Generalized string-net models and the Fibonacci string-net model

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The paper describes how to construct generalized string-net models, a class of exactly solvable lattice models that realize a large family of 2D topologically ordered phases of matter. The ground states of these models can be visualized as superpositions of different "string-net configurations". Each string-net configuration is a trivalent graph with labeled edges, drawn in the xy plane. Here, unlike the original string-net construction, one can relaxe the tetrahedral reflection symmetry requirement by specifying an appropriate gauge choice. By this generalized string-net model construction, one can analyze the Fibonacci string-net model, which is an example of a non-abelian string-net. Please see the full paper if interested DOI: 10.1103/PhysRevB.103.195155.

INTRODUCTION

One of the greatest triumphs in many-body physics is the description of Landau's theory of symmetry breaking. For many years, it was thought that Landau's theory of symmetry breaking could describe nearly all phases and phase transitions. However, in recent decades, it was realized that a new type of order, namely the topological order, is beyond the scope of Landau's theory. In order to give a description of the topological phases of matter, tensor category formalism is introduced. Within the tools of tensor category, one can analyze the topological phases of matter using a well-established method called (original) string-net construction. Here in this paper, we introduce a generalized string-net construction which has the following advantages: (i)all the properties of these constructions are using simple algebraic calculations that do not require any knowledge of tensor category; (ii) under an appropriate choice of the generalized string-net construction no longer requires the tetrahedral reflection symmetry

Moreover, we will use the generalized string-net construction to solve the Fibonacci string-net model, which is a non-abelian string-net model, then classify the quasiparticle excitations by their statistics.

GROUND STATE WAVE FUNCTION

The ground state $|\Phi\rangle = \sum_{X\in\mathcal{H}} \Phi(X)|X\rangle$ of our models is a superposition of different string-net configurations $|X\rangle$ in $\mathcal{H}$. The state $|\Phi\rangle$ is described implicitly by the constraint equations in fig[1].

These equations are defined in the Hilbert space $\mathcal{H}$ where the configurations on both sides of the equations satisfy branching rules at every vertex. Here $a, b, c, ...$ are arbitrary string types (including the null string types) and the shaded regions represent arbitrary string-net configurations which are not changed from one side of the equation to the other. The symbol $\delta_{c,d} = 1$ if $c = d$ and $\delta_{c,d} = 0$ otherwise. The parameters $F^{abc}_{def}, \tilde{F}^{abc}_{def}$ are complex numbers that depend on 6 string types $a, b, ..., f$

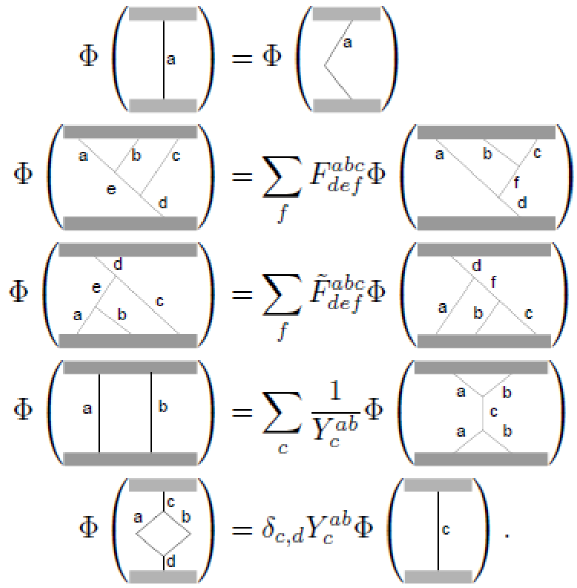

FIG. 1: Local constraint equations

obeying the appropriate branching rules: $\delta^{ab}_{e} = \delta^{ec}_{d} = \delta^{bc}_{f} = \delta^{af}_{d} = 1$. Likewise, Y^{ab}_{c} is a complex number that depends on three string types a, b, c obeying the branching rule $\delta^{ab}_{c} = 1$.

Then, by using the convention that:

$$\Phi(vacuum) = 1 \tag{1}$$

One can relate the amplitude of any string-net configurations to the amplitude of the vacuum using the operation introduced in fig[1]. Thus, once the parameters $\{F^{abc}_{def}, \tilde{F}^{abc}_{def}, Y^{ab}_{c}\}$ are given, the rules determine the wave function completely.

Here, the input data $\{F^{abc}_{def}, \tilde{F}^{abc}_{def}, Y^{ab}_{c}\}$ must satisfies the *self-consistent* conditions which can be understood graphically:

Algebraically, the *self-consistent* rules can be written as:

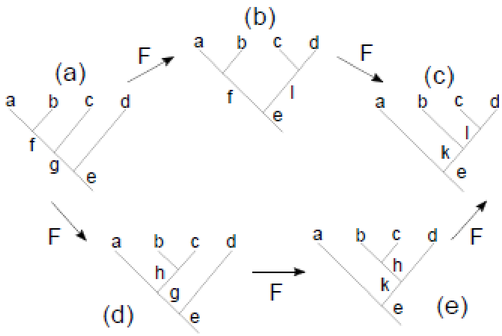

FIG. 2: Two dfferent ways to relate the amplitude of (a) to the amplitude of (c). Consistency requires the two sequences of operations give the same result.

$$F^{fcd}_{egl} F^{abl}_{efk} = \sum_h F^{abc}_{gfh} F^{ahd}_{egk} F^{bcd}_{khl} \tag{2}$$

$$\tilde{F}^{abc}_{def} = (F^{abc}_d)^{-1}_{fe} \frac{Y^{ab}_e Y^{ec}_d}{Y^{bc}_f Y^{af}_d} \tag{3}$$

$$F^{abc}_{def} = \tilde{F}^{abc}_{def} = 1 \quad if\ a\ or\ b\ or\ c = 0 \tag{4}$$

$$Y^{ab}_c = 1 \quad if\ a\ or\ b = 0 \tag{5}$$

Eqs.(2) (3) (4) (5) are the only conditions that we will impose on$\{F^{abc}_{def}, \tilde{F}^{abc}_{def}, Y^{ab}_c\}$.

QUASIPARTICLE EXCITATIONS

Once the parameters $\{F^{abc}_{def}, \tilde{F}^{abc}_{def}, Y^{ab}_c\}$ is given. One can identify a set of *stringoperators* $\{W_\alpha(P)\}$, which act along oriented paths P. We require each string operator to act on the string-net ground state in a way that is path independent. When $\{W_\alpha(P)\}$ is applied to a string-net state $\langle X|$, its action is described graphically by adding a string labeled by α along the path P *under* the preexisting string-nets:

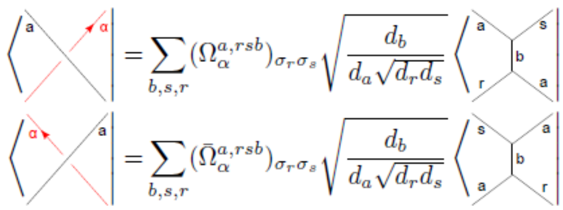

FIG. 3: Path operation rules.

After finding the quasiparticles, one can compute the braiding statistics, which can be described by the S matrix $S_{\alpha\beta}$ and the topological spins $\theta_{(\alpha)}$. In terms of string operator:

$$S_{\alpha\beta} = \frac{1}{D} \sum_{stb} \mathrm{Tr}(\bar{\Omega}^{t,ssb}_\alpha) \mathrm{Tr}(\bar{\Omega}^{t,ttb}_\beta) d_b \tag{6}$$

Here, we have used a formula that expresses the quantum dimension of α in terms of string operator data, namely:

$$d_\alpha = \sum_s n_{\alpha,s} d_s \tag{7}$$

RELATIONSHIP WITH ORIGINAL STRING-NET CONSTRUCTION AND THE TETRAHEDRAL REFLECTION SYMMETRY

In this section, we relate this general string-net construction to the original string-net construction introduced in Ref.[1]. More precisely, we can find a relationship between the input data $\{F, Y\}$ introduced here and $\{\bar{F}, \bar{d}\}$ from the original construction method. Under a special gauge choice $\omega_a \sqrt{d_a} = \nu_a$, the relationship is given by:

$$\begin{aligned} Y^{ab}_c &= \frac{\nu_a \nu_b}{\nu_c} \\ F^{a\bar{b}b}_{ac0} &= \frac{\nu_c}{\nu_a \nu_b} \\ F^{abc}_{def} &= F^{\bar{e}b\bar{f}}_{\bar{d}\bar{a}\bar{c}} \frac{\nu_e \nu_f}{\nu_a \nu_c} = F^{\bar{d}cb}_{\bar{a}\bar{e}f} = F^{ba\bar{d}}_{\bar{c}e\bar{f}} \end{aligned} \tag{8}$$

The last equation of eqs.(8) could be understand graphically. Under the original string-net construction, the ground state amplitudes are required to be invariant under the tetra hedral reflection fig[4]:

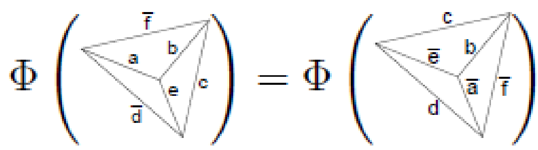

FIG. 4: Tetrahedral reflection of the string-net configuration

Algebraically, this can be written as:

$$F^{abc}_{def} Y bc_f Y^{af}_d Y^{d\bar{d}}_0 = F^{\bar{e}b\bar{f}}_{\bar{d}\bar{a}\bar{c}} Y^{b\bar{f}}_{\bar{c}} Y^{\bar{e}\bar{c}}_{\bar{d}} Y^{d\bar{d}}_0 \tag{9}$$

Choosing the gauge $Y^{ab}_c = \sqrt{\frac{d_a d_b}{d_c} \frac{\nu_a \nu_b}{\nu_c}}$ then plug this into eq.(9), one can find that eq.(9) and the last equation of eqs.(8) are identical. In the original string-net construction, one must assume that the string-net model is invariant under the tetrahedron rotation, however, here we can relax this condition by fixing an appropriate gauge. Therefore, tetrahedral reflection symmetry is no longer assumed.

EXAMPLE:FIBONACCI STRING-NET MODEL

Consider a non-abelian example, namely the Fibonacci string-net model. The string types in the Fibonacci string net are $\{0,1\}$ where 0 is the vacuum string and $1=\bar{1}$ is self dual. The allowed branching rules are $\{(0,0:0),(0,1:1),(1,0:1),(1,1:0),(1,1:1)\}$. The solution is given by

$$\begin{aligned}
&[F_1^{111}]_{ef} = \begin{bmatrix} \frac{1}{d} & \frac{1}{\sqrt{d}} \\ \frac{1}{\sqrt{d}} & -\frac{1}{d} \end{bmatrix}_{ef} \\
&Y_0^{11} = d, \qquad Y_1^{11} = \sqrt{d}, \quad other F, Y = 1 \\
&d = \frac{1+\sqrt{5}}{2}
\end{aligned} \tag{10}$$

where $e,f=0,1$. This is expected, as the Fibonacci string net can be realized by the original construction.

To find the quasiparticle excitations, we get four irreducible solutions which correspond to four distinct quasiparticles:

$$\begin{aligned}
&\alpha = 1\ : (n_{\alpha,0}, n_{\alpha,1}) = (1,0) \\
&\qquad \Omega_\alpha^{1,001} = 1 \\
&\alpha = 2\ : (n_{\alpha,0}, n_{\alpha,1}) = (0,1) \\
&\qquad \Omega_\alpha^{1,110} = e^{-i4\pi/5}, \quad \Omega_\alpha^{1,111} = e^{i3\pi/5} \\
&\alpha = 3\ : (n_{\alpha,0}, n_{\alpha,1}) = (0,1) \\
&\qquad \Omega_\alpha^{1,110} = e^{i4\pi/5}, \quad \Omega_\alpha^{1,111} = e^{-i3\pi/5} \\
&\alpha = 4\ : (n_{\alpha,0}, n_{\alpha,1}) = (1,1) \\
&\qquad \Omega_\alpha^{1,110} = 1, \quad \Omega_\alpha^{1,001} = -d^{-2}, \quad \Omega_\alpha^{1,111} = d^{-2} \\
&\qquad \Omega_\alpha^{1,101} = (\Omega_\alpha^{1,101})^* = \sqrt{3d-4}e^{-3i\pi/10}
\end{aligned} \tag{11}$$

All quasiparticles are self-dual $\alpha=\bar{\alpha}$. And also, the quantum dimensions of the quasiparticles are $d_1=1$ and $d_2=d_3=d$ and $d_4=d^2$. The topological spins and the S matrix can be computed:

$$\begin{aligned}
&e^{i\theta_1} = 1, \quad e^{i\theta_2} = e^{-i4\pi/5}, \quad e^{i\theta_3} = e^{i4\pi/5}, \quad e^{i\theta_4} = 1 \\
&S = \frac{1}{1+d^2}\begin{bmatrix} 1 & d & d & d^2 \\ d & -1 & d^2 & -d \\ d & d^2 & -1 & -d \\ d^2 & -d & -d & 1 \end{bmatrix}
\end{aligned} \tag{12}$$

The same result was found in Ref.[1].

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