When does a gapped, two-dimensional electron system without symmetries have protected, gapless edge modes?

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In this paper, we review some criteria for when an abelian, gapped, two-dimensional ($2D$) electron system without any symmetries has protected, gapless edge modes [1]. The primary result is that such systems only have *gapped* boundaries if there is a collection of bulk quasi-particles that have trivial mutual statistics with eachother, but non-trivial mutual statistics with particles outside of the collection. If no such collection exists, then the boundary must have protected, gapless edge modes.

I. INTRODUCTION

The general phenomena of gapless, protected edge modes at the boundary of gapped phases is ubiquitous in the condensed matter literature, occuring, for example, in topological insulators and superconductors, and in quantum Hall states [2–5]. These different topological materials seemingly have disparate mechanisms and criteria for the existence, and robustness, of their edge modes. What was less well known prior to the publication of [1] was exactly when a $2D$ gapped electron system, without any symmetries, has protected edge modes. A partial answer can be found by studying the thermal hall conductivity K_H at low temperatures,

$$K_H = (n_L - n_R) \cdot \frac{\pi^2 k_B^2}{3h} T$$

where n_L and n_R are the number of modes of left and right chirality, respectively. Hence, when $K_H \neq 0$, the edge is protected because in general any perturbation will gap left and right moving modes in equal numbers (assuming energy conservation is not broken). More interestingly, it turns out that systems with $K_H = 0$ can still have protected edge modes, provided that they support fractionalized quasiparticles. The main result of [1] is that a gapped edge is possible (in a electron system with abelian statistics and $K_H = 0$) if and only if there exists a set of quasi-particle types $\mathcal{M}$ (called a *Lagrangian subgroup* of $\mathcal{L}$, the full bulk quasi-particle spectrum) satisfying two properties:

C1: The particles in $\mathcal{M}$ have trivial mutual statistics.

C2: Any particle that is not in $\mathcal{M}$ has non-trivial mutual statistics with at least one particle in $\mathcal{M}$.

Conversely, if there exists no such $\mathcal{M}$, then a gapped edge is impossible i.e. the system has protected, gapless edge modes.

As an example, consider the fractional quantum hall states $\nu = 2/3$ and $\nu = 8/9$, each coupled separately to a superconductor, thereby breaking charge conservation at the edge. Both systems have $K_H = 0$. However, only the $\nu = 2/3$ edge is protected because the fractionalized quasi-particle types in the bulk don't allow for a Lagrangian subgroup that meets the above two conditions, whereas for $\nu = 8/9$ they do.

This paper will be structured as follows: in II, we will summarize the three main arguments put forward in [1] to prove the above claim on when a gapped edge is possible. The first argument follows from a microscopic analysis of the edge and proves the criteria are sufficient but not necessary for a gapped edge. The next two arguments follow from quasi-particle braiding relations and modular invariance, and they prove that the criteria is necessary for a gapped edge but not sufficient. Together, the three answers prove the desired result. In III, we refer to some articles that extend the work presented in this paper.

II. THREE ARGUMENTS

First, let us show that the conditions $C1, C2$ are sufficient for a gapped edge using a microscopic analysis. The low-energy physics is described by a $U(1)^p$ Chern-Simons (CS) theory with $p \times p$ integer K-matrix. We assume $p = 2N$ and K has zero signature (otherwise the edge is known to be protected). The quasi-particle excitations of this system can be labelled by their p-component, integer charge vectors l under the CS gauge fields, and their mutual stastics are given by $\theta_{ll'} = 2\pi l^T K^{-1} l'$. Excitations with an integer number of electrons are given by charge vector $K\Lambda$ where $\Lambda \in \mathbb{Z}^p$ (notice that these always have ± 1 exchange statistics). Since this is a system made out of electrons, it should host a topologically trivial quasi-particle with π exchange statistics, which means one of the diagonal elements of K should be odd.

We can now translate the conditions $C1, C2$ into conditions on the charge vectors $\mathcal{M} = \{m\}$. Essentially, the first condition says that $m^T K^{-1} m' \in \mathbb{Z}\ \forall m, m' \in \mathcal{M}$, and the second condition says $m^T K^{-1} l \notin \mathbb{Z}$ if l is not equivalent to any vector in $\mathcal{M}$, and $m \in \mathcal{M}$. To prove that these conditions are sufficient for a gapped boundary, let us assume that there exists such an $\mathcal{M}$. Then there exists a set of N integer vectors Λ_i such that $\Lambda_i^T K \Lambda_j = 0$ for all i, j, and we can gap out the edge by adding backscattering terms to the boundary [1].

$$\sum_{i=1}^{N} U(\Lambda_i), \qquad U(\Lambda) = U(x)\cos(\Lambda^T K\phi - \alpha(x))$$

The microscopic approach relies on abstract mathematical facts, and so now we turn to a braiding statistics argument that shows that the criteria are necessary and also provides some physical intuition. Consider a gapped $2D$ electron system with a gapped edge, and let $\mathcal{M}$ be the set of bulk quasi-particle excitations that can be annihilated (or condensed) at the edge with some local operators. Suppose we start in the ground state $|\Psi\rangle$, where we create $m, \bar{m} \in \mathcal{M}$ in the bulk, transport them along a curve β to two distant sites a, b on the edge, and then act with local operators U_a, U_b to annihilate them, returning us to the ground state. Let $\mathbb{W}_{m\beta} = U_a U_b W_{m\beta}$ implement this on the groundstate.

Now suppose we repeat this process with two sets of anyons $m, \bar{m}$ and $m', \bar{m}'$ along paths β and γ, respectively, as indicated in Fig. 1. On the ground state, $\mathbb{W}_{m\beta}$ and $\mathbb{W}_{m'\gamma}$ commute since they both act trivially. However, commuting $\mathbb{W}_{m\beta}$ and $\mathbb{W}_{m'\gamma}$ generally will produce a factor related to their mutual statistics $e^{i\theta_{mm'}}$. That means if both m, m' annihilate on the boundary, they must have trivial mutual statistics.

Suppose further that anyon $l \notin \mathcal{M}$ is a nontrivial quasiparticle and does not annihilate on the boundary. If l has trivial statistics with each $m \in \mathcal{M}$, how would we detect it away from the boundary? Both the edge and bulk are, by assumption, gapped, and therefore have a finite correlation length. Anyons are only detectable from far away via Aharanov-Bohm measurements, so if it has trivial statistics with every boundary excitation, we would have no way of knowing it's there. Therefore, it must have nontrivial mutual statistics with at least one $m \in \mathcal{M}$. The experiment that would detect this phase is indicated in Fig. 2. This concludes the argument that a gapped $2D$ electron system with gapped boundary must satisfy conditions $C1, C2$.

The last argument presented in [1] involves modular invariance–the strategy is to use the fact that any $1D$ CFT realized by a lattice model must be modular invariant to prove the existence of $\mathcal{M}$ for a gapped electron system. Let us recall the meaning of modular invariance in CFTs. Consider a $2D$ conformal field theory defined on a torus of "shape" $\tau \in \mathbb{C}$, so that $z \equiv z+1 \equiv z+\tau$. Since two tori of shapes τ and $-1/\tau$ are conformally equivalent to one another, the CFT partition function must satisfy $Z(\tau) = Z(-1/\tau)$ [6]. This puts various constraints on the operator content of the theory.

To use CFT, we put the system in a semi-infinite plane with $-\infty < x < \infty, -L_y/2 < y < L_y/2$, and make the top edge gapless and the bottom gapped. Again, take $\mathcal{M}$ to be the set of quasi-particles that can be annihilated at the gapped edge. The gapless edge can be modelled by

$$L = \frac{1}{4\pi}\partial_x\phi_I(K_{IJ}\partial_t\phi_J - V_{IJ}\partial_x\phi_J)$$

where K is again a $2N \times 2N$ integer, symmetric matrix describing the bulk theory. We can change variables and tune the interactions on the edge so that so that all the modes propogate at the same speed $|v|$ and the top edge is described by a conformal field theory.

What are the allowed scaling operators in this system? One set of operators involves derivatives of the primary fields, and can be written as $\mathcal{O}_{n_{J,k}} = \prod_{J=1}^{2N}\prod_{k=1}^{\infty}(\partial_x^k\phi_J)^{n_{J,k}}$. These operators are charge neutral. However, we can also have charged operators $e^{il^T\phi}$, provided that l is equivalent to some $m \in \mathcal{M}$. The reason is that such charged operators implement annihilation (or creation) of fractionalized quasiparticles, which in general is not allowed in the low energy theory, unless that excitation can be deposited at the lower gapped edge as part of a tunneling process. Therefore, the most general scaling operators describing this system are $e^{i(m+K\Lambda)^T\phi}\mathcal{O}_{n_{J,k}}$.

With this insight we can now apply the machinery of CFT and modular invariance to put conditions on $\mathcal{M}$. The first step is to decompose the partition function into sums over operators labelled by m, and then to use a transformation law and modular invariance to get an equation in terms of the modular S matrix, which we assume on general grounds is equivalent to the topological S matrix whose matrix elements characterize the bulk quasi-particle braiding statistics.

$$Z(\tau) = \sum_{m\in\mathcal{M}} Z_m(\tau) = Z(-1/\tau) = \sum_{m\in\mathcal{M}, l\in\mathcal{L}} S_{ml}Z_l(\tau)$$

The above equation implies that $\sum_{m\in M} S_{ml} = \delta_{l\in\mathcal{M}}$ from which the conditions $C1, C2$ immediately follow [1].

III. CONCLUSION

In this paper, we presented some criteria for when a gapped $2D$ electron system has a gapped boundary; taking the contrapositive, when know when it has protected edge modes. There are various extensions of the above argument. The extension to bosonic systems is presented in an appendix in [1] and is argumentatively similar. The main focus of this paper was for abelian systems, but one can also consider the non-abelian case. Gapped boundaries and domain walls in abelian and non-abelian topological orders in $2D$ can be examined in terms of a tunnelling matrix for anyon types, or through the Frobenius algebra of the boundary condensate [7–10]. The general framework for exploring more general topological phases and topological phase transitions can be found in Levin-Wen models [11]. The previous arguments precluded systems with symmetry. Incorporating symmetry, the existence and robustness of edge modes falls under the study of symmetry-protected topological phases [12], a vast and rich subject that is continuing to develop.

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Appendix A: Figures

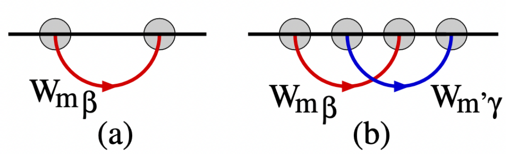

FIG. 1. Figure three in [1]. Image (a) depicts a process where a quasi-particle and its inverse $m, \bar{m}$ are created in the bulk, transported along β to the gapped boundary, and annihilated by local operators U_a, U_b supported on the grey circles. We assume that this is possible using local operators because the bulk and the boundary are gapped. Image (b) depicts a process where two bulk quasiparticles which annihilate on the boundary are created in pairs, $m, \bar{m}$ and $m', \bar{m}'$. The operators that implement this process only intersect once on the worldlines of the anyons, which is why their commutator involves the mutual statistics of m, m'.

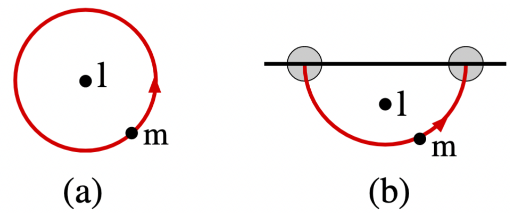

FIG. 2. Figure four in [1]. Image (a) depicts a property of non-trivial bulk quasi-particles–if l cannot be annihilated in the bulk, then it must have non-trivial mutual statistics with some bulk quasi-particle m. Otherwise, it would not be detectable from far away. Image (b) depicts a version of that property for the boundary–if l cannot be annihilated at the edge, it must have non-trivial mutual statistics with some quasi-particle m which can be. Otherwise, there is no way to detect it at the edge from far away.