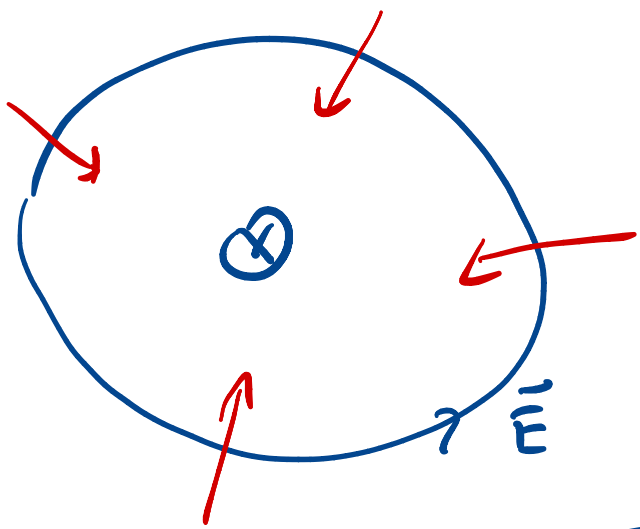
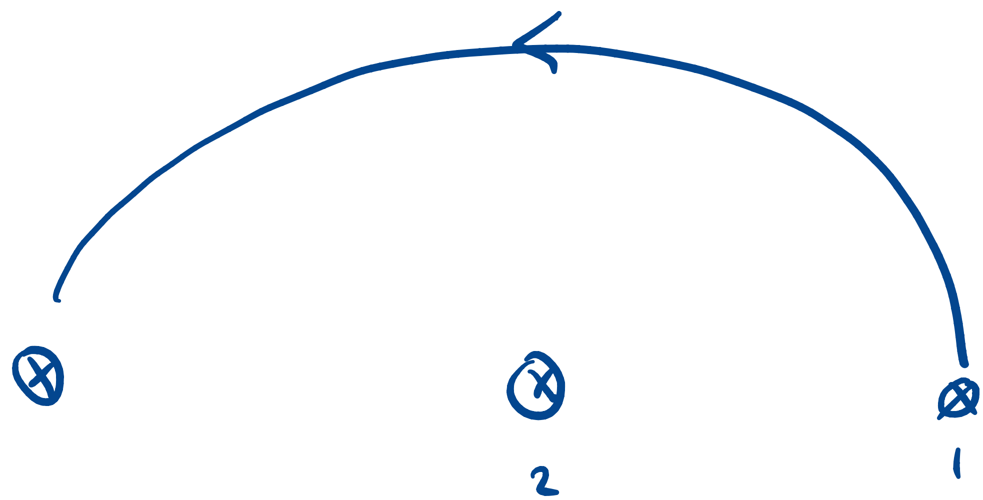
Last time: A gapped 2+1d system w/ a U(1) sym. and $\sigma^{xy} = \frac{e^2}{h}\nu$

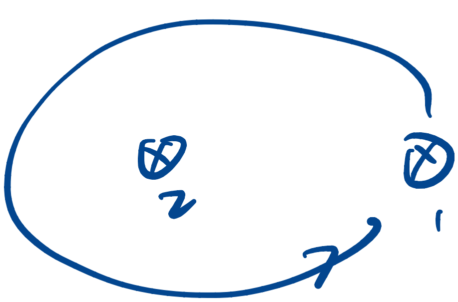
has a quasiparticle w/ charge νe,



and statistics angle $\nu\pi$.



If we move 1 all the way around 2



the phase would be

$$e^{i\nu\oint A} = e^{i\nu 2\pi}$$

→ exchange costs a phase $e^{i\pi\nu}$.

No fractionalization → $\nu \in \mathbb{Z}$ for fermions

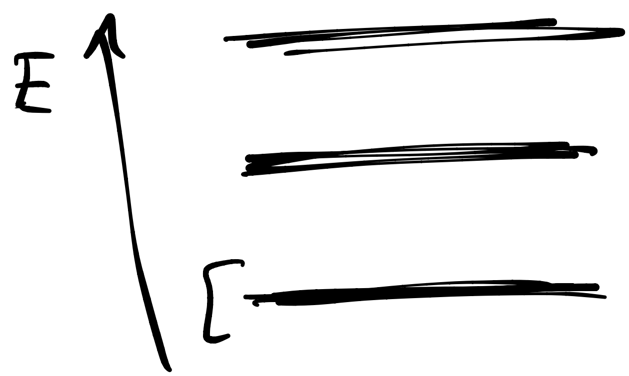
$\underline{\nu \in 2\mathbb{Z}}$ for bosons

<u>Roles of topology</u>: A quantum Hall insulator has

$$\sigma^{xy} = \frac{p}{q}\frac{e^2}{h} \qquad p, q \in \mathbb{Z}$$

<u>IQH</u>: $\underline{q=1}$. can happen for <u>free</u> fermions

$p \in \mathbb{Z}$ ⇐ <u>band</u> topology.



<u>one</u> e^- in 2D in B

or Chern insulator

<u>Not</u> top. order

"top. insulator".

FQHE: $q > 1$ requires interactions
T.O.
fractionalize

So far:
assuming
2+1 D, U(1) $\Longrightarrow$ $\sigma^{xy} = \frac{p}{q}\frac{e^2}{h}$
gap

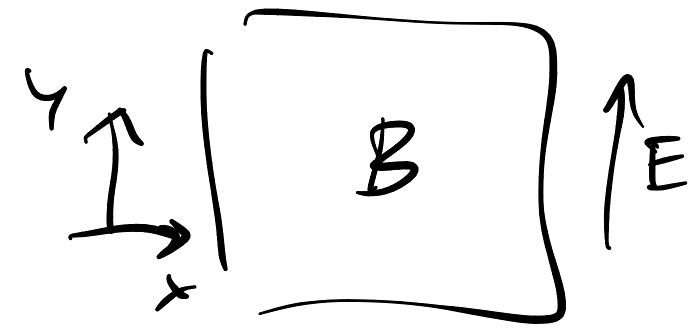
w/ q pr q charge p/q
statistics $\pi\, p/q$.

Translation inv $\Longrightarrow$ $\sigma^{xy} = \frac{\rho e c}{B}$.

Argument: $H = \sum_i K(\Pi_i) + \sum_{i<j} V(r_i - r_j)$

$$\vec{\Pi}_i = \vec{p}_i + \frac{e}{c}\vec{A}(r_i, t).$$

$$\vec{A}(r,t) = (-c\underline{E}t + Bx)\hat{y}$$



$$\begin{cases} \vec{r}_i' = r_i - c\frac{E}{B}t\hat{x} \\ \vec{p}' = p \\ t' = t \end{cases}$$

$$\underline{j_x' = -\rho e\langle \dot{x}' \rangle}$$

$P_y' = P_y = \sum_j (p_j)_y$ is conserved

If $\langle \dot{x}' \rangle \neq 0$

$$\Pi_y' = \sum_i (p_i')_y + \underline{\underline{\frac{e}{c}BX'}}$$

$$= P_y' + \frac{e}{c}BX'$$

would blow up.

$$\Rightarrow j_x' = 0 \Rightarrow \underbrace{j'(r')}_{=0} = j(r) + ec\frac{E}{B}\rho\hat{x}$$

$$\Rightarrow j_x = -\rho e c\frac{E}{B} = \sigma^{xy}E^y$$

2.2 Abelian Chern-Simons theory

$$S_0[a_I] = \sum_{I,J}^{n} \frac{K_{IJ}}{4\pi} \int a_I \wedge da_J$$

Dynamical variables. whence?

Our system has a $U(1)$ symmetry $\hat{j}_\mu$

$\partial^\mu j_\mu = 0$ in $D = 2+1$.

Solve by $j^\mu = \sum_I t^I \epsilon^{\mu\nu\rho} \partial_\nu a^I_\rho / 2\pi$

a smooth $\Longrightarrow \partial_\mu j^\mu = 0$.

charge quantization of j^0 $\longleftrightarrow$ flux quantization of a

$$\oint_S da \in 2\pi\mathbb{Z}.$$

Note: j is inv't under gauge transfs of a.

$$S[a] = \int \left[\underline{\frac{k}{4\pi} a \wedge da} + \underline{\frac{1}{4M} f_{\mu\nu} f^{\mu\nu}} + \dots \right]$$

$[k] = 0 \qquad [M] = 1. \qquad f = da$

$$0 = \frac{\delta S}{\delta a_\lambda} = \frac{k}{2\pi} \epsilon^{\lambda\rho\nu} f_{\rho\nu} + \frac{\partial_\mu f^{\mu\lambda}}{M}$$

$$f^\lambda \equiv \epsilon^{\lambda\rho\sigma} f_{\rho\sigma}$$

$$\Rightarrow \epsilon^{\mu\nu\rho} \partial_\nu f_\rho + \frac{Mk}{2\pi} f^\mu = 0$$

$\underline{\epsilon_{\mu\alpha\beta} \partial^\alpha (\text{RHS})} \qquad \epsilon_{\mu\dots} \epsilon^{\mu\dots} = \delta\delta - \delta\delta$

$$\Rightarrow \left(\partial_\mu \partial^\mu - \left(M \frac{k}{2\pi} \right)^2 \right) f_\rho = 0 .$$

massive excitation w/ mass $\frac{Mk}{2\pi}$

"topologically massive".

At $E \ll M$ ignore maxwell.

$0 = \frac{\delta S_0}{\delta a} = f \Rightarrow$ no local d.o.f.s!

Legendre $\Longrightarrow$ $H = 0$. Theory of groundstate.

Two more pieces of data:

(1) How is it coupled to A, EM?

$$j^I_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial_\nu a^I_\rho$$

$$j_\mu = \sum_I^n t_I j^I_\mu \qquad t_I \in \mathbb{Z} \quad \text{"charge vector"}$$

$$S_{EM}[a, A] = \int A^\mu t_I j^I_\mu .$$

(2) $$S_{qp}[a, j_{qp}] = \int a_I j^I_{qp}$$

OR: include Wilson lines $e^{i\oint_W a^I q_I}$ is gauge inv't.

for some closed curve W, if $q_I \in \mathbb{Z}$.

$$A^2 = \underline{\underline{(\partial\phi + A)^2}}\Big|_{\phi=0}$$

$$\Phi = v e^{i\phi}$$

<u>eg</u>: $S_0[a] = \frac{k}{4\pi}\int a \wedge da$

<u>claim</u>: this describes Laughlin $\nu = \frac{1}{k}$ state.

e^{iS_0} <u>is invt under large $U(1)$ gauge trafs</u>

$\Leftrightarrow$ <u>$k \in \mathbb{Z}$</u>, <u>"level"</u>

$\Rightarrow$ no running

more generally $\underline{\underline{K_{IJ} \in \mathbb{Z}}}$. $\underline{K_{IJ} \overset{\text{IBP}}{=} K_{JI}}$.

Under T or P, $k \to -k$.

<u>item 1</u> : fractional statistics.

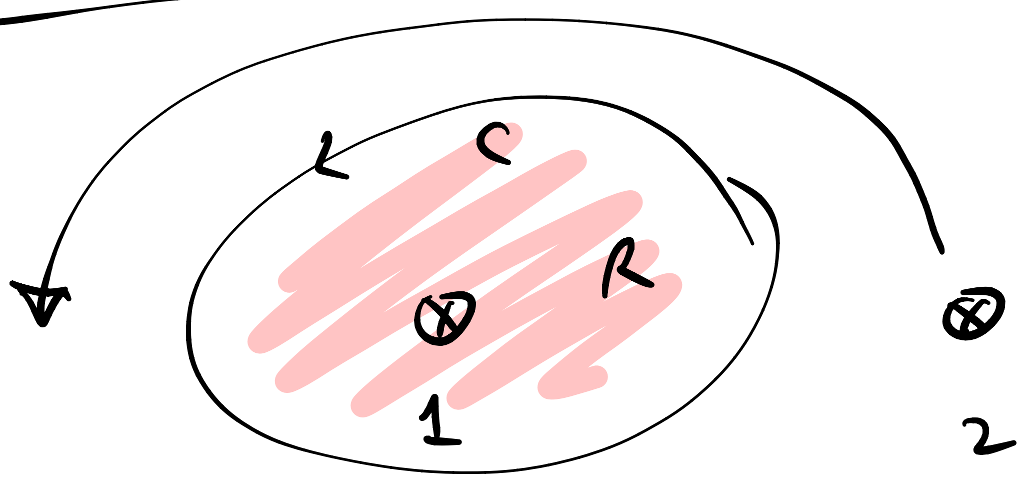
choose $j^{\mu}_{qp}(x) = \int ds\, \delta^3(x^\mu - X^\mu(s))$

$X^\mu(s) = (s, 0, 0)$

<u>eom</u> : $\frac{\delta S}{\delta a^\mu} \sim -\frac{k}{2\pi}\epsilon_{\mu\nu\rho} f_{\nu\rho} + j^{qp}_{\mu}$

<u>$\mu = t$</u> : $\Rightarrow \quad b = \epsilon_{ij} f_{ij} = \frac{2\pi}{k} \rho^{qp}$

FLUX ATTACHMENT.



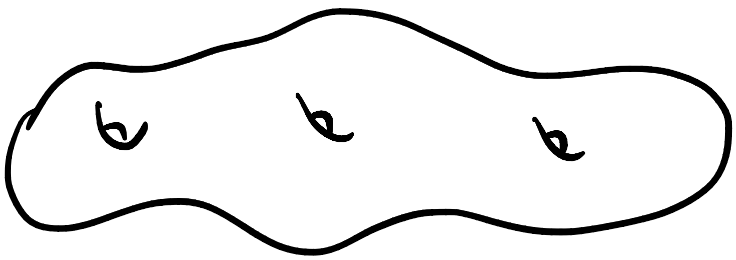
$$2\Delta\varphi_{12} = q_1 \oint_C a = q_1 \int_{R,\, \partial R = C} b = q_1 \frac{2\pi}{k} q_2$$

exchange phase is $\Delta\varphi_{12} = q_1 \frac{\pi}{k} q_2$

~~To describe electrons~~

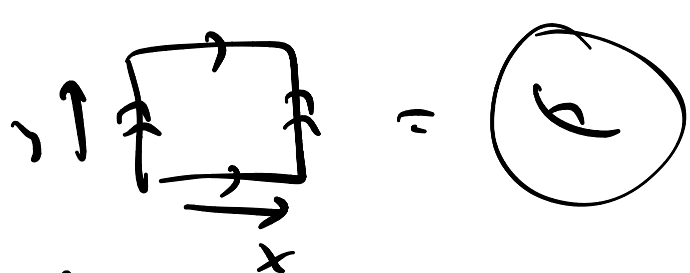
item #2: groundstate degeneracy

$$\#\text{ of groundstates on } \Sigma_g = |\det(\mathbf{K})|^g$$



$g = 3$

simple eg: $K = k$. $\Sigma_g = \Sigma_1 = T^2 = S^1 \times S^1$.



Gauge invt ops: $F_x = e^{i\oint_x a}$, $F_y = e^{i\oint_y a}$

$$S_0[a] = \frac{k}{4\pi}\int dt \int dx\, dx \quad a_x \dot{a}_y$$

(choose $a_0 = 0$ gauge)

$$\frac{\partial L}{\partial \dot{a}_y} = \frac{k}{4\pi} a_x = \pi_{a_y}$$

$$[\pi_{a_y}, a_y] = i\delta$$

$$[a_x, a_y] = \frac{2\pi i \delta}{k}$$

$$\boxed{F_x F_y = F_y F_x e^{2\pi i/k}}$$

$F^{+}_{x,y} F_{x,y} = 1\!\!1$ unitary.

if $F_x |0\rangle = |0\rangle$

then $F_x(\underbrace{F_y |0\rangle}_{|\frac{1}{k}\rangle}) = e^{2\pi i/k}(F_y|0\rangle)$

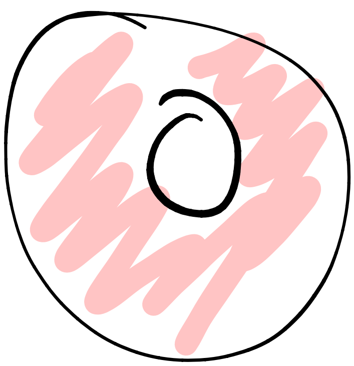
$\Rightarrow$ at least k groundstate.

$$e^{iq\oint_c a} \overset{!}{=} e^{iq\left(\oint_c (a + i g^{-1} dg)\right)} \iff q \in \mathbb{Z}.$$

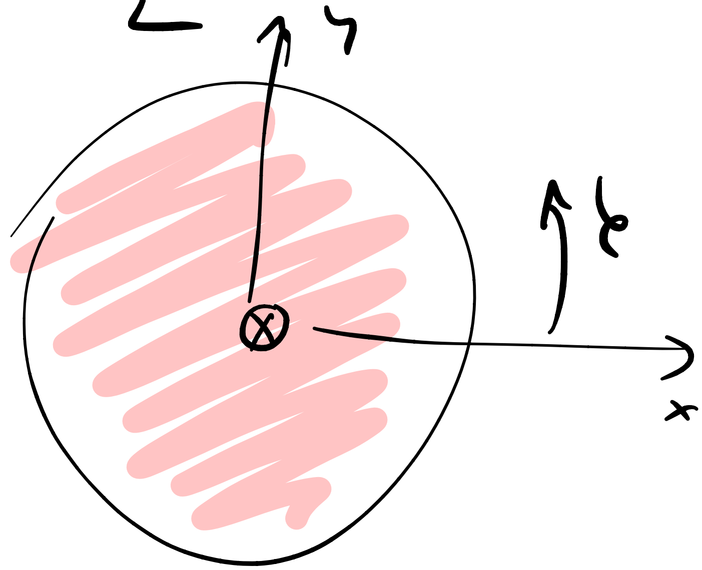
$$S[a] = k\int a \wedge da \wedge da + 0\cdot\int f_{\mu\nu}f^{\mu\nu}$$

[extended state $\Rightarrow \sigma_{xx} \sim \frac{1}{L}$]

$$W = \int \mathcal{E}\cdot j \sim \frac{L}{L}$$



vs



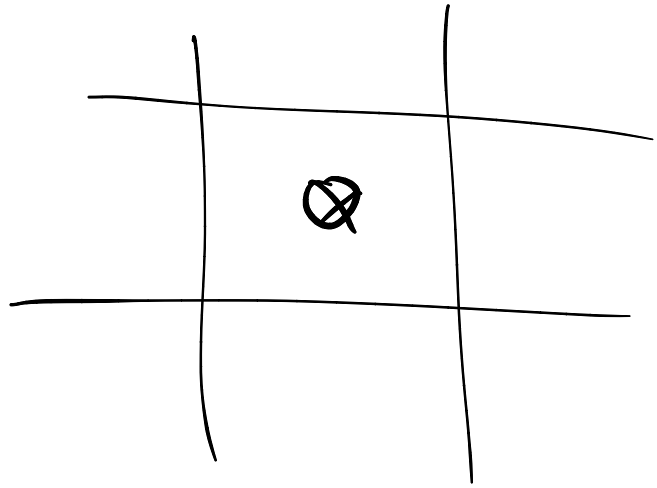
$$A = \Phi_0 \frac{d\varphi}{2\pi}$$

$$A \to A - i g^{-1} dg, \quad \underline{g = e^{i\varphi}}$$

$= 0.$

p wavef'n

$$\Psi \to \Psi \prod_{j=1}^{N} e^{i\varphi_j}$$



$\nu = 1/m$

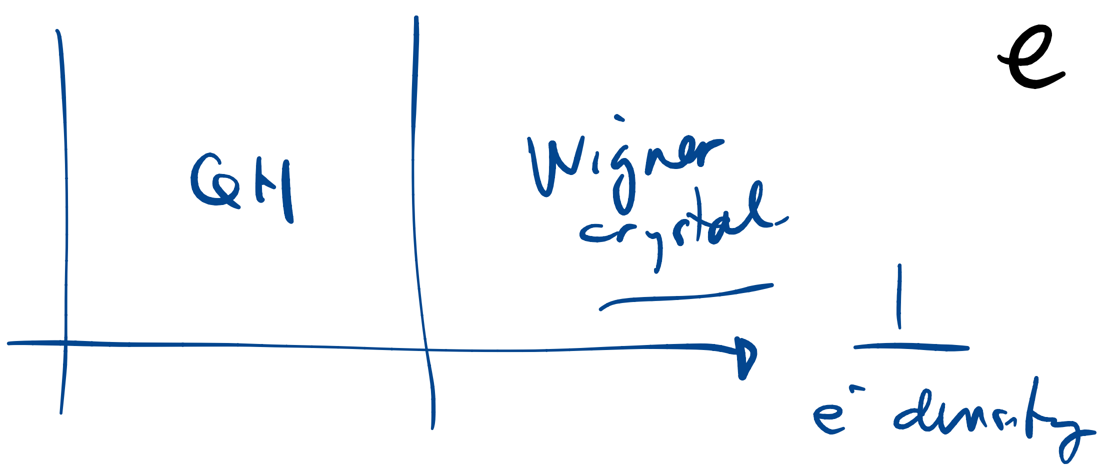
filled Landau Level

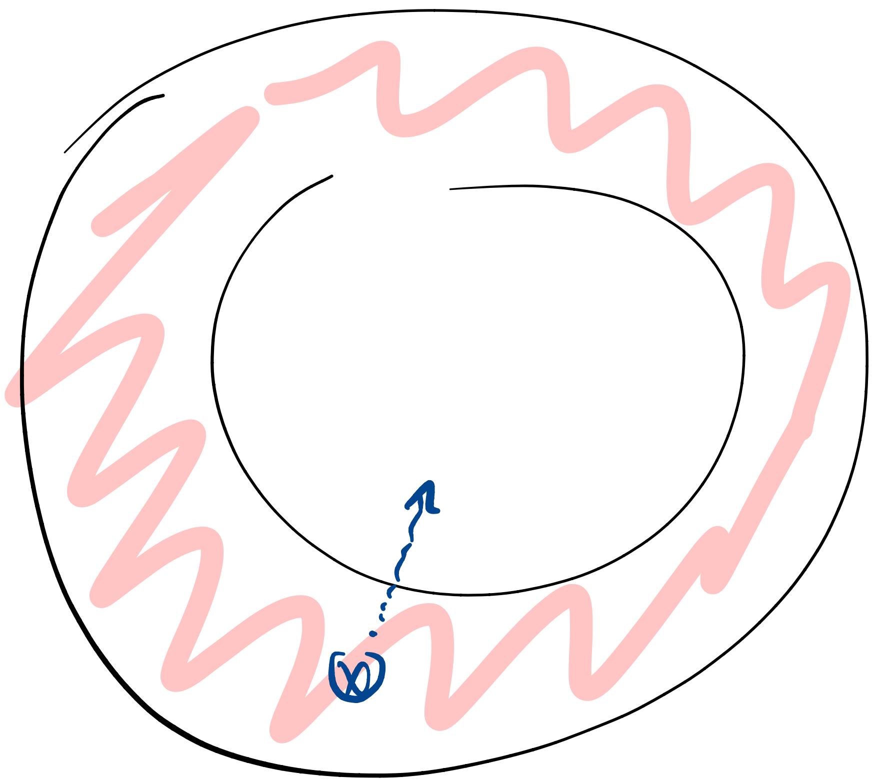
$$|\Psi_m(z)|^2 = \prod_{i<j} \underbrace{|z_{ij}|^{2m}}_{} e^{-\sum_i |z_i|^2/4\ell^2}$$

$$= \exp\left[-\sum \frac{|z_i|^2}{4\ell^2} + 2m \ln |z_i - z_j|\right]$$

looks like RMT

$$\int dM^{N^2} e^{-\mathrm{tr}\, M^2} = \int \prod^N d\lambda_i \overbrace{\prod_{i<j}^N (\lambda_i - \lambda_j)^2}^{} e^{-\sum \lambda_i^2}$$





$$\langle b(r)\, b(0)\rangle \sim \frac{1}{r^3} \Rightarrow [b] = \frac{3}{2}.$$

$$= \int \mathcal{D}a \; e^{-\int (\partial a)^2} \; b(r)\, b(0)$$

$$a \sim a + d\lambda$$

NONCOMPACT.

$$\neq a + g^{-1} dg \Rightarrow [a] = 1.$$

$$P: \quad a \leftrightarrow b$$

$$S[a,b] = \int a \wedge db + \int b \wedge da + \int_{e\text{-particles}} a + \int_{m\text{-particles}} b$$

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$H^2(T^2, \mathbb{Z}) = \mathbb{Z}.$$

$$\exists \int F/_{2\pi} = n$$

$$F = \frac{dx}{L_x} \wedge \frac{dy}{L_y} \, 2\pi n$$

$$A = x \, dy \, \frac{2\pi}{L_x L_y} n$$

$$e^{i \oint A} \ \checkmark$$

$$A \to A + i g^{-1} dg$$

$$g = e^{\frac{2\pi i x}{L_x} n}$$