = \int \langle I \rangle d\Phi \propto \int \left(\frac{d\Phi}{dt}\right)^2 dt \longrightarrow 0$

$\uparrow$ longitudinal (φ) current

thermodyn limit.

$$\text{g.s.} \xrightarrow[\text{threading}]{\text{flux}} \text{g.s.}$$

Conclusion:

W/o fractionalization charge is carried only by (electrons) charge-e objects.

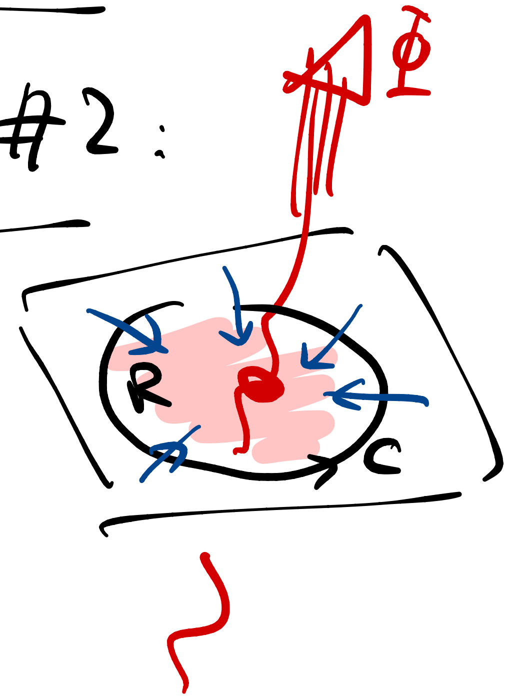
$$\longrightarrow \mathbb{Z} \ni \Delta Q = \nu$$

If we can label state by single-fermion occupation #s $|\Psi_0\rangle$ & $|\Psi_1\rangle$

- have same E
- differ by occupation #s of states localized at edges
- " " " of states at the fermi level.

$\Rightarrow$ gapless edge modes.

flux-threading argument #2:



In the plane / sphere.

Thread 2π flux through a thin solenoid at $\rho = 0$.

$$2\pi = \Delta\Phi = \int dt\, \partial_t \left(\int_R B \cdot a \right) \overset{\text{Faraday}}{=} - \int dt \oint_C \vec{E} \cdot d\vec{l}$$

$$\overset{j_r = \sigma_{xy} E_\varphi}{=} -\frac{1}{\sigma_{xy}} \underbrace{\int dt\, j_r}_{= \Delta Q}$$

$$\Rightarrow \boxed{\Delta Q = \nu e}$$

<u>claim</u>: 2π flux is invisible! $\Rightarrow$ this is a quasiparticle!

$D = 2+1$:

$$\int A \wedge F \equiv \int d^3x \; \epsilon_{\mu\nu\rho} A_\mu F_{\nu\rho} \cancel{\frac{1}{2}}$$

$F = dA \leftarrow$ 2-form.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

under $\left\{ \begin{array}{l} A \to A - i g^{-1} dg \\ F \to F \end{array} \right]$

$$\to S_v[A, \theta] = \int_{d+1} A \wedge \underbrace{d\theta^1 \wedge \ldots \wedge d\theta^d}_{d+1 \text{ form}}$$

$D = 3+1$:

$$S[A] = \left[\int \cancel{A \wedge A \wedge A \wedge A} + \int \cancel{A \wedge A \wedge F} \right.$$

?

$$\left. + \frac{\theta}{16\pi^2} \int F \wedge F \right]$$

$$\frac{\delta}{\delta A_\mu(x)} \int F \wedge F = 2\int d(\delta(\,)\wedge dA$$

$$\underset{\text{IBP}}{=} \underbrace{\int d(\delta(\,)\wedge dA)}_{} - 2\int \delta(\,)\, \cancel{d^2 A} \to 0$$

$$= 0.$$

$$\int \frac{F\wedge F}{16\pi^2} \in \mathbb{Z} \quad . \quad \Rightarrow \theta \simeq \theta + 2\pi .$$

gauge transf:

$$\boxed{A \simeq A - i g^{-1} dg}$$

for any smooth f'n g : spacetime $\to U(1)$

$$dg = \frac{\partial g}{\partial x^\mu} dx^\mu$$

small gauge transf: $A_\mu \simeq A_\mu - \partial_\mu \lambda$

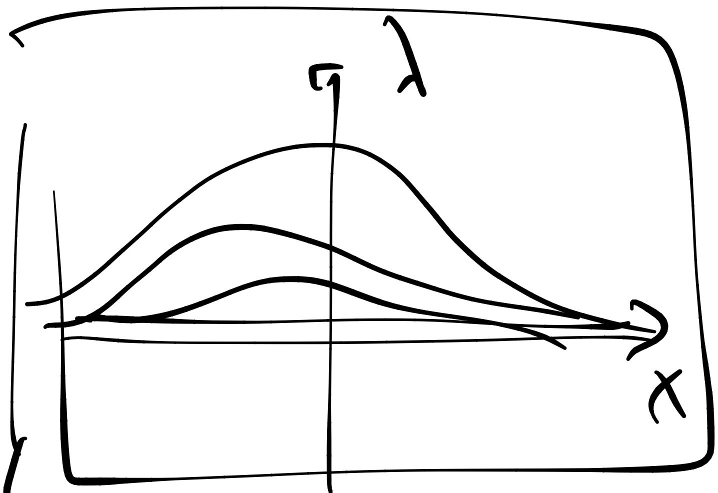
λ : spacetime $\to \mathbb{R}$

Large g is not homotopic to constant map.

eg: suppose spacetime $= S^1_x \times \dots$

($x \simeq x + R$.)

$g(x) = e^{i 2\pi x/R \, n}$ $\quad n \in \mathbb{Z}$.

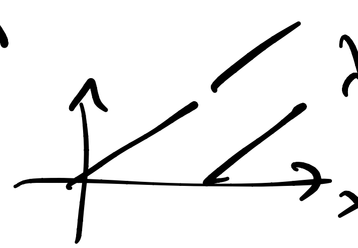


$$\Phi = \oint A \longrightarrow \oint A + 2\pi n$$

flux through x circle

If $g = e^{i\lambda}$ w λ smooth function on S^1_x

then $-i g^{-1} dg = d\lambda$



vs: $\lambda = x$ is not a smooth function.

$g : M \to G$

small : $g \simeq$ constant map to $\mathbb{1}$.

homotopic to

large : all others.