How Non-Abelian topology exists in momentum space of free fermion systems

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Nodal-line metal is a field which attracts numerous attentions while many detail aspects of it is still remain unknown. Recent study shows that some special Nodal-line system can be described by topological charges. Some general cases also displays non-Abelian topology. In this short essay, I summarize the result from multiple publication which describe such system, using the language of homotopy group.

I. INTRODUCTION

The topic of nodal-line and nodal-chain metals had been widely discussed in the past. However, many of the properties of nodal-line and nodal-chain metals are still unknown. A type of topological invariant, non-Abelian charge of nodal lines has been discovery under momentum space of metals with weak spin-orbit coupling with time-reversal ($\mathcal{T}$) and parity ($\mathcal{P}$) symmetry has been discovered [1]. The author in this paper claims that this type of non-Abelian topology in k-space is different from the non-Abelian statistics of anions since the former one does not depend on any kind of interactions, nor superconductivity. Furthermore, the analogy between the nodal-line under $\mathcal{PT}$ symmetry and non-Abelian vortex lines in biaxial nematic liquid has also been made. In order to discuss non-abelian charge generated from the system we firstly need to understand the basic definitions.

A. Nodal Line Degeneracy

Nodal line is the line degeneracy of electron energy band near the Fermi energy. We may start from a very basic two-band model to explain it and furthermore set the foundation for the discussion of non-Abelian topology mentioned in previous paragraph: two-band system.

In general, for any two-band system, we may write the Hamiltonian in k-space as

$$\mathcal{H}(\boldsymbol{k}) = h_0(\boldsymbol{k})\mathbb{1} + \vec{h}(\boldsymbol{k}) \cdot \vec{\sigma} \tag{1}$$

where $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ composed of 3 Pauli matrices and $\mathbb{1}$ is 2×2 unit matrix. Moreover we have a constraint on $\vec{h} = (h_x, h_y, h_z)$ that three components are all real functions. In the introduction part it is mentioned that we are interested in the system with spin-orbit coupling absent. Under this scenario, we can set $\mathcal{PT} = \mathcal{K}$, which is the complex conjugation. The system with such symmetry should include constraint that removes σ^y term from original Hamiltonian (Eq.1). Therefore in order to obtain degeneracy, we need the constraint that

$$h_x(\boldsymbol{k}) = h_z(\boldsymbol{k}) = 0 \tag{2}$$

which includes two independent constraints and solution should be a line in 3-d momentum space.

We may go further with some elaboration on the 2-band system we are discussion in previous section, with a mirror symmetry added: $m_z : z \mapsto -z$, which corresponds to operator $\hat{m}_z = \sigma^z$, which makes the h_x in Eq.1 an odd function and h_z an even function of k_z. The paper [1] offers a set of examples:

$$h_x(\boldsymbol{k}) = k_x k_z \ , \ h_z(\boldsymbol{k}) = \pm k_x^2 + k_y^2 \pm k_z^2 - b^2 \tag{3}$$

with b a constant. By solving the equation 2 we may obtain the nodal line for this system, visually shown in Fig.1 A-D (In the figure b is set to be 2). In the figure we may identify that there are two nodal-lines(marked in red in each subfigure) crossing with each other: one in symmetry plane (xy plane) and another outside.

II. RELATION TO HOMOTOPY THEORY

We may normalized the general Hamiltonian in previous system as

$$\mathcal{H} = \mathbb{1} - 2\left|u_{\boldsymbol{k}}^{\circ}\right\rangle\left\langle u_{\boldsymbol{k}}^{\circ}\right| \tag{4}$$

with $|u_{\boldsymbol{k}}^{\circ}\rangle$ the cell-periodic component of the two-component Bloch function corresponding to the lower band. Therefore we may construct the Hamiltonian in this format using planar vector. However the symmetry $\mathcal{K} = \mathcal{PT}$ added a "gauge redundancy" upon the sign of state or the vector (mapping $|u_{\boldsymbol{k}}^{\circ}\rangle \mapsto -|u_{\boldsymbol{k}}^{\circ}\rangle$ will not change anything). Therefore we may identify the order-parameter space as $M_{(1,1)} = S^1/\mathbb{Z}_2 \simeq S^1$ and the loop around linear defect under such space is described by homotopy group

$$\pi_1(M_{(1,1)}) = \mathbb{Z} \tag{5}$$

in this case is exactly the winding number. Meanwhile, since mirror symmetry $\hat{m}_z = \sigma^z$ separate regions in parameter space into two sets with different $\hat{m}_z$ eigenvalue of $|u_{\boldsymbol{k}}^{\circ}\rangle$, which can only be $\pm 1 = \mathbb{Z}_2$. Taking $\lambda_{\boldsymbol{k}}^{\circ}$ as the eigenvalue with $\boldsymbol{k}$ specified so we may cut any loop going through the symmetry plane into two halves, each has two endpoints $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ on symmetry plane. By setting $\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{k}} = 1$ we may also set a winding number (imagining closing from trivial line inside symmetry plane) to each half loop. The closed image generated by connecting the half-loop with its self reflection will for a complete loop with twice of winding number [2].

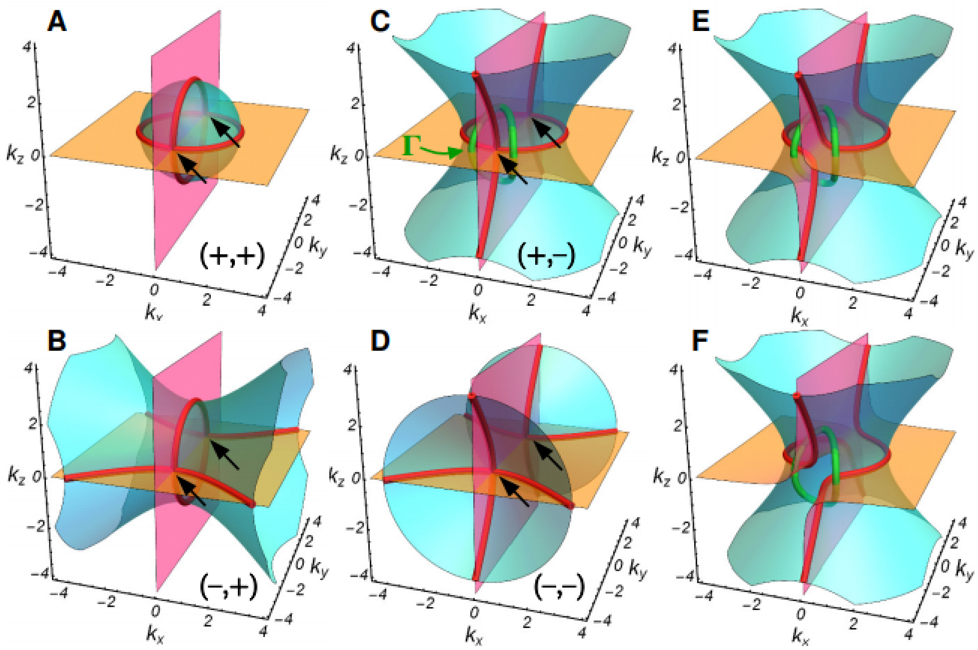

FIG. 1. Figure A-D: Plot for nodal lines under different sign for h_z (in lower-right corner); Figure E,F: Plot for nodal line without mirror symmetry

III. NON-ABELIAN TOPOLOGY

Similarly to what we did in previous section, we can expand to case of general N-band scenario: Let's set $N = 3$. By spectrum decomposition we may write hamiltonian as

$$\mathcal{H}_N(\boldsymbol{k}) = \sum_{j=1}^{N} \varepsilon_j \left| u_{\boldsymbol{k}}^j \right\rangle \left\langle u_{\boldsymbol{k}}^j \right| \tag{6}$$

which forms a parameter space spanned by unit vector chosen in basis $\{u_{\boldsymbol{k}}^j\}$. Also similar to last section, we notice that transformation $\left| u_{\boldsymbol{k}}^j \right\rangle \mapsto - \left| u_{\boldsymbol{k}}^j \right\rangle$ also keep Hamiltonian unchanged so the space is written as

$$M_3 = SO(3)/D_2 \tag{7}$$

and the fundamental group reads

$$\pi_1(M_3) = Q = \{\pm 1, \pm j, \pm i, \pm k\} \tag{8}$$

which is the quarternion group.

IV. CONCLUSION

In this paper, I summarized the definition of Nodal-line degeneracy which happens in many metals and their relation with topology charges discovered in some research [1–3]. Moreover in this paper I have also introduced the difference between topological charges embedded in system with two band and three band and the non-Abelian nature of 3-band case.

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